Dedicated Technical Progress with a Non-renewable Resource : Efficiency and Optimality

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Abstract

We consider the implications of a non renewable resource constraint over the growth possibilities of an economy performing dedicated R&D activity improving the efficiency of the natural resource use. We first derive a general rule for dynamic efficiency in R&D in such a context. We then focus on the characteristics of balanced growth paths. We show that for a full employment BGP to exhibit non constant consumption level, the production function must be of the Cobb Douglas class. We next show that the knowledge generation function must be a decomposable and linear function of the present state of knowledge for a BGP to exist. We then focus on the special case of an economy where the production function, the knowledge generation function and the utility function are all of the constant elasticity class. We consider more systematically the CES and the Cobb Douglas cases and allow either for increasing, constant or decreasing returns in the knowledge generation function. In the CES case, whatever the returns over research efforts, a constant positive consumption level may be sustained in the long run and a unique optimal regular path with positive consumption exists provided that the rate of impatience is not too high. With Cobb Douglas functions, the possibility of efficient regular paths corresponding to pure cake eating policies and constant decline of the consumption level towards zero arises with increasing returns over R&D efforts. Concerning optimality, in the decreasing returns over R&D case, there exists only one optimum corresponding to an active research policy. The same arises in the constant returns case provided that the rate of impatience is not too high. In the increasing returns case, depending upon the level of impatience, the optimal path corresponds either to an active R&D policy or to a cake eating policy without research efforts.

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Contents

1 Introduction

That the technical progress could be dedicated to improve the efficiency of some specific factors of production is not a quite new idea. Its first formalization goes back at least to Kennedy (1964)[1] and as far as optimal endogenous growth is concerned to the Uzawa (1965) fundamental paper.

Next the idea has been exploited in resources economics[2]. Curiously, in the famous Symposium Issue of the Review of Economic Studies (1974), although dedicated technical progress or factor augmenting technical progress is briefly mentioned by Dasgupta and Heal (see also Dasgupta and Heal (1979)), their seminal paper and the paper of Stiglitz are essentially oriented towards unbiased exogenous technical progress[3], and the Solow paper, towards Harrod's neutral exogenous progress.

More recently the idea got some revival for explaining the long run evolution of distribution and the changes in demand for skilled and unskilled labor (see Acemoglu (2002), (2003-a) and (2003-b) for well documented surveys) and again in resources economics (see for example, André and Smulders (2003), Eriksson (2004), Grimaud and Rougé (2003), Smulders (1996)).

A lot of papers in resource economics adopted the framework of the so-called new growth theory, with a proliferation of either intermediate goods or final consumption goods, in which the technological knowledge is embodied[4]. Although this formulation could appear as an inescapable detour for a positive theory, it tends to blur more fundamental relationships which were enlightened in the Uzawa (1965) paper. We choose in the present paper to go back to the way pioneered by Uzawa in order to isolate the specificity of dedicated investment process in relation to the non renewable resource problem, bypassing the intermediate good sector and assuming that some Hicksian aggregate consumption good can be produced directly from labor and some non renewable resource[5].

In the present model there exist two assets, a stock of non renewable natural resource and a stock of dedicated technical knowledge. The non renewable resource is an essential input in the consumption good sector. Thus this stock is necessarily decreasing through time as far as the consumption rate is positive. The stock of knowledge is a capital stock which can be accumulated like a physical capital and for which a higher rate of accumulation implies that some consumption has to be given up, because labor has to be employed in the research sector, labor which is no more available for the consumption good production sector. But this is a kind of capital the production function of which presents special features.

There are two extreme conceptions of the world of (fruitful) ideas[6]. According to the first one, the set of ideas is finite or at least bounded, and its exploration is more and more costly. According to the second one, the set of ideas is unbounded and more accumulated ideas facilitate the discovery of new ones, so that there is some learning by doing in the research sector but without the deceleration of the cumulative effects of the traditional learning by doing theory (see Arrow (1962)), so that indefinite improvements of the efficiency of the primary production factors can be sustained. This is the assumption retained in the present paper, like in most papers on endogenous growth theory.

A first objective of the paper is to disantangle carefully efficiency issues from optimality issues. In some sense efficiency problems are more fundamental and less questioned problems than optimality problems. We mean that there exists a large consensus about the fact that, whatever the rule permitting to select amongst different consumptions paths, the choice should be restricted to the set of efficient paths. As far as optimality is concerned we use the standard criterion of maximizing the sum of the discounted utilities because this is the most widely used criterion, thus facilitating the comparisons with the largest part of the literature.

A second objective is to characterize the balanced paths, efficient or not.

Balancedness is imposing strong requirements. Together with the efficiency condition it is drastically reducing the set of admissible paths by putting severe restrictions on the functional form of the dedicated knowledge generating function. Assuming that this function is depending upon both the stock of knowledge and the R&D effort, we show that it must be a linear function of the stock of knowledge. This leaves open the question of the returns of the R&D efforts.

The third objective is to provide a thorough analysis of the implications of different assumptions about the returns of the R&D effort. The problem is twofold. We show first that whatever the returns of this effort, the economy could either follow a balanced and efficient path along which the consumption is either non decreasing, in some cases increasing, or a pure cake eating path along which the consumption is regularly decreasing. The optimal path is depending upon the optimality criterion, more precisely upon the impatience rate.

Excepted for the most general properties, that is efficiency and optimality which are studied under fairly general assumptions, the most part of the analysis is laid down under constant elasticity assumptions. Concerning the production function of the consumption good, the case of the CES function and its limit, the Cobb-Douglas case in which the technical progress is no more dedicated, are strongly contrasted. With a constant population, a constant consumption level can be sustained in the long run along a regular optimal path provided that the impatience rate be not too high, even under decreasing returns in the R&D sector. This is a key point of the sustainability problem in an economy in which a non renewable resource without any renewable substitute is an essential input. In the CES case a constant consumption path is the only optimal regular path when such a path exists. Things are quite different in the limit Cobb-Douglas case. Then there may exist optimal regular paths along with the consumption is either decreasing, constant or increasing according to the impatience rate is either high, medium or low.

The paper is organized as follows. The model is laid down in section 2. Efficient paths are characterized in section 3 and optimal paths in section 4. The constant elasticity economies are introduced in section 5. Efficient regular paths and optimal regular paths of the constant elasticity economies are characterized in sections 6 and 7 respectively. We conclude in section 8.

2 The Model

We consider an economy in which the population is constant over time and the labor supply is inelastic. Without loss of generality we may assume that the labor endowment is equal to unity.

The economy produces an aggregate consumption good from labor and some non-renewable resource. Let q be the instantaneous production level of the consumption good, and l and s be respectively the amount of labor and resource inputs used in this sector.

The input efficiencies are depending upon specific technological knowledge which can be accumulated through dedicated R&D efforts. Let A be the stock of knowledge determining the efficiency of the labor input and B the stock of knowledge determining the efficiency of the resource input so that, denoting respectively by x and y the amounts of labor and resource inputs measured in efficiency units. Let us denote respectively by x^f and y^f the efficiency functions :

$$x = x^f(A, l) \quad \text{and} \quad y = y^f(B, s).$$

Assumption E.1 The efficiency functions x^f and y^f, each one $R^2_+ \rightarrow R_+$, are C^2 functions strictly increasing in each argument and such that :

$$\lim_{A\downarrow 0} x^f(A,l) = 0,\ \forall\, l > 0 \quad \text{and} \quad \lim_{l\downarrow 0} x^f(A,l) = 0,\ \forall\, A > 0$$

and

$$\lim_{B\downarrow 0} y^f(B,s) = 0,\ \forall\, s > 0 \quad \text{and} \quad \lim_{s\downarrow 0} y^f(B,s) = 0,\ \forall\, B > 0.$$

Let F be the production function of the consumption good. We assume that :

Assumption F.1 The consumption good production function, $F : R_+^2 \to R_+$, is a C^2 function strictly increasing in each argument, strictly quasi-concave and such that :

$$\lim_{x\downarrow 0} F(x,y) = 0,\ \forall\, y > 0 \quad \text{and} \quad \lim_{y\downarrow 0} F(x,y) = 0,\ \forall\, x > 0.$$

A stronger assumption although fairly standard is :

Assumption F.2 The production function F satisfies F.1 and is homogeneous.

For the sake of simplicity, we assume that only B can be increased. Let n be the employment in the R&D sector aiming at improving the resource efficiency. The instantaneous rate at which B can be increased is positively related to the research effort and the previously accumulated stock of knowledge. Furthermore there is no exogenous technical progress coming down from heaven, and no learning effect generated by the use of the resource through the cumulated production. Let us denote by b the knowledge accumulation function.

Assumption B.1 The knowledge accumulation function, $b: R^2_+ \to R_+$, is a C^2 function strictly increasing over R^2_{++} such that :

$$\lim_{n \downarrow 0} b(n, B) = 0, \ \forall B > 0 \quad \text{and} \quad \lim_{B \downarrow 0} b(n, B) = 0, \ \forall n > 0.$$

Labor is homogenous[7] and can be instantaneously and freely transferred from any sector to the other one. Hence the employment constraints are :

$$1 - l - n \geq 0, \ l \geq 0 \quad \text{and} \quad n \geq 0 \, .$$

Resource extraction is assumed to be costless. Alternatively F may be understood as describing an integrated production process, the primary inputs of which are labor, resource and the accumulated technological knowledge. The dynamics of the resource stock S is given by :

$$\dot{S} = -s$$

The instantaneous utility or surplus function u is a strictly concave function of the instantaneous consumption level c.

Assumption U.1 The utility function, $u: R_+ \to R$, is a C^2 function strictly increasing and strictly concave, satisfying the Inada condition : $\lim_{c \downarrow 0} u'(c) = +\infty$.

Welfare is the sum of discounted utilities at some positive social rate of discount ρ, assumed to be constant through time. The benevolent social planner maximizes the welfare subject to the above constraints.

3 Efficiency

A standard definition of efficiency is that it is not possible to increase the consumption over any time interval $[t_1, t_2]$ without having to reduce it over some part of the complementary interval $[0, \infty) \setminus [t_1, t_2]$. In the present context an equivalent definition is the following one. For any time interval $[t_1, t_2]$ let $\{c_t^*, t \in [t_1, t_2]\}$ be a consumption path having to be achieved over the interval and let $\{(l_t^*, n_t^*, s_t^*), t \in [t_1, t_2]\}$ be a feasible policy sustaining this consumption path starting from the initial value $B_{t_1}^*$ of B at t_1 and ending at the terminal value $B_{t_2}^*$ at t_2[8]. Then efficiency requires that the cumulated resource extraction over the interval is minimized, that is $\{(l_t^*, n_t^*, s_t^*), t \in [t_1, t_2]\}$ is a solution of the following problem (E).

$$(E) \qquad \max_{(l,n,s)} - \int_{t_1}^{t_2} s \, dt$$

$$\begin{aligned}
\text{s.t} \quad & F(x^f(A, l), y^f(B, s)) - c^* \geq 0 \\
& \dot{B} = b(n, B) \quad , \quad B_{t_1}^* > 0 \text{ given} \\
& 1 - l - n \geq 0 \; ; \\
& l \geq 0, n \geq 0 \quad \text{and} \quad s \geq 0 \\
& B_{t_2} - B_{t_2}^* \geq 0
\end{aligned}$$

To go to the core of the argument, let us assume that $c_t^* > 0, t \in [t_1, t_2]$. Then, under E.1 and F.1, we must have both $l > 0$ and $s > 0$ and we may delete the corresponding non negativity constraints. Let $\mathcal{L}^E$ be the Lagrangian :

$$\begin{aligned}
\mathcal{L}^E \quad = \quad & -s + \pi^E [F(x^f(A, l), y^f(B, s)) - c^*] + \nu^E b(n, B) \\
& + \omega^E [1 - l - n] + \gamma^E n.
\end{aligned}$$

The first order conditions are :

$$\partial \mathcal{L}^E / \partial l = 0 \;\; \Leftrightarrow \;\; \pi^E F_l = \omega^E \tag{3.1}$$

$$\partial \mathcal{L}^E / \partial n = 0 \;\; \Leftrightarrow \;\; \nu^E b_n = \omega^E - \gamma^E \tag{3.2}$$

$$\partial \mathcal{L}^E / \partial s = 0 \;\; \Leftrightarrow \;\; \pi^E F_s = 1 \tag{3.3}$$

where $F_l \equiv F_x\, x_e^f$ and $F_s \equiv F_y\, y_s^f$, together with the complementary slackness conditions :

$$\pi^E \geq 0, \quad \text{and} \quad \pi^E[F(x^f(A,l), y^f(B,s)) - c^*] = 0 \tag{3.4}$$
$$\omega^E \geq 0, \quad \text{and} \quad \omega^E[1 - l - n] = 0 \tag{3.5}$$
$$\gamma^E \geq 0, \quad \text{and} \quad \gamma^E n = 0 \tag{3.6}$$

The dynamics of the costate variable is given by :

$$\dot{\nu}^E \quad = \quad -\partial \mathcal{L}^E / \partial B \Leftrightarrow \dot{\nu}^E = -\pi^E F_B - \nu^E b_B \tag{3.7}$$

where $F_B = F_y g_B^f$.
The transversality condition is :

$$\nu_{t_2}^E \geq 0 \quad \text{and} \quad \nu_{t_2}^E[B_{t_2} - B^*] = 0 \tag{3.8}$$

Note that would t_2 be equal to ∞, the terminal constraint $B_{t_2} - B^* \geq 0$ would have to be deleted, that is B_∞ would have to be free, and the transversality condition at infinity would be :

$$\lim_{t \uparrow \infty} \nu^E B = 0. \tag{3.9}$$

This alternative characterization for the case $t_2 = \infty$ will be useful later (see the proof of the Proposition 2, section 5).

Let us consider a time sub-interval within which $n > 0$ so that $\gamma^E = 0$. Then by (3.1) $\cup$ (3.2), noting that $\pi^E = F_s^{-1}$ by (3.3), we get :

$$n > 0 \rightarrow F_l F_s^{-1} = \nu^E b_n. \tag{3.10}$$

Time differentiating this equation while making use of (3.1), (3.2) and (3.7), we obtain :

$$\frac{\dot{F}_l}{F_l} - \frac{\dot{F}_s}{F_s} = \frac{\dot{\nu}^E}{\nu^E} + \frac{\dot{b}_n}{b_n} = -b_n \frac{F_B}{F_l} - b_B + \frac{\dot{b}_n}{b_n}. \tag{3.11}$$

Last note that under E.1 and F.1, would the resource stock not be exhausted at infinity, then it could be possible to increase the consumption on any arbitrary non degenerate time interval. Thus we conclude :

Proposition 1 *Under E.1 and F.1, efficiency requires that :*

i. Over any time interval within which c>0 :

$$n > 0 \longrightarrow \frac{\dot{F}_s}{F_s} = \frac{\dot{F}_l}{F_l} + b_n \frac{F_B}{F_l} + b_B - \frac{\dot{b}_n}{b_n}. \tag{3.12}$$

ii. The resource stock should be depleted over $[0, \infty)$ *:*

$$S_0 = \int_0^{\infty} s_t dt.$$

The meaning of the above condition (3.12) is the following. Let $(c^*, l^*, n^*, B^*, S^*)$ be an efficient trajectory and $\Theta = [t, t+h+dt[, h > dt > 0$, be a time interval during which $n^* > 0$. Consider the following perturbation of both the research policy n^* and of the resource extraction path s^* over the three subsequent subintervals $\Theta_1 = [t, t+dt[, \Theta_2 = [t+dt, t+h[$ and $\Theta_3 = [t+h, t+h+dt[$, sustaining the same consumption path c^*.

Along the first subinterval Θ_1, the society increases its research effort by some constant $dn > 0$ at each point of time and decreases by the same amount the employment in the consumption good production sector while keeping the consumption at its reference level c^* thus increasing the use of the resource input. At $t + dt$, the end of the first subinterval, the resource productivity has been increased say by $d_1 B > 0$ and the resource stock is lower than along the reference path, say by $d_1 S < 0$.

During the second subinterval Θ_2 the difference $B_\tau - B^*_\tau$ is kept constant and equal to $d_1 B$. This allows for a decrease of the research effort by an amount $dn_\tau < 0, \tau \in \Theta_2$, which is not constant over the subinterval. The released labor is allocated to the consumption good production sector allowing for a decrease of the resource extraction rate with respect to s^* while keeping the consumption level equal to its reference level c^*. Furthermore, the resource productivity increases by $d_1 B$, which allows for a further decrease of the extraction rate. Let $d_2 S > 0$ be the total amount of resource saved during the second subinterval.

During the third subinterval Θ_3 the society drives back B_τ to its reference level at the end of the subinterval, $B_{t+h+dt} = B^*_{t+h+dt}$, cutting the research effort from its reference level by $dn < 0$, constant over the subinterval. The saved labor is once again allocated to the production of the consumption good for another period of reduced extraction rate with respect to $s^*, ds_\tau < 0, \tau \in \Theta_3$, where ds_τ is not constant. Let $d_3 S$ be the amount of resource saved over this third subinterval.

Since the perturbation is assumed to be feasible, we must have $dS \equiv d_1 S + d_2 S + d_3 S \leq 0$. But, where the inequality be strict, it would mean that we would have built a policy sustaining c^* and using less resource than $\int_0^\infty s^* dt$, a contradiction since the initial policy was assumed to be an efficient policy. Thus we must have $dS = 0$. We show in Appendix A.1 that (3.12) is nothing but that this condition $dS = 0$.

The above discussion shows that any perturbation of an efficient policy over some finite time interval, starting from efficient levels of the state variables B^* and S^* at the beginning of the interval and recovering efficient levels at the end of the interval, could not reduce the use of the resource. This implies that (3.12) should hold within any finite time interval within which $n > 0$. A recurrence argument would show that any sequence of perturbations of the same kind could not save natural resource while sustaining the consumption reference path. Hence (3.12) must hold along any efficient trajectory whenever $n > 0$. A trivial argument would prove that (3.12) and the depletion condition of Proposition 1 is equivalent to the more common place efficiency definition of Dasgupta and Heal (1974, 1979: chapter 7).

The previous discussion has shown how to interpret (3.12) as an arbitrage condition between the use of labor either in production or either in R&D activity. An equivalent interpretation in terms of opportunity costs may be described as follows. First remark that (3.12) may be rewritten as :

$$\frac{\dot{F}_s}{F_s} = \frac{\dot{F}_e}{F_e} - \left\{ \frac{\dot{b}_n}{b_n} - b_n \frac{F_B}{F_e} - b_B \right\}$$

where the right hand side expresses the net opportunity cost of labor in production. The term into brackets stands for the value of an increase of the R&D effort in term of resource savings possibilities. Rearranging terms and multiplying both sides by $F_l(F_s)^{-1}$, we get an equivalent expression of (3.12) :

$$\left(\frac{\dot{F}_l}{F_s} \right) = \dot{b}_n \frac{F_l}{b_n} \frac{1}{F_s} - b_n \frac{F_B}{F_s} - b_n \frac{b_B}{b_n} \frac{F_l}{F_s}$$

The $l.h.s$ is the local variation of the value of productive labor in terms of resource savings possibilities. Efficiency requires that at each time this variation should be equal to the variation of the value of labor in research also in terms of resource savings possibilities. This value is composed of the three elements appearing in the $r.h.s$. The first term measures the impact of the variation of the marginal productivity of labor in research upon the ratio of marginal productivities of labor either in production or in research. The second term is the induced effect of research labor upon the marginal rate of substitution between knowledge and resource through the knowledge increase this labor allows. The third term is a consequence of the public good characteristics of knowledge : knowledge is a non rival good used both to enhance production and research possibilities. This term describes the effect of labor in R&D upon the marginal rate of substitution between knowledge and labor in the research sector.

4 Optimality

Let (SP) be the problem of the social planner, that is :

$$
\begin{aligned}
(SP) \qquad & \max_{(c,l,n,s)} \int_0^\infty u(c)e^{-\rho t}\, dt \\
\text{s.t} \quad & F(x^f(A,l), y^f(B,s)) - c \geq 0 \\
& \dot{S} = -s \quad , \quad S_0 > 0 \text{ given} \\
& \dot{B} = b(n,B) \quad , \quad B_0 > 0 \text{ given} \\
& 1 - l - n \geq 0 \; ; \\
& c \geq 0,\; l \geq 0,\; n \geq 0 \quad \text{and} \quad s \geq 0 \\
& S \geq 0
\end{aligned}
$$

Since under U.1, clearly c must be positive at each point of time, then under E.1 and F.1, l and s have to be positive too, hence $S > 0$. Thus we may delete the corresponding non negativity constraints in the Lagrangian $\mathcal{L}$ of the problem (SP)[9] :

$$
\begin{aligned}
\mathcal{L} = \quad & u(c)e^{-\rho t} + \pi[F(x^f(A,l), y^f(B,s)) - c] + \nu b(n,B) - \lambda s \\
& + \omega[1 - l - n] + \gamma n
\end{aligned}
$$

The first order conditions are :

$$\partial\mathcal{L}/\partial c = 0 \;\Leftrightarrow\; u'e^{-\rho t} = \pi \tag{4.1}$$

$$\partial\mathcal{L}/\partial l = 0 \;\Leftrightarrow\; \pi F_l = \omega \tag{4.2}$$

$$\partial\mathcal{L}/\partial s = 0 \;\Leftrightarrow\; \pi F_s = \lambda \tag{4.3}$$

$$\partial\mathcal{L}/\partial n = 0 \;\Leftrightarrow\; \nu b_n = \omega - \gamma \tag{4.4}$$

together with the complementary slackness conditions :

$$\pi \geq 0, \quad \text{and} \quad \pi[F(x^f(A,l), y^f(B,s)) - c] = 0 \tag{4.5}$$

$$\omega \geq 0, \quad \text{and} \quad \omega[1 - l - n] = 0 \tag{4.6}$$

$$\gamma \geq 0, \quad \text{and} \quad \gamma n = 0 \tag{4.7}$$

The dynamics of the costate variables must satisfy :

$$\dot{\nu} = -\partial\mathcal{L}/\partial B \Leftrightarrow \dot{\nu} = -\pi F_B - \nu b_B \tag{4.8}$$

$$\dot{\lambda} = -\partial\mathcal{L}/\partial S \Leftrightarrow \dot{\lambda} = 0 \Leftrightarrow \lambda \text{ constant} \tag{4.9}$$

Last the transversality conditions at infinity are :

$$\lambda \lim_{t\uparrow\infty} S = 0 \tag{4.10}$$

$$\lim_{t\uparrow\infty} \nu B = 0 \tag{4.11}$$

The implications of the transversality conditions are different for S and for B, due to the different natures of the two stocks. Since clearly $\lambda > 0$, then we must have $\lim_{t\uparrow\infty} S = 0$. S is decreasing and must be exhausted. The stock of knowledge B is positive and non decreasing implying that $\lim_{t\uparrow\infty} \nu = 0$. Its imputed shadow price must decrease at a sufficiently high rate in the long run, so that the imputed value of the dedicated knowledge capital νB is reduced down to zero at infinity.

From (4.1) and (4.3) we get :

$$u' e^{-\rho t} F_s = \lambda \ , \quad t \geq 0 \tag{4.12}$$

Along an optimal path the marginal social benefit, in terms of discounted utility, from extracting one more unit of resource must be the same at each point of time. This is the standard arbitrage condition similar to the Hotelling rule in partial equilibrium analysis.

Time differentiating this arbitrage condition, we obtain :

$$\rho - \frac{u"}{u'} \dot{c} \ = \ \frac{\dot{F}_s}{F_s} \tag{4.13}$$

5 The C.E Economy: Definitions and General Properties

In order to get a precise characterization of both efficient and optimal plans we consider from now C.E. economies that is economies in which all the functional relationships are constant elasticity functions.

What we want to put in the forefront is the central role of the kind of returns in the research effort in conjunction with the type of production function in the consumption good sector.

Assumption E.2 The efficiency functions x^f and y^f are the product functions :

$$x^f(A, l) = Al \quad \text{and} \quad y^f(B, s) = Bs.$$

Assumption F.3 The production function of the consumption good, F, is the C.E.S function :

$$q = [\alpha_1 x^{-\eta} + \alpha_2 y^{-\eta}]^{-1/\eta} \ , 0 < \eta \ , 0 < \alpha_1 < 1 \quad \text{and} \quad \alpha_2 = 1 - \alpha_1.$$

Assumption F.3 is referred to as the *general case* in what follows. As we shall show, for the Cobb-Douglas function, the limit case of the C.E.S function for $\eta \to 0$, the qualitative properties of the model are drastically different, some sort of bifurcation occurring at the limit when the dedicated aspect of the research effort is vanishing.

Assumption F.4 The production function F of the consumption good is the Cobb-Douglas function :

$$q = x^{\alpha_1} y^{\alpha_2} \ , 0 < \alpha_1 < 1 \quad \text{and} \quad \alpha_2 = 1 - \alpha_1.$$

Assumption U.2 The instantaneous utility function is the isoelastic function :

$$u(c) = (1 - \varepsilon)^{-1} c^{1-\varepsilon} \ , \varepsilon > 0 \quad \text{and} \quad \varepsilon \neq 1.$$

Last concerning the production function of new technological knowledge we consider both cases of decreasing and increasing returns of the research effort, with a correction to take into account the size of the population although the labor force has been normalized to 1.
As we shall see in section 6 (Proposition 6) a necessary condition for the existence of regular paths is that $b(n, B) = b\phi(n)B$. Hence the only way to obtain regular paths with an isoelastic b function is to postulate B.2.

Assumption B.2 The knowledge accumulation function is the isoelastic function :

$$\dot{B} = b(n\bar{n}^{-1})^{\beta} B \ , b > 0 \ , \beta > 0 \quad \text{and} \quad \bar{n} > 0.$$

The parameter β is characterizing the returns to scale of the research effort. The marginal and average productivities of n are increasing, constant or decreasing according to β is higher, equal or lower than 1. Note that with isoelastic functions the both productivities of n, for $n = 0$, are respectively equal to 0 in the increasing returns case and infinite in the decreasing returns case. Thus in this last case it is always optimal to have an active R&D sector, although small in some circumstances, while in the former case a nil research effort cannot be excluded. As we shall show it may be optimal to allocate all the available labor to the consumption good production sector in this last case.

The parameter $\bar{n}$ is a population scaling factor balancing the assumption of an active population normalized to 1. The lower is $\bar{n}$ the higher is the productivity of any given effort $n > 0$, so that for any $B > 0$:

$$\lim_{\bar{n}\downarrow 0} \dot{B} = +\infty \ , \lim_{\bar{n}\downarrow 0} \frac{\partial \dot{B}}{\partial n} = +\infty \quad \text{and} \quad \lim_{\bar{n}\downarrow 0} \frac{\dot{B}}{n} = +\infty$$

and

$$\lim_{\bar{n}\uparrow \infty} \dot{B} = 0 \ , \lim_{\bar{n}\uparrow \infty} \frac{\partial \dot{B}}{\partial n} = 0 \quad \text{and} \quad \lim_{\bar{n}\uparrow \infty} \frac{\dot{B}}{n} = 0.$$

This type of knowledge accumulation function is implicitely assuming some kind of global increasing returns with respect to the whole set of factors generating new knowledge since it is homogeneous of degree $1+\beta$. As pointed out by Jones (1994, 1995) in a slightly different context, it leads to a global scale effect which is here necessary to obtain sustainable steady states, although sustainable steady states (along which the growth rate of the consumption is non negative) are not necessarily the optimal ones (see section 7).

We determine now some general properties of the efficient and optimal plans in economies satisfying E.2 and B.2 under the very general assumption F.1.

Proposition 2 *Under E.2, F.1 and B.2 along any efficient path, the marginal rate of substitution between l and s is depending upon the current stock of resource S and the employment in the research sector n, over any time interval within which the research sector is active :*

$$n > 0 \quad \Rightarrow \quad F_l F_s^{-1} = b\beta n^{\beta-1}\bar{n}^{-\beta}S \; , \tag{5.1}$$

so that in the constant returns case $\beta = 1$:

$$n > 0 \quad \Rightarrow \quad F_l F_s^{-1} = b\bar{n}^{-1}S. \tag{5.2}$$

Proof Consider a path solution of (E). Under B.2, (3.7) is now $\dot{\nu}^E = -\pi^E F_B - \nu^E b n^\beta \bar{n}^{-\beta}$. Multiplying the both sides by B we get $\dot{\nu}^E B + \nu^E b n^\beta \bar{n}^{-\beta} B = -\pi^E F_B B$, that is[10] :

$$(\dot{\nu^E B}) \;\; = \;\; -\pi^E F_B B.$$

Under E.2, $F_B = F_y s$ and $F_s = F_y B$. Thus $F_B B = F_s s$. By (3.3), $\pi^E F_s = 1$ hence $\pi^E F_B B = s$, so that :

$$-(\dot{\nu^E B}) \;\; = \;\; s.$$

Integrating over $[t, \infty)$ and using the transversality condition (3.9) results in :

$$-\int_t^\infty (\dot{\nu_\tau^E B_\tau}) d\tau = -\lim_{\tau \uparrow \infty} \nu_\tau^E B_\tau + \nu_t^E B_t = \nu_t^E B_t = \int_t^\infty s_\tau d\tau = S_t.$$

Under B.2, $\nu^E b_n = \nu^E b\beta n^{\beta-1}\bar{n}^{-\beta}B$, hence $\nu^E b_n = b\beta n^{\beta-1}\bar{n}^{-\beta}S$. Next from (3.1) and (3.3), we get $F_l F_s^{-1} = \omega^E$, and for $n > 0$, by (3.2), $\nu^E b_n = \omega^E$. Thus we conclude that $F_l F_s^{-1} = b\beta n^{\beta-1}\bar{n}^{-\beta}S$ provided that $n > 0$. ■

The point to be emphasized is that this characterization does not require that F be homogenous. Under the strong assumption of linear knowledge accumulation function, the marginal rate of substitution between productive labor and natural resource is proportional to the current stock of resource S, the proportionality coefficient being the index of the productivity in the research sector, $b\bar{n}^{-1}$. (5.2) states what may be called a pure substitution effect.

A second strong implication of E.2 and B.2 is that the stock of resource S and the stock of dedicated knowledge B must have the same imputed value at each point of time along an optimal path. This must hold under F.1. Thus there again we do not need that F be homogenous.

Proposition 3 *Under E.2, F.1, B.2 and U.1 the shadow value of the stock of resource and the shadow value of the stock of technological knowledge must be the same at each point of time along any optimal path :*

$$\lambda S \;=\; \nu B. \tag{5.3}$$

Proof Consider a solution path of $(S.P)$. Under B.2, (4.9) is given as $\dot{\nu} = -\pi F_B - \nu b n^{\beta} \bar{n}^{-\beta}$. Multiplying both sides by B, we obtain $\dot{\nu} B = -\pi F_B B - \nu b n^{\beta} \bar{n}^{-\beta} B$, that is :

$$-(\dot{\nu B}) \;=\; \pi F_B B.$$

Under E.2, $F_B = F_y s$ and $F_s = F_y B$ imply that $BF_B = sF_s$, and according to (4.3) $\pi F_s = \lambda$, hence :

$$-(\dot{\nu B}) \;=\; \lambda s.$$

Integrating over $[t, \infty)$ and making use of the transversality condition (4.11) we obtain (5.3). ■

A look at the proof of the proposition shows that the result does not depend upon the isoelastic specification of the knowledge generation function. Proposition 3 remains valid for any function of the form $\dot{B} = b\phi(n)B$.
This link between the imputed values of the two stock variables B and S suggests that there should exist a way to reduce the two state variables problem $(S.P)$ to a one state variable problem. We shall show that in C.E economies the optimal dynamics can be described as one state variable dynamics, the unique state variable being the product BS we call the *resource potential* of the economy and we denote by $R : R \equiv BS$. At any time R is nothing but the resource stock measured in efficiency units like Bs is the resource input flow in the production good sector, measured in efficiency units.

Last we characterize the optimal consumption proportional growth rates[11].

Proposition 4 *Along any optimal path :*

i. : Under E.2, F.1 and U.2 :

$$\varepsilon g^c + \rho = \dot{F}_s F_s^{-1} = g^{Fs} \tag{5.4}$$

ii. Under E.2, F.3 (general case) and U.2, denoting by z the input ratio in efficiency units, $z = xy^{-1}$:

$$\varepsilon g^c + \rho = g^B + (1+\eta)(1+\alpha_1^{-1}\alpha_2 z^{\eta})^{-1} g^z. \tag{5.5}$$

iii. Under E.2, F.4 (Cobb-Douglas case) and U.2 :

$$\varepsilon g^c + \rho = g^B + \alpha_1 g^z. \tag{5.6}$$

Proof (5.4) is a direct implication of (4.13) under U.2 which implies that $u'' \dot{c} u'^{-1} = -\varepsilon g^c$.

Under F.3, $F_s = \alpha_2 B(Bs)^{-(1+\eta)}c^{(1+\eta)}$ and $c = \alpha_2^{-1/\eta}Bs(1+\alpha_1\alpha_2^{-1}z^{-\eta})^{-1/\eta}$ resulting in $F_s = \alpha_2^{-1/\eta}B(1+\alpha_1\alpha_2^{-1}z^{-\eta})^{-(1+\eta)/\eta}$. Time differentiating and using (5.4) we obtain (5.5).
In the Cobb-Douglas case $F_s = \alpha_2 cs^{-1} = \alpha_2 B(Bs)^{-\alpha_1}(Al)^{\alpha_1} = \alpha_2 Bz^{\alpha_1}$. Again time differentiating and using (5.4) results in (5.6). ■

6 Efficient Regular Paths

By regular path we mean a path of the economy along which both the flow variables l, n, s and c and the stock variables B and S are all growing at the some constant proportional growth rates although not necessarily the same for all the variables.

Let us define a Full Employment path (in brief a F.E. path) as a path along which first all the primary physical factors are fully used at each point of time, $l + n = 1$ and $\dot{S} = -s$, second these factors are exploited at their full potential given the state of knowledge, $x = Al$ and $y = Bs$, and last the initial endowment in the non renewable resource is fully exploited that is $\lim_{t\uparrow\infty} S_t = 0$. To focus on the main point we restrict the analysis to the regular paths satisfying these minimum efficiency requisites.

Although the F.E. requirement could seem at first sight a rather loose condition, the weakest non waste condition, it happens that together with the regularity assumption it is imposing severe restrictions about the fundamental relationships of the model. We first explore the restrictions induced by regularity and next the more stringent restrictions implied by both regularity and full efficiency.

Since $l + n \leq 1$ an immediate implication of regularity in a stationary population model is that $g^l \leq 0$ and $g^n \leq 0$. Under the additional F.E assumption we must have $g^l = g^n = 0$.

Next note that $g^s = -sS^{-1}$ so that since g^s is constant, then $\dot{g}^s = -\dot{s}S^{-1} + s^2S^{-2} = 0$, hence $\dot{s}s^{-1} = sS^{-1}$ that is $g^s = g^S$.

We can show now that along a F.E regular path, if the growth rate of consumption is not zero and if the consumption good production function is homogeneous, then this function must be a Cobb-Douglas function.

Proposition 5 *Under E.2 and F.2 a necessary condition for the existence of a F.E regular path along which $g^c \neq 0$ is that the consumption good production function be a Cobb-Douglas function.*

Proof Under E.2 and F.2 we may write the consumption good production function in the intensive following form :

$$c = Bsf(z) \tag{6.1}$$

where $f' > 0$. Time differentiating (6.1) we get :

$$g^c = g^B + g^s + \frac{f'(z)z}{f(z)}g^z.$$

Since $g^l = 0$ and $g^s = g^S$ along any regular path, then :

$$g^z = -g^B - g^s = -g^B - g^S \quad \Rightarrow \quad g^c = (g^B + g^S)\left(1 - \frac{f'(z)z}{f(z)}\right) \tag{6.2}$$

Now assume that $g^c \neq 0$, then both $g^B + g^S \neq 0$ and $f'(z)zf(z)^{-1} \neq 1$.

Let $1 - \alpha = g^c(g^B + g^S)^{-1}$ so that under E.2 and F.2 :

$$\frac{f'(z)z}{f(z)} = \alpha \ , \quad \text{with } 0 < \alpha < 1 \ , \quad \Rightarrow \quad \frac{df(z)}{f(z)} = \alpha\frac{dz}{z} \quad \Rightarrow \quad lnf(z) = \alpha lnz + k \ ,$$

where k is the integration constant. Let $K = lnk$. If $g^c \neq 0$, then :

$$f(z) = Kz^{\alpha}.$$

Substituting for $f(z)$ into (6.1), we obtain :

$$c = Bsf(z) = BsKz^{\alpha} = BsK(Al)^{\alpha}(Bs)^{-\alpha} = K(Al)^{\alpha}(Bs)^{1-\alpha},$$

that is F is nothing but than the Cobb-Douglas function. ■

Note that in this Cobb-Douglas case, since $1-\alpha > 0$, we get from (6.2) :

$$\text{(F.4)} \;\Rightarrow \qquad i. \quad g^c \neq 0 \Rightarrow \text{sign}(g^B + g^S) = \;\; \text{sign } g^c \tag{6.3}$$

$$ii. \quad |g^c| < |g^B + g^S| \; . \tag{6.4}$$

Remembering that $R = BS$ is the resource potential, we show now that a direct implication of the above Proposition 5 is that this potential must be constant along any F.E path, the Cobb-Douglas case excepted.

Corollary 1 *Under E.2 and F.2, for any consumption good production function the Cobb-Douglas class excepted, then along any F.E regular path starting at time τ :*

$$i. \qquad \textit{both} \quad g^c_t = 0 \quad \textit{and} \quad g^z_t = 0 \; , \quad t \geq \tau \tag{6.5}$$

$$ii. \qquad g^B_t + g^s_t = g^B_t + g^S_t = 0 \quad \Rightarrow \quad R_t = R_{\tau} \; , \quad t \geq \tau \tag{6.6}$$

Proof Concerning the first point $i, g^c = 0$ is an immediate implication of the above Proposition 5. Next, from (6.2), we get that either $g^B + g^s = 0$ so that $g^z = 0$ because $g^l = 0$ under F.E, or $f'(z)zf(z)^{-1} = 1$, or both. Assume that $g^z \neq 0$ so that $f'(z)zf(z)^{-1} = 0$ which is equivalent to $f(z) = Kz$ from which we obtain :

$$F(Al, Bs) \;=\; KBsAl(Bs)^{-1} = KAl,$$

contradicting F.2, more precisely $F_y = \partial F/\partial y > 0$. Thus the only issue is $g^z = 0$.

Now concerning point ii., note that $g^z = 0$ implies that $g^B + g^s = g^B + g^S = 0$, that is (6.6). ■

The strong point of Proposition 5 and its Corollary is that the results are independent of the precise specification of the technical progress production function as far as the technical progress is a resource biased technical progress. With the exception of the Cobb-Douglas form of the consumption good production function, what has to be expected along a F.E regular path in a stationary population economy is at best a constant consumption good level. The crux of the problem is the regularity assumption[12]. As shown in the following Proposition 6, the regularity assumption is also strongly restricting the set of admissible knowledge accumulation functions. Note that this restriction is depending neither upon the type of efficiency functions nor upon the type of consumption good production function. The only required condition is that the rather weak assumption B.1 be satisfied.

Proposition 6 *Under B.1 a necessary condition for the existence of a F.E regular path along which $g^B > 0$ is that $b(n, B) = b\phi(n)B$ with $\phi'(n) > 0$.*

Proof Time differentiating $g^B = b(n, B)B^{-1}$, we get since $\dot{g}^B = 0$ and $\dot{n} = 0$ along any regular path :

$$\frac{\dot{g}^B}{g^B} = \frac{b_n\dot{n} + b_B\dot{B}}{b(n, B)} - g^B = 0 \;\Rightarrow\; g^B = b_B \quad \Leftrightarrow \quad \frac{b_B}{b(n, B)} = \frac{1}{B}.$$

Thus $b_B b(n,B)^{-1}$ is independent of n, from which we conclude that :

$$b(n,B) \quad = \quad \phi(n)\beta(B) \quad \Rightarrow \quad b_B = \phi(n)\beta'(B) = g^B$$

where β is some function of B and B only. Since $g^B > 0$, then $B_t = B_\tau e^{g^B(t-\tau)}$, $t \geq \tau$, hence $\beta'(B)$ must be some positive constant, that is $\beta(B) = bB + k, b > 0$. Under B.1, $k = 0$. ■

Let us examine now how is determined, under B.2, the set of F.E regular paths and amongst this set, the subset of efficient paths. Proposition 7 is concerning the general case, while Propositions 8-11 characterize what is happening in the Cobb-Douglas case.

Proposition 7 *Under E.2, F.3 and B.2, along any F.E regular path starting at time τ, the constant consumption level is equal to*

$$c_t = c = \left[\alpha_1 \left(A(1-n)\right)^{-\eta} + \alpha_2 \left(bn^\beta \bar{n}^{-\beta} R\right)^{-\eta}\right]^{-1/\eta} \quad , \quad t \geq \tau. \tag{6.7}$$

where c, n and R are the constant values of c_t, n_t and R_t, $t \geq \tau$. The constant value of the input ratio, measured in efficiency units, is given by :

$$z_t = z = A(1-n)(bn^\beta \bar{n}^{-\beta} R)^{-1} \; , \quad t \geq \tau. \tag{6.8}$$

There exists some function $\hat{R}(n,c)$ giving the constant resource potential required to follow the path $n_t = n$ and $c_t = c$, $t \geq \tau$. For any feasible c, as a function of $n, \hat{R}$ is first decreasing and next increasing over the range $(0,1)$. The locus of pairs (R,n) sustaining the efficient paths (at which R is minimized for given $c's$) is given by :

$$\tilde{R}(n) \quad = \quad \left[(\alpha_2 \beta A^\eta (1-n)^{1+\eta})(\alpha_1 (b\bar{n}^{-\beta})^\eta n^{1+\beta\eta})^{-1}\right]^{1/\eta}. \tag{6.9}$$

Proof Along a F.E regular path, in the general case F.3, $g^c = 0$ and $g^B = -g^S$ (cf (6.5) - (6.6)). Under B.2 this last equality results in : $bn^\beta \bar{n}^{-\beta} = sS^{-1}$. Multiplying both sides by B, we obtain :

$$Bs = bn^\beta \bar{n}^{-\beta} BS = bn^\beta \bar{n}^{-\beta} R. \tag{6.10}$$

Substituting for Bs its expression (6.10) into the expression of c under F.3, we get (6.7).

Equation (6.7) is defining the set of all the feasible F.E. regular paths in the space (c, n, R). Let $\hat{c}(n, R)$ be the left hand side of (6.7). For any feasible c we define $\hat{R}(n, c)$ as the required resource potential having to be available from the start at time τ and constant along the path, as a function of n and c. Clearly :

$$\frac{d\hat{R}}{dn} = -\frac{\partial \hat{c}(n, R)/\partial n}{\partial \hat{c}(n, R)/\partial R}.$$

Standard manipulations lead to :

$$\frac{\partial \hat{c}}{\partial n} = c^{1+\eta} \left[-\frac{\alpha_1}{A^\eta (1-n)^{1+\eta}} + \frac{\alpha_2 \beta}{(b\bar{n}^{-\beta} R)^\eta n^{1+\beta\eta}} \right]$$

$$\frac{\partial \hat{c}}{\partial R} = c^{1+\eta} \left[\frac{\alpha_2}{(b\bar{n}^{-\beta} n^\beta)^\eta R^{1+\eta}} \right] > 0 \,,$$

hence

$$\frac{\partial \hat{c}}{\partial n} \leq 0 \quad \Leftrightarrow \quad \frac{(1-n)^{1+\eta}}{n^{1+\beta\eta}} \leq \frac{\alpha_1}{\alpha_2} \frac{(b\bar{n}^{-\beta} R)^\eta}{\beta A^\eta} \;.$$

Let us denote by $h(n)$ the left hand side of the above inequality and by $k(R)$ its right hand side.

Note that : $\lim_{n\downarrow 0} h(n) = +\infty$ and $\lim_{n\uparrow 1} h(n) = 0$ and since $n \in (0,1)$:

$$\frac{dh}{dn} = -\frac{(1+\eta)(1-n)^{\eta}n^{1+\beta\eta} + (1+\beta\eta)(1-n)^{1+\eta}n^{\beta\eta}}{n^{2(1+\beta\eta)}} < 0.$$

This function is illustrated in the North-East quadrant of Figure 1. The function $k(R)$ is illustrated in the North-West quadrant (the curve $k(R)$ is drawn assuming $\eta = 1$ for the sake of simplicity, hence k is linear in R). In the South-East quadrant we obtain the curve $\tilde{R}(n)$, the locus of pairs (R, n) such that $h(n) = k(R)$,

$$\tilde{R}(n) = Ab^{-1}\bar{n}^{\beta}\left[\frac{\alpha_2\beta(1-n)^{1+\eta}}{\alpha_1 n^{1+\beta\eta}}\right]^{1/\eta}$$

Figure 1 here

Consider the points E', E'' and E''' located on the curve $\tilde{R}(n)$. Each one is on some curve $\hat{R}(n, c)$ determining the resource potential required to follow the F.E regular path along which $c_t = c$ and $n_t = n$. Since above the curve $\tilde{R}(n), \partial\hat{c}/\partial n > 0$, and under the curve $\partial\hat{c}/\partial n < 0$, then for a given $c, \hat{R}(n, c)$ is minimized when crossing the curve $\tilde{R}(n)$, because at the crossing point $\partial\hat{R}(n, c)/\partial n = 0$. For given n', n'' and n''' with $n' > n'' > n'''$ the crossing points E', E'' and E''' are corresponding to consumption levels c', c'' and $c''', c' < c'' < c'''$, a direct implication of $\partial\tilde{R}/\partial n < 0$. Thus all the points of the $\tilde{R}(n)$ curve are corresponding to points at which, for different consumption levels having to be sustained along a F.E regular path, the required constant resource potential is minimized. Equivalently, for given values of R, the maximum constant consumption which can be sustained along a F.E regular path is the consumption level corresponding to an allocation of labor to the R&D sector n such that $\tilde{R}(n) = R$. Hence $\tilde{R}(n)$ is the locus of efficient paths in the (R,n) space.

It remains to check that along an efficient path solution of the problem (E), which is also a regular path, R should be given by $\tilde{R}$. Under E.2, F.2 and B.2, the efficiency condition characterizing the value of $F_l F_s^{-1}$ is given by (5.1), so that under the stronger assumption F.3, (5.1) results in :

$$\frac{F_l}{F_s} = \frac{\alpha_1 A^{-\eta} l^{-(1+\eta)}}{\alpha_2 B^{-\eta} s^{-(1+\eta)}} = b\beta n^{\beta-1}\bar{n}^{-\beta} S$$

That is :

$$\frac{\alpha_1 A^{-\eta} l^{-(1+\eta)}}{\alpha_2 (Bs)^{-(1+\eta)}} = b\beta n^{\beta-1}\bar{n}^{-\beta} BS.$$

Along a F.E regular path, Bs is given by (6.10). Substituting in the above expression, we obtain :

$$\frac{\alpha_1 (bn^{\beta}\bar{n}^{-\beta})^{1+\eta} R^{1+\eta}}{\alpha_2 A^{\eta}(1-n)^{1+\eta}} = b\beta n^{\beta-1}\bar{n}^{-\beta} R\ ,$$

resulting in the following expression of R^{η} :

$$R^{\eta} = \frac{\alpha_2 A^{\eta}(1-n)^{1+\eta}(bn^{\beta}\bar{n}^{-\beta})\beta}{\alpha_1 (bn^{\beta}\bar{n}^{-\beta})^{1+\eta} n} = \frac{\alpha_2 \beta A^{\eta}(1-n)^{1+\eta}}{\alpha_1 (b\bar{n}^{-\beta})^{\eta} n^{1+\beta\eta}} = (\tilde{R}(n))^{\eta}.\blacksquare$$

The main implication of Proposition 7 is that in the general case, whatever the rate of return of the research effort, a constant consumption can be sustained in the long run. The strong point is that it holds even under decreasing returns of the research effort, $\beta < 1$.

In the general case the qualitative properties of the both functions $\hat{R}(n,c)$ and $\tilde{R}(n)$ are independent of the value of β, the coefficient of returns to scale of the research effort, and $\bar{n}$, the scaling parameter. The picture is quite different in the Cobb-Douglas case. In this case g^c may be different from 0, hence the F.E. regular consumption paths must be characterized by two parameters, the growth rate g^c and the consumption level c_τ at the beginning of the path. Proposition 8 characterizes general properties of F. E. regular paths in this case, while Propositions 9-11 characterize the efficient paths, according to the nature of returns to scale in the research sector.

Proposition 8 *Under E.2, F.4 and B.2, the set of constant consumption growth rates g^c which can be sustained along a F.E. regular path is bounded from above by $\bar{g}^c$:*

$$\bar{g}^c = \alpha_2 b \bar{n}^{-\beta} \tag{6.11}$$

Each feasible consumption rate g^c, $g^c < \bar{g}^c$, can be sustained by any pair (g^S, n), $g^S = \hat{g}^S(n, g^c)$ and $n \in (\underline{n}(g^c), 1)$ where :

$$\hat{g}^S(n, g^c) \quad = \quad \alpha_2^{-1} g^c - b n^\beta \bar{n}^{-\beta} = \alpha_2^{-1}[g^c - \bar{g}^c n^\beta], \; n \in [\underline{n}(g^c), 1) \tag{6.12}$$

$$\underline{n}(g^c) = \begin{cases} \bar{n}[g^c(\alpha_2 b)^{-1}]^{1/\beta} & , \; g^c \in (0, \bar{g}^c) \\ 0 & , \; g^c \in (-\infty, 0] \end{cases} \tag{6.13}$$

Proof Under E.2 and F.4 according to (6.2) and taking into account that $g^s = g^S$, we get :

$$g^c = \alpha_2[g^B + g^s] = \alpha_2[g^B + g^S]$$

so that, under the additional assumption B.2 :

$$g^c = \alpha_2[b n^\beta \bar{n}^{-\beta} + g^S].$$

Note that if either $g^S = 0$ or $n = 1$, or both, the consumption good production level is nil whatever B. Hence given that we must have $c \geq 0$ along the F.E path, we obtain :

$$\lim_{g^S \uparrow 0} \lim_{n \uparrow 1} g^c = \lim_{n \uparrow 1} \lim_{g^S \uparrow 0} g^c = \alpha_2 b \bar{n}^{-\beta} \equiv \bar{g}^c.$$

In the (g^S, n) space, the iso g^c curves are defined by the function :

$$g^S = \alpha_2^{-1} g^c - b n^\beta \bar{n}^{-\beta} \equiv \hat{g}^S(n, g^c) \tag{6.14}$$

Since we must have $g^S \leq 0$, then either $g^c \leq 0$ and the condition is satisfied, or $g^c > 0$ in which case a necessary condition is that :

$$\bar{n}[g^c(\alpha_2 b)^{-1}]^{1/\beta} \leq n. \qquad \blacksquare$$

As a function of the scaling factor $\bar{n}$, the upper bound of the feasible consumption growth rates $\bar{g}^c$ is decreasing from $+\infty$ for $\bar{n} = 0$ an infinite scaled population, down to 0 for $\bar{n} = +\infty$, a scaled population having shrunk down to 0.

The function $\hat{g}^S$ is decreasing in n, for any given g^c. Increasing the research effort is allowing a higher extraction rate of the non renewable resource, whatever be the value of β, because in the long run this higher depletion rate of the resource in balanced by the more efficient use of the resource generated by the increase of β. $\hat{g}^S$, as a function of n, is respectively concave, linear and convex according to the returns of the research efforts are decreasing, constant or increasing. These functions are illustrated in the South parts of Figures 2, 3 and 4.

At the time τ at which the economy begins to evolve along a F.E regular path, under F.4 :

$$c_\tau = [A(1-n)]^{\alpha_1}[B_\tau s_\tau]^{\alpha_2} = [A(1-n)]^{\alpha_1}[s_\tau S_\tau^{-1}]^{\alpha_2} R_\tau^{\alpha_2}.$$

Substituting for $s_\tau S_\tau^{-1} = -g^S$, the constant value of g^S given by (6.14) we obtain c_τ as a function of n, g^c and R_τ. Let $\hat{c}_\tau$ be this function :

$$\hat{c}_\tau(n, g^c, R_\tau) = [A(1-n)]^{\alpha_1}[bn^\beta \bar{n}^{-\beta} - \alpha_2^{-1} g^c]^{\alpha_2} R_\tau^{\alpha_2} \quad n \in [\underline{n}(g^c), 1]. \quad (6.15)$$

Clearly there is some tradeoff between the growth rate of the consumption and its initial level, given any n and R_τ :

$$\frac{\partial \hat{c}_\tau}{\partial g^c} = -c_\tau[bn^\beta \bar{n}^{-\beta} - \alpha_2^{-1} g^c]^{-1} < 0 \ , \quad g^c < \bar{g}^c. \quad (6.16)$$

Efficient paths are those paths for which, given g^c and R_τ, c_τ is maximized by choosing the best R&D effort. A look at (6.15) shows that this efficient effort does not depend upon R_τ, $\hat{c}_\tau(n, g^c, R_\tau) \equiv C(n, g^c) R_\tau^{\alpha_\tau}$ being a decomposable function of R_τ and some function of n and g^c.

Proposition 9 *Under E.2, F.4 and B.2, and assuming decreasing returns in the research effort, $\beta < 1$, then $\hat{c}_\tau$, as function of n, is first increasing over some interval $[\underline{n}(g^c), \tilde{n}(g^c))$, $0 \leq \underline{n}(g^c) < \tilde{n}(g^c) < 1$ and next decreasing over the interval $(\tilde{n}(g^c), 1]$, where :*

$$\frac{d\tilde{n}}{dg^c} > 0 \ , \ \lim_{g^c \downarrow -\infty} \tilde{n}(g^c) = 0 \quad \textit{and} \quad \lim_{g^c \uparrow \bar{g}^c} \tilde{n}(g^c) = 1. \quad (6.17)$$

Let $\tilde{c}_\tau(g^c, R_\tau) \equiv \hat{c}_\tau(\tilde{n}(g^c), g^c, R_\tau)$ be the maximal value of c_τ. Then for any given R_τ there is a tradeoff between the growth rate of the consumption and its initial level, that is :

$$\frac{\partial \tilde{c}_\tau}{\partial g^c} < 0 \ , \ \lim_{g^c \downarrow -\infty} \tilde{c}_\tau(g^c, R_\tau) = +\infty \quad \textit{and} \quad \lim_{g^c \uparrow \bar{g}^c} \tilde{c}_\tau(g^c, R_\tau) = 0. \quad (6.18)$$

The efficient proportional rate of exhaustion of the non renewable resource stock is an increasing function of the consumption growth rate, we denote by $\tilde{g}^S$ $: \tilde{g}^S(g^c) \equiv g^S(\tilde{n}(g^c), g^c)$, *and*

$$\frac{d\tilde{g}^S}{dg^c} > 0 \; , \; \lim_{g^c \downarrow -\infty} \tilde{g}^S(g^c) = -\infty \quad \textit{and} \quad \lim_{g^c \uparrow \bar{g}^c} \tilde{g}^S(g^c) = 0. \tag{6.19}$$

Proof See Appendix A.2 ■

The functions $\hat{c}_\tau$ and $\hat{g}^S$, and the loci $\{\tilde{c}_\tau(g^c, R_\tau), g^c < \bar{g}^c\}$ and $\{\tilde{g}^S(g^c), g^c < \bar{g}^c\}$ are illustrated in the below Figure 2.

Figure 2 here

Figure 2 is drawn for some given R_τ. The research effort maximizing $\hat{c}_\tau, \tilde{n}(g^c)$, is independent of R_τ. But $\hat{c}_\tau$ is depending upon R_τ. For two different values of R_τ, R'_τ and R''_τ such that $R''_\tau = \vartheta R'_\tau, \vartheta > 0$, then $\hat{c}_\tau(n, g^c, R''_\tau) = \vartheta^{\alpha_2}\hat{c}_\tau(n, g^c, R'_\tau)$. The curves $\hat{g}^S$ do not depend upon R_τ. The points E', E_0 and E'' on the locus $\{\tilde{c}_\tau(g^c, R_\tau), g^c < \bar{g}^c\}$ and the points F', F_0 and F'' on the locus $\{\tilde{g}^S(g^c), g^c < \bar{g}^c\}$ correspond respectively to consumption growth rates $g^{c'} > 0, 0$ and $g^{c''} < 0$.

A feature having to be pointed out is that for $g^c < 0, \hat{c}_\tau(0, g^c, R_\tau) > 0$. For $n = 0$ the problem is a pure cake eating problem since $B_t = B_\tau, t \geq \tau$, so that the instantaneous proportional rate of decrease of the resource potential is nothing but that the instantaneous proportional rate of exhaustion of the non-renewable resource. A first difference with a pure cake eating problem is that here, the cake has to be cooked via the consumption good production function. A second difference is that the kitchen is equipped with a device boosting the low research efforts.

Due to the fact that the marginal productivity of the research effort is very high in the neighborhood of $n = 0$, tending to infinity with n tending to 0 when $\beta < 1$, the efficient research effort is kept away from 0, slightly compensating for the pure exhaustion effect which would be the unique effect in a pure cake eating problem.

Proposition 10 *Under E.2, F.4 and B.2 and assuming constant returns to scale of the research efforts, $\beta = 1$, there exists a critical level of $g^c, g^c = \underline{g}^c = -\alpha_1^{-1}\alpha_2\bar{g}^c < 0$, such that :*

i. For $g^c \in (\underline{g}^c, \bar{g}^c)$ the qualitative properties of $\hat{c}_\tau$ are the same that in the case of decreasing returns, that is $\tilde{n}(g^c) \in (0,1), d\tilde{n}/dg^c > 0$ and $\lim_{g^c \uparrow \bar{g}^c} \tilde{n}(g^c) = 1$, but here $\lim_{g^c \downarrow \underline{g}^c} \tilde{n}(g^c) = 0$.

ii. For $g^c < \underline{g}^c, \hat{c}_\tau$ is decreasing over the whole range $(0,1)$ so that $\tilde{n}(g^c) = 0$.

iii. The function $\tilde{g}^S$ is given by :

$$\tilde{g}^S(g^c) = \begin{cases} g^c - \bar{g}^c & , \textit{ for } \quad g^c \in (\underline{g}^c, \bar{g}^c) \\ \alpha_2^{-1} g^c & , \textit{ for } \quad g^c \in (-\infty, \underline{g}^c) \end{cases} \tag{6.20}$$

Proof See Appendix A.2 ■

The functions $\hat{c}_\tau$ and $\hat{g}^S$, and the loci $\{\tilde{c}_\tau(g^c, R_\tau), g^c < \bar{g}^c\}$ and $\{\tilde{g}^S(g^c), g^c < \bar{g}^c\}$ are illustrated in Figure 3.

Figure 3 here

Proposition 11 *Under E.2, F.4 and B.2, and assuming increasing returns in the research effort, $\beta > 1$, then there exist three critical levels of g^c : $0, \underline{g}_1^c$ and $\underline{g}_2^c$, $0 > \underline{g}_1^c > \underline{g}_2^c$, such that :*

i. *For* $g^c \in [0, \bar{g}^c)$*, the qualitative properties of* $\hat{c}_\tau$ *as a function of* n *are the same as in the case of decreasing returns of the research effort.*

ii. *For* $g^c \in (\underline{g}_2^c, 0)$*, there exist two research effort levels* $n_1(g^c)$ *and* $n_2(g^c)$*,* $0 < n_1(g^c) < n_2(g^c) < 1$*, such that* $\hat{c}_\tau$ *is first decreasing over the interval* $(0, n_1(g^c))$*, next increasing over the interval* $(n_1(g^c), n_2(g^c))$ *and last decreasing again over* $(n_2(g^c), 1)$*, with :*

$$\hat{c}(0, g^c, R_\tau) >, =, < \hat{c}_\tau(n_2(g^c), g^c, R_\tau) \quad \Leftrightarrow \quad g^c <, =, > \underline{g}_1^c \tag{6.21}$$

hence the efficient research effort is given by :

$$\tilde{n}(g^c) = \begin{cases} n_2(g^c) & , \ for \quad g^c \in (\underline{g}_1^c, 0) \\ \{0, n_2(g^c)\} & , \ for \quad g^c = \underline{g}_1^c \\ 0 & , \ for \quad g^c \in (\underline{g}_2^c, \underline{g}_1^c). \end{cases} \tag{6.22}$$

iii. *For* $g^c \in (-\infty, \underline{g}_2^c], \hat{c}_\tau$ *is decreasing ever the whole range* $[0, 1]$ *so that* $\tilde{n}(g^c) = 0$.

Thus the function $\tilde{n}(g^c)$ *is constant and equal to 0 over the interval* $(-\infty, \underline{g}_1^c)$*, bivalued and jumping upwards at* $g^c = \underline{g}_1^c$ *and next increasing and tending towards 1 over the interval* $(\underline{g}_1^c, \bar{g}^c)$*. The function* $\tilde{c}_\tau(g^c, R_\tau)$ *is continuous and decreasing over the whole range* $(-\infty, \bar{g}^c)$*, from* $+\infty$ *down to 0, although not differentiable at* $g^c = \underline{g}_1^c$*.*

iv. *The function* $\tilde{g}^S$ *is increasing, equal to* $\alpha_2^{-1} g^c$*, for* $g^c \in (-\infty, \underline{g}_1^c)$*. For* $g^c \in (\underline{g}_1^c, \bar{g}^c)$ *:*

$$\frac{d\tilde{g}^S}{dg^c} >, =, < 0 \quad \Leftrightarrow \quad \tilde{n}(g^c) \ >, =, < (\beta - 1)\beta^{-1}. \tag{6.23}$$

In any case $\tilde{g}^S$ *is discontinuous at* $g^c = \underline{g}_1^c$*, falling from* $\alpha_2^{-1} \underline{g}_1^c$ *down to* $\alpha_2^{-1} \underline{g}_1^c - b n_2(\underline{g}_1^c)^\beta \bar{n}^{-\beta} = \alpha_2^{-1} [\underline{g}_1^c - n_2(\underline{g}_1^c) \bar{g}^c]$.

Proof See Appendix A.2 ■

The functions $\hat{c}_\tau$ and $\hat{g}^S$, the locus $\{\tilde{c}_\tau(g^c, R_\tau), g^c < \bar{g}^c\}$ and the locus $\{\hat{g}^S(g^c), g^c < \bar{g}^c\}$ are illustrated in Figure 4.

Figure 4 here

Note that since $(\beta - 1)\beta^{-1} < 1$ and $\lim_{g_c \uparrow \bar{g}^c} \tilde{n}(g^c) = 1$, then $\tilde{g}^S(g^c)$ is increasing in the left neighborhood of $\bar{g}^c$. But $\tilde{g}^S$ is not necessarily monotonic over the range $(\underline{g}_1^c, \bar{g}^c)$, although the curve has been drawn increasing in Figure 4 for the easiness of the drawing.

A second point to note is that, for growth rates slightly higher than $\underline{g}_1^c$, the stock of resource decreases at a significantly higher rate than at $\underline{g}_1^c$ while the decrease of the initial consumption level is relatively small, as illustrated in Figure 4. This is the counterpart of the discontinuity of $\tilde{n}(g^c)$, the efficient research effort jumping upwards at $g^c = \underline{g}_1^c$ while $\tilde{c}(g^c, R_\tau)$, the initial efficient consumption level, is continuous. Thus the labor allocated to the consumption good sector must fall. Since the amount of consumption good is roughly the same, then the use of the resource must be boosted upwards too, that is the rate of exhaustion of the resource stock must be significantly increased. In the long run this exhaustion acceleration is balanced by the jumping upwards of the rate of knowledge accumulation.

7 Optimal Regular Paths

Amongst the continuum of efficient regular paths, the optimal paths must satisfy the conditions laid down in Proposition 4 which hold along any optimal path.

7.1 The General Case

In the general case we know from Corollary 1 (cf. 6.5) that, along any F.E regular path, $g^z = g^c = 0$. Thus (5.5) in Proposition 4 is reduced to :

$$\rho = g^B \quad \Leftrightarrow \quad n = (\rho b^{-1})^{1/\beta} \bar{n}.$$

Since clearly we must have $n < 1$, then there exists an optimal regular path iff $b\bar{n}^{-\beta} > \rho$. The productivity of the research effort must be higher than the social rate of discount. We denote by n^* the above optimal research effort.

In order to evolve along this path from τ onwards, the economy must be endowed with a sufficiently large initial resource potential R_τ. Since an optimal path is an efficient path, the required initial resource potential is given by $R^* = \tilde{R}(n^*)$, where $\tilde{R}$ is the function characterized in (6.9). Given R^*, both the constant consumption level c^* and the constant input ratio z^* can be calculated by using respectively (6.7) and (6.8). Thus we conclude :

Proposition 12 *Under E.2, F.3, B.2 and U.2, there exists an optimal regular path iff $\rho < b\bar{n}^{-\beta}$, in which case it is unique and sustained by the constant research effort :*

$$n^* \;\; = \;\; (\rho b^{-1})^{1/\beta} \bar{n}. \tag{7.1}$$

The comparative statics of n^* is rather straightforward. The higher is the population scale effect, i.e. the lower is $\bar{n}$, the lower is the optimal research effort. The higher is the rate of return of the research effort, i.e. the higher is β, the lower is n^*.

Note that the optimal research effort does not depend upon ϵ. A look at either (5.4) or (5.5) characterizing the optimal paths, shows that for optimal paths along which the consumption level is constant, the term ϵg^c disappears because $g^c = 0$. However, would the initial resource potential be different from R^*, then the economy would have to follow a path along which $g^c \neq 0$, hence the intertemporal elasticity of substitution would have to enter the scene to determine the optimal trajectory of the economy.

7.2 The Cobb-Douglas Case

In the Cobb-Douglas case the consumption growth rate may be either positive, nil or negative. Thus we organize the discussion so as to determine the optimal constant consumption growth rate g^{c*}. First we state the fundamental equation that g^{c*} must satisfy. Next we discuss its solutions according to the returns of the research efforts are decreasing, constant or increasing.

The Fundamental Equation of the Optimal Regular Paths

From $z \equiv (Al)(Bs)^{-1}$ and taking into account that l is constant we get $g^z = -g^B - g^s$. As pointed out at the beginning of section 6, $g^s = g^S$, thus the condition (5.6) of Proposition 4 characterizing an optimal path in the Cobb-Douglas case, must be written for a regular path as :

$$\varepsilon g^c + \rho = \alpha_2 g^B - \alpha_1 g^S \tag{7.2}$$

But in the Cobb-Douglas case, along a regular path, another expression of g^c is :

$$g^c = \alpha_2 g^B + \alpha_2 g^S.$$

Substituting for g^c in (7.2) results in :

$$\rho = (1 - \varepsilon) g^c - g^S.$$

Last, remark that since an optimal path must be an efficient path, then g^S must be given by $\tilde{g}^S(g^c)$, so that the optimal consumption growth rate must solve :

$$\rho = (1 - \varepsilon) g^c - \tilde{g}(g^c). \tag{7.3}$$

Let us denote by $r(g^c)$ the $r.h.s$ of (7.3). The existence problem is reduced to the study of the properties of $r(g^c)$. Its asymptotic properties are independent of the returns of the research effort (cf. Appendix A.3), that is :

$$\lim_{g^c \downarrow -\infty} r(g^c) = +\infty \quad \text{and} \quad \lim_{g^c \uparrow \bar{g}^c} r(g^c) = (1 - \varepsilon)\bar{g}^c. \tag{7.4}$$

We know that along an efficient regular path, if $g^c = \bar{g}^c$, then the whole available labor must be allocated to the research sector. Hence the solution of (7.3) must be strictly lower than $\bar{g}^c$. If $\varepsilon > 1$, then $r(\bar{g}^c) < 0$, and any solution of (7.3) is lower than $\bar{g}^c$. But if $\varepsilon < 1$, then $r(\bar{g}^c) > 0$ and such solution exists iff $\rho > (1 - \varepsilon)\bar{g}^c$. For $\varepsilon < 1$ the society is not losing too much instantaneous utility for very low consumption levels. If furthermore it is not discounting the future utilities at a not too high discount rate, it is favoring low initial consumption levels in exchange of high future consumption rates generated by intensive research efforts. The only bound to this type of arbitrage favoring future consumption is that it has to obtain rewarding positive future consumption levels. For $n = 1$ the consumption is nil forever : the reward disappears and the arbitrage collapses.

Proposition 13 *Under F.2, F.4 and B.2, whatever the returns of the research efforts, if $\varepsilon < 1$, then a necessary condition for the existence of an optimal regular path, is that :*

$$\rho \geq (1 - \varepsilon)\bar{g}^c. \tag{7.5}$$

For the non extreme values of g^c the behavior of $r(g^c)$ is strongly dependent upon the type of returns β.

The Decreasing and Constant Return Cases

In the strictly decreasing case, $\beta < 1, dr/dg^c < 0$. Hence the higher is the social rate of discount, the lower is g^{c*}, and for sufficiently high rates g^{c*} is negative. Remember that for $\beta < 1$, whatever $g^c < \bar{g}^c$, then $\tilde{n}(g^c) > 0$ (cf. Proposition 9 supra). Thus even if the consumption is strongly decreasing, the optimal research effort $n^* = \tilde{n}(g^{c*})$ must be positive, due to the fact that the marginal productivity of the effort is very high for small efforts.

In case of constant returns $\beta = 1, dr/dg^c$ is a step function till negative, though r is continuous. The first main difference is that now $\tilde{n}(g^c) = 0$ for $g^c \leq \underline{g}^c$ (cf. Proposition 10 supra) so that for high values of the social discount rate the optimal regular path is a *pure cake eating* path. The second difference is that optimal paths are necessarily regular paths so that, whatever R_τ, the economy can evolve along its optimal regular trajectory from τ onwards.

Proposition 14 *Under F.2, F.4 and B.2, assuming non increasing returns of the research efforts and either $\varepsilon > 1$ or (7.5) if $\varepsilon \leq 1$, then there exists a unique optimal regular path. Furthermore :*

$$g^{c*} \; > , = , < 0 \quad \Leftrightarrow \quad \rho \; < , = , > b\bar{n}^{-\beta} \left[\alpha_2\beta(\alpha_1 + \alpha_2\beta)^{-1}\right]^\beta . \qquad (7.6)$$

Under strictly decreasing returns the optimal research effort is necessarily positive whereas under constant returns :

$$n^* > 0 \quad \Leftrightarrow \quad \rho < (1 + \varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c \, , \quad \textit{for} \quad \beta = 1 \qquad (7.7)$$

In case of constant returns any optimal path is a regular path.

Proof See Appendix A.3 ■

In both cases the triplet (g^{c*}, g^{S*}, n^*) is a continuous function of all the parameters of the model.

The Increasing Returns Case

In the case $\beta > 1$ we know from Proposition 11 that $\tilde{g}^S(g^c)$ is falling downwards at $g^c = \underline{g}_1^c$ corresponding to an upwards jump of $\tilde{n}(g^c)$ from 0 to some positive level. Thus $r(g^c)$ is jumping upwards at $g^c = \underline{g}_1^c$. For $g^c < \underline{g}_1^c$, then $\tilde{g}^S(g^c) = \alpha_2^{-1} g^c$ so that $r(g^c) = (1-\varepsilon-\alpha_2^{-1})g^c$ with $1-\varepsilon-\alpha_2^{-1} < 0$. We show in Appendix A.3 that for $g^c > \underline{g}_1^c$ two cases may appear corresponding to the fact that $r(g^c)$ is maximized either at $g^c = \underline{g}_1^c$ or at some value $g^c = \breve{g}^c \in (\underline{g}_1^c, \bar{g}^c)$. The first case is illustrated in Figure 5 below and the second case in Figure 6[13]. Let $\underline{r}(\underline{g}_1^c)$ and $\bar{r}(\underline{g}_1^c)$ denote respectively the lowest limit (*lhs*) and the highest limit (*rhs*) of $r(g^c)$ at $g^c = \underline{g}_1^c$. For $\rho \in [\underline{r}(\underline{g}_1^c), \bar{r}(\underline{g}_1^c))$ equation (7.3) has two roots in both cases illustrated in Figures 5 and 6, and for the Figure 5 case it holds also for $\rho = \bar{r}(\underline{g}_1^c)$. In case of Figure 6, equation (7.3) has three roots for $\rho \in [\bar{r}(\underline{g}_1^c), r(\breve{g}^c))$ and only two in the limit case $\rho = r(\breve{g}^c)$.

Figure 5 here

Figure 6 here

We show in Appendix A.3 that there exists some critical value of the rate of discount, denoted by $\bar{\rho}$, with $\underline{r}(\underline{g}_1^c) \leq \bar{\rho} \leq \bar{r}(\underline{g}_1^c)$ in case of Figure 5 and $\underline{r}(\underline{g}_1^c) \leq \bar{\rho} \leq r(\breve{g}^c)$[14] in case of Figure 6, at which the optimal policy switches from the highest root of (7.3) when $\rho < \bar{\rho}$ to the lowest root of (7.3) when $\rho > \bar{\rho}$. Since the highest root is higher than $\underline{g}_1^c$ while the lowest root is lower than $\underline{g}_1^c$, then the optimal research effort is strictly positive for $\rho < \bar{\rho}$ and falls discontinuously down to 0 for $\rho > \bar{\rho}$.

Proposition 15 *Under E.2, F.4, B.2, assuming increasing returns of the research effort, $\beta > 1$, and either $\varepsilon > 1$ or (7.5) if $\varepsilon < 1$, then there exists some critical value of the social rate of discount, $\bar{\rho}$, with $\underline{r}(\underline{g}_1^c) \leq \bar{\rho} \leq \bar{r}(\underline{g}_1^c)$ if $\breve{g}^c \leq \underline{g}_1^c$ and $\underline{r}(\underline{g}_1^c) \leq \bar{\rho} \leq r(\breve{g}^c)$ if $\breve{g}^c > \underline{g}_1^c$, such that :*

i. if $\rho > \bar{\rho}$ then the optimal policy is unique and is a pure cake eating policy :

$$\begin{aligned} n^* &= 0, g^{c*} = \rho[1 - \varepsilon - \alpha_2^{-1}]^{-1} < \underline{g}_1^c \\ g^{S*} &= g^{s*} = \rho[(1 - \varepsilon)\alpha_2 - 1]^{-1} \end{aligned} \tag{7.8}$$

ii. if $\rho < \bar{\rho}$ then the optimal policy is unique and is an active research policy where g^{c*} is the largest root of the equation (7.3) in case of multiple roots :

$$g^{c*} >, =, < 0 \quad \Leftrightarrow \quad \rho <, =, > \tilde{g}^S(0) = b\bar{n}^{-\beta}[\alpha_2\beta(\alpha_1 + \alpha_2\beta)^{-1}]^{\beta} \tag{7.9}$$

$$n^* = \tilde{n}(g^{c*}) > 0 \tag{7.10}$$

iii. for the critical value $\overline{\rho}$ the society is indifferent between choosing a pure cake eating policy with $n^* = 0$ and $g^{c*} < \underline{g}_1^c$ or an active research policy with $n^* > 0$ and $g^{c*} > \underline{g}_1^c$, the first one corresponds to the lowest root of (7.3) and the second one to its highest root.

Proof See Appendix A.3 ■

8 Conclusion

Recent contributions in the growth literature have revived interest about directed or "biased" technical change approaches, a view of technical progress pioneered by the Uzawa (65) famous work. In such a dedicated technical change framework where R&D efforts are aimed to improve the productive efficiency of an essential exhaustible resource, we have shown that it is always optimal to sustain at least a strictly positive consumption level in the long run under poor substitution conditions between productive inputs and even under strictly decreasing marginal returns of R&D efforts. The presence of scale effects in R&D activities is generally acknowledged as a main cause of sustained growth. We have shown in this paper that a multiplicative form of the knowledge generation function, linear in the previously accumulated stock of knowledge, is a prerequisite for the existence of balanced paths under mild assumptions.
Such a multiplicative form exhibits of course increasing returns to scale with respect to labor and to knowledge inputs in the generation of new knowledge. Our study shows that whatever the marginal returns over R&D effort or whatever the possible labor scaling returns over these effects, a balanced path is sustainable in the sense of allowing for a positive consumption level in the long run.

It appears that the Cobb-Douglas specification of the aggregate production possibility frontier, although widely used in the growth literature entails very specific features of the efficient and optimal balanced paths. First, positive growth in the long run now becomes possible even under decreasing returns in R&D efforts. Second, contrary to the CES case with poor substitution between the natural resource inputs and other inputs, it may be non optimal for the society to improve the resource efficiency even if it is an essential input and even if the R&D technology exhibits increasing marginal returns.

Increasing returns in production is a standard case for multiplicity of optima or for non-existence of such optima. We proved that if increasing returns applies to the knowledge production function, a unique optimal balanced path exists. The possibility of at most two distinct optimal paths, one of them corresponding to a cake eating path without resource efficiency improvement and the other one to some active research policy is confined to a unique generic value of the rate of impatience.

The possibility of sustained growth through labor efficiency improving knowledge generation and physical capital accumulation was one of the main conclusions of the Uzawa (65) seminal work. Consideration for physical and human capital accumulation possibilities in our present framework would most probably enhance growth possibilities but would come at a cost, either in terms of foregone consumption or labor allocation to the improvement of the human capital stock. Our analysis has shown that it should always be possible for the society to design an efficient resource management policy able to sustain in the long run at least a constant consumption level, even under the constraint of an essential exhaustible resource. That such an outcome be optimal from the society point of view depends as usual upon the level of the social rate of impatience. Such a result stands in contrast to the classical analysis of Dasgupta and Heal (1974) where man made capital accumulation alone could not prevent an asymptotic decline of the consumption level towards zero without a sufficient level of exogenous technical progress.

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Appendix

A.1 Interpretation of the efficiency condition (3.12)

The increment $d_1B(>0)$ of the resource productivity factor (w.r.t the reference path) over the first subinterval Θ_1, that is at $t+dt$, and the decrement $d_1S(<0)$ of the stock of resource induced by the perturbation of the policy are given by:

$$d_1B \simeq b^*_{n,t}dndt \quad and \quad d_1S \simeq -(F^*_{l,t}/F^*_{s,t})dndt$$

where a star means that the function is evaluated along the reference path.

Since over the second sub-interval Θ_2, the difference $B_\tau - B^*_\tau$ is kept constant and equal to d_1B, the research effort can be relaxed by an amount equal to $dn_\tau = -(b^*_{B,\tau}/b^*_{n,\tau})\ d_1B < 0, \quad \tau \in \Theta_2$. Assuming that this labor is now allocated to the consumption good production sector, and taking into account that the productivity of the resource factor is now higher, the society can save the resource. Let d_2S be the amount of resource saved over the sub-interval. We get:

$$d_2S = d_1B \int_{t+dt}^{t+h} \left[\frac{F^*_{l,\tau}}{F^*_{s,\tau}} \frac{b^*_{B,\tau}}{b^*_{n,\tau}} + \frac{F^*_{B,\tau}}{F^*_{s,\tau}} \right] \quad d\tau$$

For h sufficiently small, hence for dt small too, we obtain:

$$\begin{aligned} d_2S &\simeq d_1B \left[\frac{F^*_{l,t+dt}}{F^*_{s,t+dt}} \frac{b^*_{B,t+dt}}{b^*_{n,t+dt}} + \frac{F^*_{B,t+dt}}{F^*_{s,t+dt}} \right] (h-dt) \\ &= \left[\frac{F^*_{l,t+dt}}{F^*_{s,t+dt}} \frac{b^*_{B,t+dt}}{b^*_{n,t+dt}} + \frac{F^*_{B,t+dt}}{F^*_{s,t+dt}} \right] (h-dt) b^*_{n,t}dndt \end{aligned}$$

Over the third sub-interval Θ_3, the society reduces the research effort, $dn_\tau = -(b^*_{n,t}/b^*_{n,t+h})\quad dn, \tau \in \Theta_3$ for driving back B_t to its reference level at the end of the sub-interval, $B_{t+h+dt} = B^*_{t+h+dt}$. $-dn_\tau$ is allocated to the physical good production sector while keeping its production level to its reference level. Thus the resource saved amounts to:

$$d_3S \simeq \frac{b^*_{n,t}}{b^*_{n,t+h}} \frac{F^*_{l,t+h}}{F^*_{s,t+h}} dndt$$

Let $dS = d_1S + d_2S + d_3S$, be the amount of resource saved w.r.t. the reference path. Adding and substracting $(F^*_{l,t+h}/F^*_{s,t+h})$ to the expression of dS results in:

$$dS \simeq \left[\frac{F^*_{l,t+h}}{F^*_{s,t+h}} - \frac{F^*_{l,t}}{F^*_{s,t}} - \frac{b^*_{n,t+h} - b^*_{n,t}}{b^*_{n,t+h}}\frac{F^*_{l,t+h}}{F^*_{s,t+h}} + b^*_{n,t}\left(\frac{F^*_{l,t+dt}}{F^*_{s,t+dt}}\frac{b^*_{B,t+dt}}{b^*_{n,t+dt}} + \frac{F^*_{B,t+dt}}{F^*_{s,t+dt}}\right)(h-dt)\right] dndt$$

For h sufficiently small and dt infinitely smaller than h we get the following approximations:

$$h - dt \simeq h$$

$$\frac{b^*_{B,t+dt}}{b^*_{n,t+dt}} \simeq \frac{b^*_{B,t}}{b^*_{n,t}} \quad \text{and} \quad \frac{F^*_{B,t+dt}}{F^*_{s,t+dt}} \simeq \frac{F^*_{B,t}}{F^*_{s,t}}$$

$$\frac{F^*_{l,t+h}}{F^*_{s,t+h}} \simeq \frac{F^*_{l,t}}{F^*_{s,t}} + \left(\dot{\frac{F^*_{l,t}}{F^*_{s,t}}}\right) h^{15} \quad \text{and} \quad b^*_{n,t+h} \simeq b^*_{n,t} + \dot{b}^*_{n,t}h$$

$$\frac{F^*_{l,t+h}}{F^*_{s,t+h}} / \frac{F^*_{l,t}}{F^*_{s,t}} \simeq 1 + \left(\dot{\frac{F^*_{l,t}}{F^*_{s,t}}}\right) h / \frac{F^*_{l,t}}{F^*_{s,t}} \quad \text{and} \quad \frac{b^*_{n,t}}{b^*_{n,t+h}} \simeq 1 - \frac{\dot{b}^*_{n,t}h}{b^*_{n,t+h}}$$

hence:

$$dS \simeq \frac{F^*_{l,t}}{F^*_{s,t}}\left[\left\{\frac{\dot{F}^*_{l,t}}{F^*_{l,t}} - \frac{\dot{F}^*_{s,t}}{F^*_{s,t}} - \frac{\dot{b}^*_{n,t}}{b^*_{n,t}} + b^*_{B,t} + b^*_{n,t}\frac{F^*_{B,t}}{F^*_{l,t}}\right\} h\ dn\ dt - \left\{\frac{\dot{b}^*_{n,t}}{b^*_{n,t}}[E_1 - E_2h]\right\} h^2\ dn\ dt\right]$$

where:

$$E_1 = \left[\left(\dot{\frac{F^*_{l,t}}{F^*_{s,t}}}\right) / \frac{F^*_{l,t}}{F^*_{s,t}}\right] - \frac{\dot{b}^*_{n,t}}{b^*_{n,t+h}} \quad \text{and} \quad E_2 = \frac{\dot{b}^*_{n,t}}{b^*_{n,t+h}}\left(\dot{\frac{F^*_{l,t}}{F^*_{s,t}}}\right) / \frac{F^*_{l,t}}{F^*_{s,t}}$$

Letting $h \downarrow 0$, the second term of the above sum converges to 0 before the first term.

Now note that since the policy is the policy of the reference path over the interval $[0, t) \cup [t + dt + h, \infty)$ then $\int_0^t s^*_\tau d\tau + \int_{t+dt+h}^{\infty} s^*_\tau d\tau$ is not affected by the policy perturbation. Thus if the perturbation is feasible, we must have:

$$dS = \int_t^{t+h+dt} s_\tau d\tau - \int_t^{t+h+dt} s^*_\tau d\tau \leq 0.$$

But clearly $dS < 0$ would imply that the reference path is not efficient, hence we must have $dS = 0$, thus :

$$\frac{\dot{F}^*_{l,t}}{F^*_{l,t}} - \frac{\dot{F}^*_{s,t}}{F^*_{s,t}} + \frac{\dot{b}^*_{n,t}}{b^*_{n,t}} + b^*_{B,t} + b^*_{n,t}\frac{F^*_{B,t}}{F^*_{l,t}} = 0,$$

that is (3.12).

A.2 Proofs of Propositions 9, 10 and 11

We first determine the qualitative properties of $\tilde{c}_\tau$ and next the properties of $\tilde{g}^S$.

A.2.1 Properties of $\tilde{c}_\tau$

From (6.15) we obtain :

$$\frac{d\hat{c}_\tau}{dn} = c_\tau(1-n)^{-1}|g^S|^{-1}b\bar{n}^{-\beta}\left[\alpha_1\alpha_2^{-1}\bar{n}^\beta b^{-1}g^c - (\alpha_1 + \alpha_2\beta)n^\beta + \alpha_2\beta n^{\beta-1}\right] ,$$

hence :

$$\frac{d\hat{c}_\tau}{dn} \geq 0 \quad \Leftrightarrow \quad (\alpha_1 + \alpha_2\beta)n^\beta - \alpha_2\beta n^{\beta-1} \leq \alpha_1\alpha_2^{-1}b^{-1}\bar{n}^\beta g^c \ . \quad \text{(A.2.1)}$$

Let us denote by $h(n)$ the $l.h.s$ of this inequality and by $k(g^c)$ its $r.h.s$.

The properties of the function h are :

$$h(n) = \begin{cases} < \ 0 \quad , \quad \text{for} \quad n \in (0, n_h) \\ \\ = \ 0 \quad , \quad \text{for} \quad n = n_h \\ \\ > \ 0 \quad , \quad \text{for} \quad n \in (n_h, 1) \end{cases} \tag{A.2.2}$$

where :

$$n_h = \beta\alpha_2(\alpha_1 + \alpha_2\beta)^{-1} < 1 \ , \tag{A.2.3}$$

$$\lim_{n \uparrow 1} h(n) = \alpha_1 \ , \tag{A.2.4}$$

$$\frac{dh}{dn} = \beta n^{\beta - 2} \left[(\alpha_1 + \alpha_2\beta)n + \alpha_2(1 - \beta) \right] \ , \tag{A.2.5}$$

$$\lim_{n \uparrow 1} \frac{dh}{dn} = \beta > 0 \ . \tag{A.2.6}$$

Thanks to (6.11) $k(g^c)$ may be written as :

$$k(g^c) \quad = \quad \alpha_1 g^c / \bar{g}^c \ , \tag{A.2.7}$$

so that :

$$g^c \in [0, \bar{g}^c) \quad \Rightarrow \quad k(g^c) < \alpha_1 \quad \text{and} \quad \lim_{g^c \uparrow \bar{g}^c} k(g^c) = \alpha_1 \ , \tag{A.2.8}$$

and

$$\frac{dk}{dg^c} \quad = \quad \alpha_1 / \bar{g}^c > 0 \ . \tag{A.2.9}$$

The decreasing returns case, $\beta < 1$

In this case :

$$\lim_{n \downarrow 0} h(n) = -\infty \tag{A.2.10}$$

and (cf. A.2.5)

$$\frac{dh}{dn} > 0 \quad , \quad n \in (0,1) \tag{A.2.11}$$

The properties of $h(n)$ and $k(g^c)$ are illustrated in Figure A.2.1 :

Figure A.2.1 Case $\beta < 1$

A look at Figure A.2.1 shows clearly that the equation $h(n) = k(g^c)$, $g^c \leq \bar{g}^c$, has a unique solution $\tilde{n}(g^c) \in [0, \bar{g}^c]$ corresponding to a maximum of $\hat{c}_\tau$ with all the properties listed in Proposition 9. Since $\partial \hat{c}_\tau / \partial n = 0$ at $n = \tilde{n}(g^c)$ (envelope theorem), then :

$$\frac{\partial \tilde{c}_\tau}{\partial n} = \left. \frac{\partial \hat{c}_\tau}{\partial n} \right|_{n = \tilde{n}(g^c)} < 0.$$

The asymptotic properties of $\tilde{c}_\tau$ are, for $g^c \uparrow \bar{g}^c$, an immediate implication of $\lim_{g^c \uparrow \bar{g}^c} [1 - \tilde{n}(g^c)] = 0$, and for $g^c \downarrow -\infty$ an immediate implication of (6.15).

The constant returns case, $\beta = 1$

This is a case in which (cf. A.2.3, A.2.5) :

$$\lim_{n \downarrow 0} h(n) = -\alpha_2 \ , \ \frac{dh}{dn} = 1 \ n \in (0,1) \quad \text{and} \quad n_h = \alpha_2. \tag{A.2.12}$$

Next the solution of $h(n) = k(g^c)$, is :

$$\underline{g}^c = -\alpha_1^{-1} \alpha_2^2 b \bar{n}^{-1} = -\alpha_1^{-1} \alpha_2 \bar{g}^c < 0. \tag{A.2.13}$$

The properties of $h(n)$ and $k(g^c)$ are illustrated in the below Figure A.2.2 :

Figure A.2.2 Case $\beta = 1$

For $g^c \in [\underline{g}^c, \bar{g}^c]$ the equation $h(n) = k(g^c)$ has a unique solution $\tilde{n}(g^c) \in [0,1]$, increasing with g^c, corresponding to a maximum of $\hat{c}_\tau$. For $g^c < \underline{g}^c, h(n) > k(g^c), n \in [0,1]$. Hence $\hat{c}_\tau$ is a decreasing function of n over the whole segment $[0,1]$, from which we conclude that $\tilde{n}(g^c) = 0$, $g^c < \underline{g}^c$.

The increasing returns case, $\beta > 1$

In this case :

$$\lim_{n \downarrow 0} h(n) = 0 \tag{A.2.14}$$

and

$$\frac{dh}{dh} \begin{cases} < \ 0 \ , & \text{for} \quad n \in (0, n_\beta) \\ = \ 0 \ , & \text{for} \quad n = n_\beta \\ > \ 0 \ , & \text{for} \quad n \in (n_\beta, 1) \end{cases} \tag{A.2.15}$$

where

$$n_\beta \ = \ (\beta - 1)\alpha_2(\alpha_1 + \alpha_2\beta)^{-1} < n_h < 1 \tag{A.2.16}$$

The properties of $h(n)$ and $k(g^c)$ are illustrated in the below Figure A.2.3.

Figure A.2.3 Case $\beta > 1$

For $g^c \in [0, \bar{g}^c]$ then $k(g^c) \in [0, \alpha_1)$ and there exists a unique $\tilde{n}(g^c)$ solving $h(n) = k(g^c)$ as in the case $\beta \leq 1$ and with the same properties.

There exists a critical value of g^c solving $k(g^c) = h(n_\beta)$ we denote by $\underline{g}_2^c, \underline{g}_2^c < 0$. For $g^c = \underline{g}_2^c, \hat{c}_\tau$ is decreasing over $(0,1)$ with $\partial\hat{c}_\tau/\partial n = 0$ at $n = n_\beta$ and $\partial\hat{c}_\tau/\partial n < 0$ at $n \neq n_\beta$. For $g^c < \underline{g}_2^c$, $\partial\hat{c}_\tau/\partial n < 0, n \in (0,1)$. Thus for $g^c \leq \underline{g}_2^c$ the efficient research effort is $\tilde{n}(g^c) = 0$.

For $g^c \in (\underline{g}_2^c, 0)$ the equation $h(n) = k(g^c)$ has two roots, $n_1(g^c)$ and $n_2(g^c)$, $0 < n_1(g^c) < n_\beta < n_2(g^c) < n_h < 1$ (see Figure A.2.3). Clearly $n_1(g^c)$ corresponds to a local minimum while $n_2(g^c)$ corresponds to a local maximum. $\hat{c}_\tau$ is first decreasing over $(0, n_1(g^c))$, next increasing over $(n_1(g^c), n_2(g^c))$ and last decreasing again over $(n_2(g^c), 1)$. To determine the efficient research effort we must compare $\hat{c}_\tau(0, g^c, R_\tau)$ to $\hat{c}_\tau(n_2(g^c), g^c, R_\tau)$. Let us show that there exists a critical value of g^c, we denote by $\underline{g}_1^c$, $\underline{g}_2^c < \underline{g}_1^c < 0$, such that :

$$\hat{c}_\tau(0, g^c, R_\tau) \;>\,,\;<\; \hat{c}_\tau(n_2(g^c), g^c, R_\tau) \;\Leftrightarrow\; g^c \;<\,,\;>\; \underline{g}_1^c. \quad \text{(A.2.17)}$$

From (6.15) we get for $g^c < 0$:

$$\hat{c}_\tau(0, g^c, R_\tau) = A^{\alpha_1}(\alpha_2^{-1}|g^c|)^{\alpha_2} R_\tau^{\alpha_2} \ ,$$

$$\hat{c}_\tau(n_2(g^c), g^c, R_\tau) = A^{\alpha_1}[1 - n_2]^{\alpha_1}[bn_2^\beta \bar{n}^{-\beta} + \alpha_2^{-1}|g^c|]^{\alpha_2} R_\tau^{\alpha_2} \ ,$$

where n_2 stands for $n_2(g^c)$. Hence :

$$\begin{aligned} \delta(n_2(g^c)) \;&\equiv\; \frac{\hat{c}_\tau(0, g^c, R_\tau)}{\hat{c}_\tau(n_2(g^c), g^c, R_\tau)} = (1 - n_2)^{-\alpha_1}[b\alpha_2 n_2^\beta \bar{n}^{-\beta}|g^c|^{-1} + 1]^{-\alpha_2} \\ &= \;(1 - n_2)^{-\alpha_1} G^{-\alpha_2} \ , \qquad \text{(A.2.18)} \end{aligned}$$

where $G = b\alpha_2 n_2^\beta \bar{n}^{-\beta}|g^c|^{-1} + 1 = n_2^\beta \bar{g}^c |g^c|^{-1} + 1$.

Since at $n = n_2(g^c)$, by definition, $k(g^c) = h(n)$, then, taking (A.2.7) into account :

$$k(g^c) \;=\; \alpha_1 g^c (\bar{g}^c)^{-1} = (\alpha_1 + \alpha_2\beta) n_2^\beta - \alpha_2\beta n_2^{\beta-1} = h(n_2),$$

so that, since $g^c < 0$:

$$|g^c|^{-1} \;=\; \alpha_1(\bar{g}^c)^{-1}\left[\alpha_2\beta n_2^{\beta-1} - (\alpha_1 + \alpha_2\beta) n_2^\beta\right]^{-1}.$$

Substituting for $|g^c|^{-1}$ in G, we obtain :

$$\begin{aligned} G \;&=\; \alpha_1 n_2^\beta \left[\alpha_2\beta n_2^{\beta-1} - (\alpha_1 + \alpha_2\beta) n_2^\beta\right]^{-1} + 1 \\ &=\; \alpha_1 n_2 \left[\alpha_2\beta - (\alpha_1 + \alpha_2\beta) n_2\right]^{-1} + 1 = (1 - n_2)(1 - n_2 n_h^{-1})^{-1}. \end{aligned}$$

Next let us substitute for G in (A.2.18) to get :

$$\delta(n_2) = (1 - n_2)^{-1}(1 - n_2 n_h^{-1})^{\alpha_2} \; . \tag{A.2.19}$$

Note that :

$$\delta(0) = 1 \quad \text{and} \quad \delta(n_h) = 0. \tag{A.2.20}$$

Routine but tedious calculations lead to :

$$\begin{aligned} \frac{d\delta}{dg^c} &= \frac{d\delta}{dn_2} \cdot \frac{dn_2}{dg^c} \\ &= \left[\alpha_1 n_h^{-1}(1 - n_2)^{-2}(1 - n_2 n_h^{-1})^{-\alpha_1}(n_\beta - n_2)\right] \frac{dn_2}{dg^c} \; . \end{aligned} \tag{A.2.21}$$

Thus as a function of n_2, δ is first increasing over the interval $(0, n_\beta)$ so that $\delta(n_\beta) = \delta(n_2(\underline{g}_2^c)) > 1$. Over the interval (n_β, n_h) that is for $g^c \in (\underline{g}_2^c, 0)$ δ is decreasing from $\delta(n_2(\underline{g}_2^c)) > 1$ down to 0. Thus there exists a unique value of g^c, we denote by $\underline{g}_1^c, \underline{g}_2^c < \underline{g}_1^c < 0$, such that :

$$\delta(n_2(g^c)) > , = , < 1 \quad \Leftrightarrow \quad g^c < , = , > \underline{g}_1^c.$$

Hence, for $g^c \in (\underline{g}_2^c, \underline{g}_1^c), \tilde{n}(g^c) = 0$; for $g^c = \underline{g}_1^c, \tilde{n}(g^c) = \{0, n_2(\underline{g}_1^c)\}$; for $g^c \in (\underline{g}_1^c, 0), \tilde{n}(g^c) = n_2(g^c)$.

A.2.2 Properties of $\tilde{g}^S$

From (6.12) we know that, if $\tilde{n}(g^c) = 0$, then $\tilde{g}^S(g^c) = \alpha_2^{-1} g^c$. Thus the only problem is to determine the behavior of $\tilde{g}^S$ for $\tilde{n}(g^c) > 0$. In this case :

$$\frac{d\tilde{g}^S}{dg^c} = \frac{\partial \hat{g}^S}{\partial n}|_{n=\tilde{n}(g^c)} \cdot \frac{d\tilde{n}}{dg^c} + \frac{\partial \hat{g}^S}{\partial g^c}. \tag{A.2.22}$$

Differentiating (6.12) we obtain :

$$\frac{\partial \hat{g}^S}{dn}|_{n=\tilde{n}(g^c)} = -b\beta\tilde{n}(g^c)^{\beta-1}\bar{n}^{-\beta} < 0.$$

From the above discussion in paragraph A.2.1, we conclude that :

$$\tilde{n}(g^c) > 0 \quad \Leftarrow \quad \frac{\partial \tilde{n}}{\partial g^c} > 0.$$

Lastly from (6.12) again :

$$\frac{\partial \hat{g}^S}{\partial g^c} = \alpha_2^{-1} > 0.$$

Hence the first term of the $r.h.s$ of (A.2.22) is negative while the second term is positive. We may not conclude from purely qualitative arguments.

Since $\tilde{n}(g^c)$ is implicitely defined by $h(n) = k(g^c)$ with $k(g^c) = \alpha_1 g^c/\bar{g}^c$, then :

$$\frac{d\tilde{n}}{dg^c} = \frac{\alpha_1}{h'(\tilde{n})\bar{g}^c} \ ,$$

hence :

$$\frac{d\tilde{g}^S}{dg^c} = \left(-b\beta\tilde{n}^{\beta-1}\bar{n}^{-\beta}\right)\left(\frac{\alpha_1}{h'(\tilde{n})\bar{g}^c}\right) + \alpha_2^{-1} = \alpha_2^{-1}\left[1 - \frac{\alpha_1\beta\tilde{n}^{\beta-1}}{h'(\tilde{n})}\right] \ .$$

Let us substitute for $h'(\tilde{n})$ its value (A.2.5). We obtain :

$$\frac{d\tilde{g}^S}{dg^c} = \frac{\beta\tilde{n} + (1-\beta)}{(\alpha_1 + \alpha_2\beta)\tilde{n} + \alpha_2(1-\beta)} \ , \tag{A.2.23}$$

so that :

$$\tilde{n}(g^c) > 0 \quad \Rightarrow \quad \frac{d\tilde{g}^S}{dg^c} \begin{cases} > 0 \ , & \text{if} \quad \beta < 1 \\ = 1 \ , & \text{if} \quad \beta = 1 \end{cases} \ , \tag{A.2.24}$$

from which we conclude, since $\tilde{g}^S(g^c) = 0$, that :

$$\beta = 1 \;\; \Rightarrow \;\; \tilde{g}^S(g^c) = g^c - \alpha_2 b\bar{n}^{-1} = g^c - \bar{g}^c < 0, g^c \in (\underline{g}^c, \bar{g}^c) \quad \text{(A.2.25)}$$

Now let us remark that $d\tilde{g}^S/dg^c$ may be rewritten as :

$$\frac{d\tilde{g}^S}{dg^c} \;\; = \;\; \frac{\beta\tilde{n} - (\beta - 1)}{(\alpha_1 + \alpha_2\beta)(\tilde{n} - \alpha_2(\beta - 1)(\alpha_1 + \alpha_2\beta)^{-1})} \; ,$$

so that, taking (A.2.16) into account :

$$\frac{d\tilde{g}^S}{dg^c} \;\; = \;\; \frac{\beta\tilde{n} - (\beta - 1)}{(\alpha_1 + \alpha_2\beta)(\tilde{n} - n_\beta)}. \quad \text{(A.2.26)}$$

But as shown in § A.2.1, for $\beta > 1, \tilde{n}(g^c) > 0 \Leftrightarrow g^c > \underline{g}^c_1$ in which case $\tilde{n}(g^c) = n_2(g^c) > n_\beta$. Thus the sign of $d\tilde{g}^S/dg^c$ is the sign of the numerator of the *r.h.s* of (A.2.26), hence (6.23).

A.3 Proofs of Propositions 14 and 15

A.3.1 Limit behavior of $r(g^c) = (1 - \varepsilon)g^c - \tilde{g}^S(g^c)$

From Propositions 9-11 we know that for any $\beta > 0$:

$$\lim_{g^c \downarrow -\infty} \tilde{g}^S(g^c) = -\infty \quad \text{and} \quad \lim_{g^c \uparrow \bar{g}^c} \tilde{g}^S(g^c) = 0.$$

Thus the only indetermination is for $g^c \downarrow -\infty$ when $\varepsilon < 1$.
Note that $r(g^c)$ may be rewritten as :

$$r(g^c) = g^c \left[(1 - \varepsilon) - \tilde{g}^S(g^c)/g^c\right].$$

First $\tilde{g}^S(g^c) = \alpha_2^{-1} g^c$ for $g^c < \underline{g}^c$ if $\beta = 1$ and for $g^c < \underline{g}_1^c$ if $\beta > 1$ according to Propositions 10 and 11 respectively, hence $\tilde{g}^S(g^c)/g^c = \alpha_2^{-1}$ so that $1 - \varepsilon - \alpha_2^{-1} < 0$ and $\lim_{g^c \downarrow -\infty} r(g^c) = +\infty$.
Next, from (6.12), we may write :

$$\frac{\tilde{g}^S(g^c)}{g^c} \quad = \quad \alpha_2^{-1} \left[1 - \bar{g}^c \frac{\tilde{n}(g^c)^\beta}{g^c} \right] .$$

Since $\lim_{g^c \downarrow -\infty} \tilde{n}(g^c) = 0$ when $\beta < 1$, then $\lim_{g^c \downarrow -\infty} \tilde{g}^S(g^c)/g^c = \alpha_2^{-1}$ and again $\lim_{g^c \downarrow -\infty} r(g^c) = +\infty$.

Thus we conclude :

$$\forall\, \varepsilon,\ \forall\, \beta \qquad \lim_{g^c \uparrow \bar{g}^c} r(g^c) = (1 - \varepsilon) \bar{g}^c \begin{cases} > 0 \quad , \quad \text{if} \quad \varepsilon < 1 \\ \\ < 0 \quad , \quad \text{if} \quad \varepsilon > 1 \end{cases} . \tag{A.3.1}$$

A.3.2 Intermediate Behavior of $r(g^c)$. Case $\beta < 1$

>From (A.2.24) we know that $d\tilde{g}^S/dg^c > 0$ if $\beta < 1$. Hence $dr/dg^c < 0$ for $\varepsilon > 1$.
Consider now the case $\varepsilon < 1$. From (A.2.23), we obtain :

$$\begin{aligned} \frac{dr}{dg^c} < 0 \quad &\Leftrightarrow \quad (1 - \varepsilon) < \frac{\beta \tilde{n}(g^c) + 1 - \beta}{\alpha_1 \tilde{n}(g^c) + \alpha_2 \left[\beta \tilde{n}(g^c) + 1 - \beta \right]} \\ &\Leftrightarrow \quad \tilde{n}(g^c) \left[\frac{\alpha_1 (1 - \varepsilon)}{\alpha_1 + \alpha_2 \varepsilon} - \beta \right] < 1 - \beta \, . \end{aligned}$$

From $\varepsilon < 1$ and $\tilde{n}(g^c) < 1$ we conclude that this inequality is satisfied.

Lastly note that (cf. (6.12), (A.2.3) and Figure A.2.1) :

$$r(0) = -\tilde{g}^S(0) = b \tilde{n}(0)^\beta \bar{n}^{-\beta} = b n_h^\beta \bar{n}^{-\beta} \left[\beta \alpha_2 / (\alpha_1 + \alpha_2 \beta) \right]^\beta .$$

A.3.3 Intermediate Behavior of $r(g^c)$. Case $\beta = 1$

In this case (cf. Ap.2, § A.2.2 and A.2.24) :

$$\frac{d\tilde{g}^S}{dg^c} = \begin{cases} \alpha_2^{-1} & , \text{ for } \quad g^c \in (-\infty, \underline{g}^c) \\ \\ 1 & , \text{ for } \quad g^c \in (\underline{g}^c, \bar{g}^c) \end{cases}$$

where $\underline{g}^c = -\alpha_1^{-1}\alpha_2\bar{g}^c < 0$, so that :

$$\frac{dr}{dg^c} = \begin{cases} 1 - \varepsilon - \alpha_2^{-1} & , \text{ for } \quad g^c \in (-\infty, \underline{g}^c) \\ \\ -\varepsilon < 0 & , \text{ for } \quad g^c \in (\underline{g}^c, \bar{g}^c) \end{cases}$$

Next :

$$\tilde{n}(g^c) > 0 \quad \Leftrightarrow \quad g^c \in (\underline{g}^c, \bar{g}^c)$$

Since $\tilde{g}^S(\underline{g}^c) = -\alpha_2$ (cf. Figure A.2.2), then $r(\underline{g}^c) = (1-\varepsilon)\underline{g}^c - \alpha_2\underline{g}^c$. Let us substitute for $\underline{g}^c$ its expression as a function of $\bar{g}^c$ given by (A.2.13).
We get :

$$r(\underline{g}^c) = [1 + \varepsilon\alpha_1^{-1}\alpha_2]\bar{g}^c \ ,$$

so that :

$$n^* > 0 \quad \Leftrightarrow \quad \rho < [1 + \varepsilon\alpha_1^{-1}\alpha_2]\bar{g}^c.$$

A.3.4 Proof that any optimal path is a regular path if $\beta = 1$

First consider an optimal path along which $n = 0$ over the whole path.
In the Cobb-Douglas case :

$$g^c = \alpha_1 g^l + \alpha_2 g^B + \alpha_2 g^s \ . \tag{A.3.2}$$

If $n = 0$, then $l = 1$ and B is constant hence :

$$g^c \;=\; \alpha_2 g^s \; . \tag{A.3.3}$$

Also since $l = 1$, then :

$$z = \frac{Al}{Bs} = \frac{A}{Bs} \;\;\Rightarrow\;\; g^z = -g^s \; . \tag{A.3.4}$$

From (A.3.3) and (A.3.4) we conclude that the optimality condition (5.6) has to be written as :

$$\varepsilon \alpha_1 g^s + \rho = -\alpha_1 g^s \;\;\Rightarrow\;\; g^s = -\rho(\alpha_1 + \varepsilon\alpha_2)^{-1} \; , \tag{A.3.5}$$

implying that g^s is constant along the whole path.
Lastly :

$$S_t = \int_t^{\infty} s_t e^{g^s(\tau - t)} d\tau = -\frac{1}{g^s} s_t \;\;\Rightarrow\;\; g^S = -s_t S_t^{-1} = g^s \; , \tag{A.3.6}$$

and g^S is also constant.
Thus the proportional growth rate of all the stock and flow variables are constant and the path is regular.

Next, consider an optimal path along which $n > 0$ over the whole path. In the Cobb-Douglas case :

$$F_l F_s^{-1} = \alpha_1 s(\alpha_2 l)^{-1}.$$

If $n > 0$, then (5.2) must hold so that :

$$\alpha_1 s(\alpha_2 l)^{-1} = b\bar{n}^{-1} S \;\;=\;\; \alpha_2^{-1} \bar{g}^c S. \tag{A.3.7}$$

Multiplying the both sides of the above $l.h.s$ equality by BA^{-1} results in :

$$\frac{\alpha_1 Bs}{\alpha_2 Al} = \frac{\bar{g}^c BS}{\alpha_2 A} \;\;\Rightarrow\;\; z \equiv \frac{Al}{Bs} = \frac{\alpha_1 A}{\bar{g}^c} \cdot \frac{1}{R} \tag{A.3.8}$$

so that :

$$g^z = g^l - g^B - g^s = -g^B - g^S \quad \Rightarrow \quad g^l = g^s - g^S. \tag{A.3.9}$$

Also from (A.3.7) :

$$g^S = -sS^{-1} = -\alpha_1^{-1} l\bar{g}^c. \tag{A.3.10}$$

From the first equality (A.3.9) ($l.h.s$) and (A.3.10), we obtain :

$$\begin{aligned} g^B + \alpha_1 g^z &= g^B - \alpha_1 g^B - \alpha_1 g^S = \alpha_2 g^B + l\bar{g}^c \\ &= \alpha_2 bn\bar{n}^{-1} + l\bar{g}^c = n\bar{g}^c + l\bar{g}^c = \bar{g}^c \ . \end{aligned} \tag{A.3.11}$$

Thus (5.6) may be written as :

$$\varepsilon g^c + \rho = \bar{g}^c \quad \Rightarrow \quad g^c = \varepsilon^{-1}[\bar{g}^c - \rho] \ , \tag{A.3.12}$$

so that the consumption growth rate is constant.

The other way to write (5.6) is :

$$\begin{aligned} \varepsilon g^c + \rho &= g^B + \alpha_1 g^l - \alpha_1 g^B - \alpha_1 g^s = \alpha_1 g^l + \alpha_2 g^B - \alpha_1 g^s \\ &= \alpha_1 g^l + \alpha_2 g^B + \alpha_2 g^s - g^s = g^c - g^s \ , \end{aligned}$$

hence :

$$g^s = (1 - \varepsilon) g^c - \rho.$$

Thus g^s is constant, hence equal to g^S (cf. (A.3.5) supra), so that by (A.3.9) $g^l = 0$. All the flow and stock variables are growing at some constant proportional rate and the path is a regular path.

Last, it may be the case that an optimal path would be composed of a sequence of temporary phases where $n = 0$ and phases where $n > 0$. Let us show that such sequences cannot be optimal.
First, it may be seen that with a strictly concave utility function, c should be a continuous function of time along an optimal path. But making use of (4.12) under U.2 and F.4 :

$$\pi F_s = \lambda \;\Rightarrow\; c^{-\varepsilon} e^{-\rho t} \alpha_2 c s^{-1} = \lambda \;\Rightarrow\; s = \alpha_2 \lambda^{-1} c^{1-\varepsilon} e^{-\rho t} \; .$$

Since λ should be constant over the whole optimal path, we conclude that s should be a continuous function of time. Furthermore :

$$c = (Al)^{\alpha_1} (Bs)^{\alpha_2} \;\Rightarrow\; l = (cB^{-\alpha_2} s^{-\alpha_2})^{1/\alpha_1} A^{-1}$$

implies that l should also be a continuous function of time.
Next, through (4.4) under B.2 with $\beta = 1$:

$$\gamma = \omega - b\nu B \bar{n}^{-1} \; .$$

Making use of (4.2) and (4.3) under F.4, we get :

$$\omega = \pi F_l = \lambda F_l F_s^{-1} = \lambda \alpha_1 \alpha_2^{-1} s l^{-1} \; .$$

Since $\nu B = \lambda S$ through (5.3), we obtain :

$$\gamma = \lambda(\alpha_1 \alpha_2^{-1} s l^{-1} - bS\bar{n}^{-1}) = \lambda \alpha_2^{-1} (\alpha_1 s l^{-1} - \bar{g}^c S) \; . \qquad \text{(A.3.13)}$$

Since l, s and S are continuous functions, we conclude that γ should also be a continuous function of time over the whole optimal path.

Now consider some non degenerate interval $\mathcal{T}_0 = [t_0, t_1)$ such that $n_t = 0$, $t \in \mathcal{T}_0$. $l = 1$ for $t \in \mathcal{T}_0$ would imply :

$$\gamma = \lambda \alpha_2^{-1} (\alpha_1 s - \bar{g}^c S) = \lambda \alpha_2^{-1} S(\alpha_1 |g^S| - \bar{g}^c) \geq 0 \;\;,\; t \in \mathcal{T}_0 \; .$$

Hence $n_t = 0$ implying $\gamma_t \geq 0$, $|g^S| \geq \alpha_1^{-1} \bar{g}^c$, $t \in \mathcal{T}_0$. Differentiating through time, the growth rate of $|g^S|$ is equal to $g^s + |g^S|$.
If $|g^S| > |g^s|, |g^S|$ increases over time and thus $|g^S_{t_0}| \geq \alpha_1^{-1} \bar{g}^c \Rightarrow |g^S_t| > \alpha_1^{-1} \bar{g}^c$, $t > t_0$, $t \in \mathcal{T}_0$. We conclude that $\gamma_{t_1} > 0$, implying that such a cake eating phase cannot be followed by an active research phase, since this would result in a downward jump of γ_t at t_1, contradicting the continuity requirement over γ_t along an optimal path. If $\mathcal{T}_0 \equiv [t_0, \infty)$, some infinite duration time interval, we know that $|g^s| = |g^S|$, a contradiction with our

previous assumption. Hence $|g^S| \leq |g^s|$ is the only possibility compatible with optimality. But in this case, from (A.3.5) :

$$|g^s| = \rho(\alpha_1 + \varepsilon\alpha_2)^{-1} \geq |g^S| \geq \alpha_1^{-1}\bar{g}^c \ , \quad t \in \mathcal{T}_0$$

Next, differentiating through time the expression of γ_t :

$$\dot{\gamma} = \lambda\alpha_2^{-1}(\alpha_1\dot{s} + \bar{g}^c s) = \lambda\alpha_2^{-1}s[\bar{g}^c - \alpha_1|g^s|].$$

Since we have shown that $\rho \geq (1 + \varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c$, we conclude that :

$$\dot{\gamma} \begin{cases} < 0 & \text{if} \quad \rho > (1 + \varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c \\ \\ = 0 & \text{if} \quad \rho = (1 + \varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c \end{cases}$$

Consider first the case $\rho > (1 + \varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c$. Since γ should be a strictly decreasing function of time over $\mathcal{T}_0$, γ should be strictly positive at the beginning of this time interval. Hence an active research policy (during which $\gamma = 0$) cannot be followed by such a pure cake eating phase, since this would imply an upward jump of γ at the transition between phases, contradicting the continuity of γ over an optimal path. Thus the only remaining possibility would have to be a cake eating phase followed by an active research phase. But since such an active research phase would be followed forever, we know that l, $l < 1$, should be a constant. Hence this would imply a downward jump of l at the transition between phases, contradicting the necessary continuity of l along an optimal path.

Last, in the case $\rho = (1+\varepsilon\alpha_1^{-1}\alpha_2)\bar{g}^c$, we get for any active research phase, making use of (A.3.12) and substituting for ρ its above value :

$$g^s = (1 - \varepsilon)g^c - \rho = \frac{(1 - \varepsilon)}{\varepsilon}(\bar{g}^c - \rho) - \rho = -\bar{g}^c\alpha_1^{-1} \ .$$

Hence, from (A.3.10) and (A.3.11) we obtain :

$$g^l = g^s - g^S = -\alpha_1^{-1}\bar{g}^c(1 - l) < 0 \ .$$

We conclude that an active research phase cannot be followed by a cake eating phase since this would imply an upward jump of l at the transition between phases. A cake eating phase (where $l = 1$) followed by an active research phase (where l should be some constant, with $l < 1$) is excluded by a similar argument.

A.3.5 Proof of Proposition 15

A.3.5.1 Sketch of the proof

Following the same procedure as before, we first investigate the intermediate behavior of the function $r(g^c)$ in the case $\beta > 1$ (in subsection A.3.5.2). This study shows that the function $r(g^c)$ is discontinuous at $g^c = \underline{g}_1^c$. For $g^c < \underline{g}_1^c$, $r(g^c)$ is a decreasing linear function of g^c. At $g^c = \underline{g}_1^c$, $r(g^c)$ jumps upwards from a level $\underline{r}(\underline{g}_1^c)$ up to a level $\bar{r}(\underline{g}_1^c)$, the size of the jump being increasing in $\underline{g}_1^c$.

Next we show that there exists some critical threshold, denoted by $\breve{n}$, parametrically determined, and such that either $r(g^c)$ is a decreasing function over the open interval $(\underline{g}_1^c, \bar{g}^c)$ if $\breve{n} < \tilde{n}(\underline{g}_1^c)$, or either $r(g^c)$ is first increasing and then decreasing over the same interval if $\breve{n} > \tilde{n}(\underline{g}_1^c)$, with a maximum at $\breve{g}^c$ solution of $\tilde{n}(g^c) = \breve{n}$. We conclude that depending upon the level of ρ, the equation $r(g^c) = \rho$ may admit at most three distinct roots (see Figures 5 and 6).

Then we investigate the case of multiple roots of $r(g^c) = \rho$ (in subsection A.3.5.3). To select the solution corresponding to an optimal path amongst the possible roots, we introduce the value function $\tilde{V}_\tau$ giving the value at time τ of an efficient regular policy followed from this time τ onwards. This value is some function of g^c. We check that, depending upon the level of ρ, some root of $r(g^c) = \rho$ corresponds to a global maximum of the efficient value function, for any τ, starting from a given potential resource R_τ available at τ. For low values of ρ, the global maximum corresponds to the highest root of $r(g^c) = \rho$, and hence to an active research optimal policy. For high values of ρ, the global maximum corresponds to the lowest root of the equation $r(g^c) = \rho$, thus corresponding to a pure cake eating optimal policy.

A.3.5.2 Intermediate Behavior of $r(g^c)$. Case $\beta > 1$

Using (A.2.23), we get :

$$\frac{dr(g^c)}{dg^c} < 0 \;\;\Rightarrow\;\; 1 - \varepsilon < \frac{\beta\tilde{n}(g^c) - (\beta - 1)}{\alpha_1\tilde{n}(g^c) + \alpha_2[\beta\tilde{n}(g^c) - (\beta - 1)]} \,.$$

Note that:

$$\alpha_1\tilde{n}(g^c)+\alpha_2[\beta\tilde{n}(g^c)-(\beta-1)] = (\alpha_1+\alpha_2\beta)[\tilde{n}(g^c)-\frac{\alpha_2(\beta-1)}{\alpha_1+\alpha_2\beta}] = (\alpha_1+\alpha_2\beta)(\tilde{n}(g^c)-n_\beta)\,.$$

But since $n_\beta < \tilde{n}(\underline{g}_1^c) < \tilde{n}(g^c)$ for $g^c > \underline{g}_1^c$, we conclude that the denominator of the above ratio is strictly positive. Hence :

$$\frac{dr(g^c)}{dg^c} < 0 \;\Rightarrow\; (1-\varepsilon)[(\alpha_1+\alpha_2\beta)\tilde{n}(g^c) - \alpha_2(\beta-1)] < \beta\tilde{n}(g^c) - (\beta-1) \ ,$$

which is equivalent to :

$$\frac{dr(g^c)}{dg^c} < 0 \;\Rightarrow\; \tilde{n}(g^c) > \frac{(\beta-1)(\alpha_1+\varepsilon\alpha_2)}{(\beta-1)(\alpha_1+\varepsilon\alpha_2)+\varepsilon} \equiv \breve{n} \ . \qquad \text{(A.3.14)}$$

Hence we conclude that :

$$\frac{dr(g^c)}{dg^c} \;<,=,>\; 0 \;\Leftrightarrow\; \tilde{n}(g^c) \;>,=<\; \breve{n} \quad \text{for } g^c \neq \underline{g}_1^c \ .$$

Proof that $\tilde{n}(\underline{g}_1^c)$ may be either lower or higher than $\breve{n}$.
First note that :

$$n_h < \breve{n} \;\Leftrightarrow\; \varepsilon < \frac{\alpha_1^2(\beta-1)}{\alpha_2(\alpha_1+\alpha_2\beta)} \ .$$

Since $\tilde{n}(\underline{g}_1^c) < n_h$, we conclude that for some parameters values the function $r(g^c)$ should be first increasing and then decreasing over the range of values $(\underline{g}_1^c, \bar{g}^c)$. Let us concentrate upon the case $\breve{n} < n_h$.
Remembering that $\delta(\tilde{n}(\underline{g}_1^c)) = 1$ by definition of $\underline{g}_1^c$ and that $\delta(n_2)$ is a strictly decreasing function of n_2, we conclude that :

$$\delta(\breve{n}) \;<,=,>\; 1 \;\Leftrightarrow\; \breve{n} \;>,=,<\; \tilde{n}(\underline{g}_1^c) \ .$$

Making use of the expression (A.3.14) of $\breve{n}$ and (A.2.19) of $\delta(n_2)$, we obtain :

$$\delta(\breve{n}) \;=\; \left[1 + \frac{(\beta-1)(\alpha_1+\varepsilon\alpha_2)}{\varepsilon}\right]^{\alpha_1} \left[1 - \frac{\alpha_1}{\alpha_2\beta}\frac{(\beta-1)(\alpha_1+\varepsilon\alpha_2)}{\varepsilon}\right]^{\alpha_2} \ .$$

Let $e \equiv \varepsilon^{-1}(\beta-1)(\alpha_1+\varepsilon\alpha_2)$, some function of the parameters. Thus :

$$\delta(\breve{n}) \equiv \delta(e) = (1+e)^{\alpha_1}(1 - \frac{\alpha_1}{\alpha_2\beta}e)^{\alpha_2} \ .$$

Note that $\breve{n} < n_h$ implies that $e < \alpha_1^{-1}\alpha_2\beta \equiv \bar{e}$. Moreover $\delta(0) = 1$ and $\delta(\bar{e}) = 0$. Differentiating :

$$\frac{d\delta(e)}{de} \;=\; \delta\left[\frac{\alpha_1}{1+e} - \frac{\alpha_1\alpha_2}{\alpha_2\beta-\alpha_1 e}\right] = \frac{\alpha_1}{(1+e)(\alpha_2\beta-\alpha_1 e)}(\alpha_2(\beta-1)-e) \ .$$

Note that $\bar{e} = \alpha_1^{-1}\alpha_2\beta > \alpha_2\beta > \alpha_2(\beta - 1)$, hence $e_0 \equiv \alpha_2(\beta - 1) < \bar{e}$. As a function of the parameters, $\delta(e)$ is first increasing up to e_0 (and $\delta(e_0) > 1$) and then decreasing down to zero for $e \to \bar{e}$. This implies the existence of a unique critical value of e, denoted by e_1, solution of $\delta(e) = 1$ and such that $0 < e_0 < e_1 < \bar{e}$. Thus we conclude that $\delta(\breve{n}) > 1$ if $e \in (0, e_1)$ and $\delta(\breve{n}) < 1$ if $e \in (e_1, \bar{e})$.

The previous discussion shows that depending upon the parameter values (e. g. the parametric function e), $\breve{n}$ may be higher or lower than $\tilde{n}(\underline{g}_1^c)$. In a case where $\breve{n} < \tilde{n}(\underline{g}_1^c)$, this would imply $\breve{n} < \tilde{n}(g^c)$, for $g^c \in (\underline{g}_1^c, \bar{g}^c)$ and thus that $r(g^c)$ should be a strictly decreasing function of g^c in this interval. Conversely, if $\breve{n} > \tilde{n}(\underline{g}_1^c)$ then $r(g^c)$ should be strictly increasing up to $\breve{g}^c$ solution of $\tilde{n}(g^c) = \breve{n}$, and then decreasing over the interval $(\breve{g}^c, \bar{g}^c)$. These are the two possibilities illustrated in Figures 5 and 6.

$>$From the above discussion, we conclude that :

i) If $\breve{n} < \tilde{n}(\underline{g}_1^c)$ and :

- If $\rho > \bar{r}(\underline{g}_1^c)$ or if $\rho < \underline{r}(\underline{g}_1^c)$, then the equation $r(g^c) = \rho$ admits only one root, g^{c*}, either corresponding to a cake eating path where $n^* = 0$ (if $\rho > \bar{r}(\underline{g}_1^c)$), or corresponding to an active research policy (if $\rho < \underline{r}(\underline{g}_1^c)$), where $n^* = \tilde{n}(g^{c*})$.

- If $\rho \in [\underline{r}(\underline{g}_1^c), \bar{r}(\underline{g}_1^c)]$, then the equation $r(g^c) = \rho$ admits two roots corresponding either to a cake eating policy or to an active research policy.

ii) If $\breve{n} > \tilde{n}(\underline{g}_1^c)$ and :

- If $\rho < \underline{r}(\underline{g}_1^c)$ or if $\rho > r(\breve{g}^c)$, then the equation $r(g^c) = \rho$ admits only one root, g^{c*}, either corresponding to an active research policy or to a cake eating path.

- If $\underline{r}(\underline{g}_1^c) < \rho < \bar{r}(\underline{g}_1^c)$, then the equation $r(g^c) = \rho$ admits two distinct roots corresponding either to an active research policy or to a cake eating policy.

- If $\bar{r}(\underline{g}_1^c) < \rho < r(\breve{g}^c)$, then the equation $r(g^c) = \rho$ admits three roots, one of them corresponding to a cake eating policy, the two others to active research policies.

A.3.5.3 Optimal regular paths

We now prove that despite the possible multiplicity of roots of the equation $r(g^c) = \rho$, there exists in all cases, excepted for a critical value of ρ one and only one optimal regular path.

Value function along an efficient regular path

Let V_τ be the value function for a regular path starting from τ in current value :

$$\begin{aligned} V_\tau &= e^{\rho\tau}\int_\tau^\infty u(c_t)e^{-\rho t}dt \\ &= (1-\varepsilon)^{-1}\int_\tau^\infty c_\tau^{1-\varepsilon}e^{[(1-\varepsilon)g^c-\rho](t-\tau)}dt \\ &= \frac{c_\tau^{1-\varepsilon}}{(1-\varepsilon)(\rho-(1-\varepsilon)g^c)} \; . \end{aligned} \tag{A.3.15}$$

Hence the value function for an *efficient* regular path, $\tilde{V}_\tau(g^c)$, where $c_\tau = \tilde{c}_\tau(g^c)$, is given by :

$$\tilde{V}_\tau(g^c) = \frac{[\tilde{c}_\tau(g^c)]^{1-\varepsilon}}{(1-\epsilon)(\rho-(1-\epsilon)g^c)} \tag{A.3.16}$$

Since $\tilde{c}_\tau(g^c)$ is a continuous function of g^c, even at $g^c = \underline{g}_1^c$, as shown before, $\tilde{V}_\tau(g^c)$ is a continuous function of g^c, but not differentiable at $g^c = \underline{g}_1^c$.

Over any open interval $\mathcal{G}$ of differentiability of the function $\tilde{V}_\tau(g^c)$, we get :

$$\frac{d\tilde{V}_\tau(g^c)}{dg^c} = \frac{\tilde{c}_\tau(g^c)^{-\varepsilon}}{(\rho-(1-\varepsilon)g^c)^2}\left[(\rho-(1-\varepsilon)g^c)\frac{d\tilde{c}_\tau(g^c)}{dg^c}+\tilde{c}_\tau(g^c)\right] \; , \; g^c \in \mathcal{G} \; .$$

By (6.16) :

$$d\,\frac{\tilde{c}_\tau(g^c)}{dg^c} = \frac{\partial \hat{c}_\tau}{\partial g^c}|_{n=\tilde{n}(g^c)} \;=\; -\tilde{c}_\tau(g^c)[b\tilde{n}(g^c)^\beta\bar{n}^{-\beta}-\alpha_2^{-1}g^c]^{-1} \; ,$$

hence we obtain :

$$\frac{d\tilde{V}_\tau(g^c)}{dg^c} \;=\; \frac{\tilde{c}_\tau^{1-\varepsilon}}{(\rho-(1-\varepsilon)g^c)^2}\left[1-\frac{\rho-(1-\varepsilon)g^c}{b\bar{n}^{-\beta}\tilde{n}(g^c)^\beta-\alpha_2^{-1}g^c}\right] \; ,$$

which is equivalent to :

$$\frac{d\tilde{V}_\tau(g^c)}{dg^c} = \frac{\tilde{c}_\tau(g^c)^{1-\varepsilon}}{(\rho-(1-\varepsilon)g^c)^2}\frac{r(g^c)-\rho}{|\tilde{g}^S(g^c)|} \; . \tag{A.3.17}$$

Let us consider the most intricate case of three roots, that is the case $\tilde{n}(\underline{g}_1^c) < \breve{n}$ and $\bar{r}(\underline{g}_1^c) < \rho < r(\breve{g}^c)$. Let g_0^c, g_1^c and g_2^c be the three roots of the equation $r(g^c) = \rho$, such that $g_0^c < \underline{g}_1^c < g_1^c < \breve{g}^c < g_2^c$. Since :

$$\frac{d\tilde{V}_\tau(g^c)}{dg^c} \;<,\; =,\; >\; 0 \;\;\Leftrightarrow\;\; r(g^c) \;<,\; =,\; >\; \rho \quad g^c \neq \underline{g}_1^c \; ,$$

we conclude that $\tilde{V}_\tau(g^c)$ is first increasing over the interval $(-\infty, g_0^c)$, decreasing over the interval (g_0^c, g_1^c), increasing for $g^c \in (g_1^c, g_2^c)$ and last decreasing over the interval $(g_2^c, \bar{g}^c)$. The root g_1^c corresponds to a local minimum, and thus only the extreme roots g_0^c (which corresponds to a cake eating policy) and g_2^c (which corresponds to an active research policy) have to be compared to determine the global maximum of $\tilde{V}_\tau(g^c)$. This reasoning extends to the other cases of roots multiplicity.

Ranking of the roots of $r(g^c) = \rho$ through the value function

Let g_0^c be the root of $r(g^c) = \rho$ corresponding to a cake eating efficient regular policy (where $\tilde{n}(g_0^c) = 0$) and g_1^c be the maximal root of $r(g^c) = \rho$ corresponding to an active research efficient policy (where $\tilde{n}(g_1^c) > 0$). Let us also denote by $V_0 \equiv \tilde{V}_\tau(g_0^c)$ and $V_1 = \tilde{V}_\tau(g_1^c)$ the corresponding values of such regular efficient policies. Let $\underline{g}_0^c$ be the solution of either $r(g_0^c) = \bar{r}(\underline{g}_1^c)$ in a case where $\breve{n} \leq \tilde{n}(\underline{g}_1^c)$ or the solution of $r(g_0^c) = r(\breve{g}^c)$ in a case where $\breve{n} > \tilde{n}(\underline{g}_1^c)$. Let $\bar{g}_1^c$ be the solution of $r(g_1^c) = \underline{r}(\underline{g}_1^c)$. By construction, the problem of multiple roots of $r(g^c) = \rho$ is confined to the intervals $g_0^c \in I_0 \equiv [\underline{g}_0^c, \underline{g}_1^c]$ and $g_1^c \in I_1$, where $I_1 \equiv [\breve{g}^c, \bar{g}_1^c]$ if $\breve{n} < \tilde{n}(\underline{g}_1^c)$, or where $I_1 \equiv [\underline{g}_1^c, \bar{g}_1^c]$ if $\breve{n} > \tilde{n}(\underline{g}_1^c)$.

Since $r(g_0^c) = r(g_1^c)$ for $(g_0^c, g_1^c) \in I_0 \times I_1$, we get a first relation between g_0^c and g_1^c :

$$\begin{aligned}(1 - \varepsilon - \alpha_2^{-1})g_0^c &= (1-\varepsilon)g_1^c + |\tilde{g}^S(g_1^c)| \\ &= (1 - \varepsilon - \alpha_2^{-1})g_1^c + \alpha_2^{-1}\bar{g}^c[\tilde{n}_1(g_1^c)]^\beta.\end{aligned}$$

Let $n_1 \equiv \tilde{n}(g_1^c)$. Since $\tilde{n}(g_1^c)$ is a monotone increasing function of $g_1^c, g_1^c \in I_1$, the above relation is equivalent to the following relation between g_0^c and n_1 :

$$g_0^c = \alpha_1^{-1}\bar{g}^c h(n_1) + \frac{1}{\alpha_2(1-\varepsilon) - 1}\,\bar{g}^c n_1^\beta,$$

which, using the expression of $h(n_1)$, is equivalent to :

$$g_0^c = G_0(n_1) \equiv -\alpha_1^{-1}\alpha_2\beta\bar{g}^c(1 - k_0 n_1) \quad g_0^c \in I_0,$$

where :

$$k_0 = 1 - \frac{\alpha_1(1-\varepsilon)}{\beta(1 - \alpha_2(1-\varepsilon))}$$

Next, since $r(g^c) = \rho$, for either $g^c = g_0^c$ or $g^c = g_1^c$, $\rho - (1-\varepsilon)g^c = |\tilde{g}^S(g^c)|$, hence :

$$V_i = \frac{\tilde{c}(g_i^c)^{1-\varepsilon}}{(1-\varepsilon)|\tilde{g}^S(g_i^c)|}\,, \quad i = 0, 1\,.$$

Making use of the expressions of $\tilde{c}_\tau(g^c)$ and $\tilde{g}^S(g^c)$, V_0 and V_1 may be easily computed :

$$\begin{aligned} V_0 &= K(\varepsilon)(-\alpha_2^{-1}g_0^c)^{\alpha_2(1-\varepsilon)-1} \\ V_1 &= K(\varepsilon)(1-\tilde{n}(g_1^c))^{\alpha_1(1-\varepsilon)}[\alpha_2^{-1}(\bar{g}^c\tilde{n}(g_1^c))^\beta - g_1^c]^{\alpha_2(1-\varepsilon)-1} \end{aligned}$$

where $K(\varepsilon) = (A^{\alpha_1}R_\tau^{\alpha_2})^{1-\varepsilon}(1-\varepsilon)^{-1}$.

Since $g_1^c = \alpha_1^{-1}\bar{g}^c h(n_1)$, it is possible to rewrite V_1 as a function of $n_1 \equiv \tilde{n}(g_1^c)$:

$$\begin{aligned} V_1 &= K(\varepsilon)(1-n_1)^{\alpha_1(1-\varepsilon)}[\alpha_2^{-1}\bar{g}^c(n_1^\beta - \alpha_1^{-1}h(n_1))]^{\alpha_2(1-\varepsilon)-1} \\ &= K(\varepsilon)(1-n_1)^{\alpha_1(1-\varepsilon)}[\bar{g}^c\alpha_1^{-1}\beta n_1^{\beta-1}(1-n_1)]^{\alpha_2(1-\varepsilon)-1}, \end{aligned}$$

which after simplifications results in :

$$V_1 = K(\varepsilon)(1-n_1)^{-\varepsilon}[\alpha_1^{-1}\beta\bar{g}^c n_1^{\beta-1}]^{\alpha_2(1-\varepsilon)-1}.$$

Since $K(\varepsilon) > / < 0$ depending upon $\varepsilon < / > 1$, we conclude that :

$$(-\alpha_2^{-1}g_0^c)^{\alpha_2(1-\varepsilon)-1} \geq (1-n_1)^{-\varepsilon}[\alpha_1^{-1}\beta\bar{g}^c n_1^{\beta-1}]^{\alpha_2(1-\varepsilon)-1}$$

$$\Leftrightarrow \begin{cases} V_0 \geq V_1 & \text{if} \quad \varepsilon < 1 \\ V_0 \leq V_1 & \text{if} \quad \varepsilon > 1 \end{cases}$$

Since $\alpha_2(1-\varepsilon) - 1 < 0$, whatever the value of ε, the above is equivalent to :

$$g_0^c \geq G_1(n_1) \equiv -\alpha_1^{-1}\alpha_2\beta\bar{g}^c n_1^{\beta-1}(1-n_1)^{\frac{\varepsilon}{1-\alpha_2(1-\varepsilon)}} \Leftrightarrow \begin{cases} V_0 \geq V_1 & \text{if} \quad \varepsilon < 1 \\ V_0 \leq V_1 & \text{if} \quad \varepsilon > 1 \end{cases}$$

Now, observe that for $g_0^c \in I_0$:

$$g_0^c \geq G_1(n_1) \Leftrightarrow G_0(n_1) \geq G_1(n_1) \Leftrightarrow 1 - k_0 n_1 \leq (1-n_1)^{\frac{\varepsilon}{1-\alpha_2(1-\varepsilon)}}.$$

Next consider the function $f(n) \equiv (1-n)^{\frac{\varepsilon}{1-\alpha_2(1-\varepsilon)}}$, the right hand side of the above relation, over the range $n \in [0,1]$. Note that $k_0 < / > 1$ depending upon $\varepsilon < / > 1$. We get immediately :

$$f(0) = 1\ ,\ f(1) = 0\ ,\ f'(n) = -\frac{\varepsilon}{1-\alpha_2(1-\varepsilon)}\,(1-n)^{-\frac{\alpha_1(1-\varepsilon)}{1-\alpha_2(1-\varepsilon)}} < 0$$

$$|\lim_{n\downarrow 0} f'(n)| = \frac{\varepsilon}{1-\alpha_2(1-\varepsilon)} \begin{cases} < k_0 & \text{if} \quad \varepsilon < 1 \\ > k_0 & \text{if} \quad \varepsilon > 1 \end{cases}$$

$$f''(n) = -\frac{\alpha_1\varepsilon(1-\varepsilon)}{(1-\alpha_2(1-\varepsilon))^2}(1-n)^{\frac{\alpha_1(1-\varepsilon)}{1-\alpha_2(1-\varepsilon)}-1} \begin{cases} > 0 & \text{if} \quad \varepsilon < 1 \\ < 0 & \text{if} \quad \varepsilon > 1 \end{cases}$$

In the case $\varepsilon < 1$, $f(.)$ is a strictly decreasing concave function crossing the line $1-k_0 n$ from above at a unique point $\bar{n} \in (0,1)$ since $|\lim_{n\downarrow 0} f'(n)| < k_0$ in such a case. Hence $G_0(n) \geq G_1(n) \Leftrightarrow n \leq \bar{n}$. This implies that $V_0 \geq V_1$ for $n_1 \leq \bar{n}$. Conversely, if $\varepsilon > 1$, $f(n)$ is a strictly decreasing convex function crossing the line $1-k_0 n$ from below at a unique point $\bar{n} \in (0,1)$, since now $|\lim_{n\downarrow 0} f'(n)| > k_0$. Hence $G_0(n) \leq G_1(n) \Leftrightarrow n \leq \bar{n}$. Thus, this implies also that $V_0 \geq V_1$ for $n_1 \leq \bar{n}$.

Let $\underline{n} = \tilde{n}(\underline{g}_1^c)$ if $\breve{n} < \tilde{n}(\underline{g}_1^c)$ or $\underline{n} = \breve{n}$ if $\breve{n} \geq \tilde{n}(\underline{g}_1^c)$, and $\bar{n}_1 \equiv \tilde{n}(\bar{g}_1^c)$. We may distinguish three cases.

i) Either $\bar{n} < \underline{n} < \bar{n}_1$. In this case $V_0 \leq V_1$ for $(g_0^c, g_1^c) \in I_0 \times I_1$. We conclude that for $\rho \leq \bar{r}(\underline{g}_1^c)$ in case of either $\breve{n} < \tilde{n}(\underline{g}_1^c)$ or $\rho \leq r(\breve{g}^c)$ when $\breve{n} \geq \tilde{n}(\underline{g}_1^c)$, then the active research regular efficient path is the only regular optimal path. Letting $\bar{\rho} = \bar{r}(\underline{g}_1^c)$ if $\breve{n} < \tilde{n}(\underline{g}_1^c)$ or $\bar{\rho} = r(\breve{g}^c)$ if $\breve{n} \geq \tilde{n}(\underline{g}_1^c)$, then the optimal regular policy corresponds to a cake eating policy if $\rho > \bar{\rho}$ or to an active research policy if $\rho \leq \bar{\rho}$.

ii) Or $\bar{n} \in [\underline{n}, \bar{n}_1]$. In this case, we conclude that $V_0 < V_1$ for values of g_1^c such that $\tilde{n}(g_1^c) < \bar{n}$ and that $V_0 > V_1$ for values of g_1^c such that $\tilde{n}(g_1^c) > \bar{n}$. To the critical value $\bar{n}$ where $V_0 = V_1$ there corresponds a unique critical value of $g_1^c, \hat{g}_1^c$, such that $\tilde{n}(\hat{g}_1^c) = \bar{n}$. Let $\bar{\rho} = r(\hat{g}_1^c)$, then for $\rho > \bar{\rho}$, since $g_1^c < \hat{g}_1^c$, we get $\tilde{n}(g_1^c) < \bar{n}$ hence $V_0 > V_1$. The only optimal regular path is a pure cake eating path in this case. Conversely if $\rho < \bar{\rho}$, then $g_1^c > \hat{g}_1^c$ implies that $\tilde{n}(\hat{g}_1^c) > \bar{n}$ hence that $V_0 < V_1$. The only optimal path corresponds to an active research policy if $\rho < \bar{\rho}$. In the critical case $\rho = \bar{\rho}, V_0 = V_1$, hence there exists two optimal regular paths, the first one corresponding to a pure cake eating policy and the second one to an active research policy.

iii) Or last $\bar{n} > \bar{n}_1$. In this case $V_0 > V_1$ for $(g_0^c, g_1^c) \in I_0 \times I_1$. Let $\bar{\rho} = \underline{r}(\underline{g}_1^c)$, then for $\rho \geq \bar{\rho}$, the only optimal regular path is the cake eating path, and, for $\rho < \bar{\rho}$, is an active research path.

Hence, in all cases, there exists $\bar{\rho} \in [\underline{r}(\underline{g}_1^c), \bar{r}(\underline{g}_1^c)]$ if $\breve{n} < \tilde{n}(\underline{g}_1^c)$ or $\bar{\rho} \in [\underline{r}(\underline{g}_1^c), r(\breve{g}^c)]$ if $\breve{n} \geq \tilde{n}(\underline{g}_1^c)$ such that, when $\rho > \bar{\rho}$, the only optimal regular policy corresponds to an efficient regular path where $n = 0$ (a pure cake eating path) ; when $\rho < \bar{\rho}$, the only optimal regular policy is an active research policy, and if $\rho = \bar{\rho}$, there exists two optimal regular paths corresponding either to the cake eating path or to the active research path. ■

Notes

[1] See also Dandrakis and Phelps (1965), Nelson and Phelps (1966), Phelps (1966), Samuelson (1965), von Weizäker (1966), and the recent contributions of Acemoglu (2003-b, 2007) and Sato (2006).

[2]See Suzuki (1976), Dasgupta, Heal and Majumdar (1977), Kamein and Schwartz (1978), Davison (1978), Chiarella (1980), Takayama (1980), Robson (1980) for examples, and Dasgupta, Gilbert and Stiglitz (1980) for a partial equilibrium analysis.

[3]Stiglitz (1974) assumes Cobb-Douglas production functions so that the technical progress cannot be dedicated under the additional assumption that any factor x measured in efficiency units takes the form ax, where a is some positive efficiency index.

[4]However this proliferation would be questioned by most historians versed in technological history. To anybody which would not be convinced we suggest to visit any technological museum, on pre-industrial art and craft. The most striking evidence of such visits is that, given a level of development of general knowledge, each niche is fully exploited, giving rise to a lot of either intermediate or final goods. For example in the "Musée de l'outil", in the small town of Troyes (Champagne country, France) we denumbered sixty types of planes used by wood workers, and it is a mere sample of such capital goods because there exists a continuum of such goods, each one adapted to some specific task and to the characteristics of each wood worker (skill, morphology, and so on...).

[5]Concerning the strong differences between non renewable and renewable resource economies, see Amigues, Long and Moreaux (2004), Amigues and Moreaux (2004), and Moreaux and Ricci (2004).

[6]and a continuum of intermediate conceptions.

[7]The heterogenous labor case is explored in Amigues *and alii* (2007).

[8]Given n_t^* $t \in [t_1, t_\tau]$ and $B_{t_1}^*$, $B_{t_2}^*$ is the value at t_2 of the solution of the differential equation $\dot{B}_t = (n_t^*, B_t)$ through $B_{t_1}^*$ at t_1.

[9]The pure state constraint $S_t \geq 0$ would require to introduce a Lagrange multiplier γ_t such that along an optimal path $\gamma_t S_t = 0$, $\forall t \geq 0$. However, since U.1 will imply an infinite duration exhaustion program of the natural resource, then $\gamma_t = 0, t \geq 0$ along the optimal path. Because the transversality condition (4.10) implies the same kind of requirement for optimality, we dispense from introducing explicitly the constraint $S \geq 0$ into the expression of the Lagrangian, a standard procedure in such contexts.

[10]For any pair of time functions x and y, we denote by $(\dot{xy})$ the time derivative of the product xy : $(\dot{xy}) = \dot{x}y + x\dot{y}$.

[11]For any variable x function of time, we denote by g^x its instantaneous proportional rate of growth : $g_t^x = \dot{x}_t x_t^{-1}$.

[12]Clearly without the F.E condition what could happen could not be better since some parts of the primary resources could possibly be wasted.

[13]Figures 5 and 6 are drawn assuming that $\varepsilon < 1$ so that $r(\bar{g}^c) > 0$. But it could happen that $r(\bar{g}^c) < 0$ for $\varepsilon > 1$. The highest root of (7.3) for $\rho = \bar{\rho}$, denoted by $\bar{g}^{c*}(\bar{\rho})$, is always higher than $\underline{g}_1^c < 0$. But it may be higher or lower than 0, whatever ε.

[14]In case of Figure 6, $\bar{\rho}$ may be either higher or lower than $\bar{r}(\underline{g}_1^c)$ depending upon the values of the parameters of the model.

[15]For any ratio $(\frac{x}{y})$, we denote by $(\frac{\dot{x}}{y})$ the time derivative $\frac{d}{dt}(\frac{x}{y})$

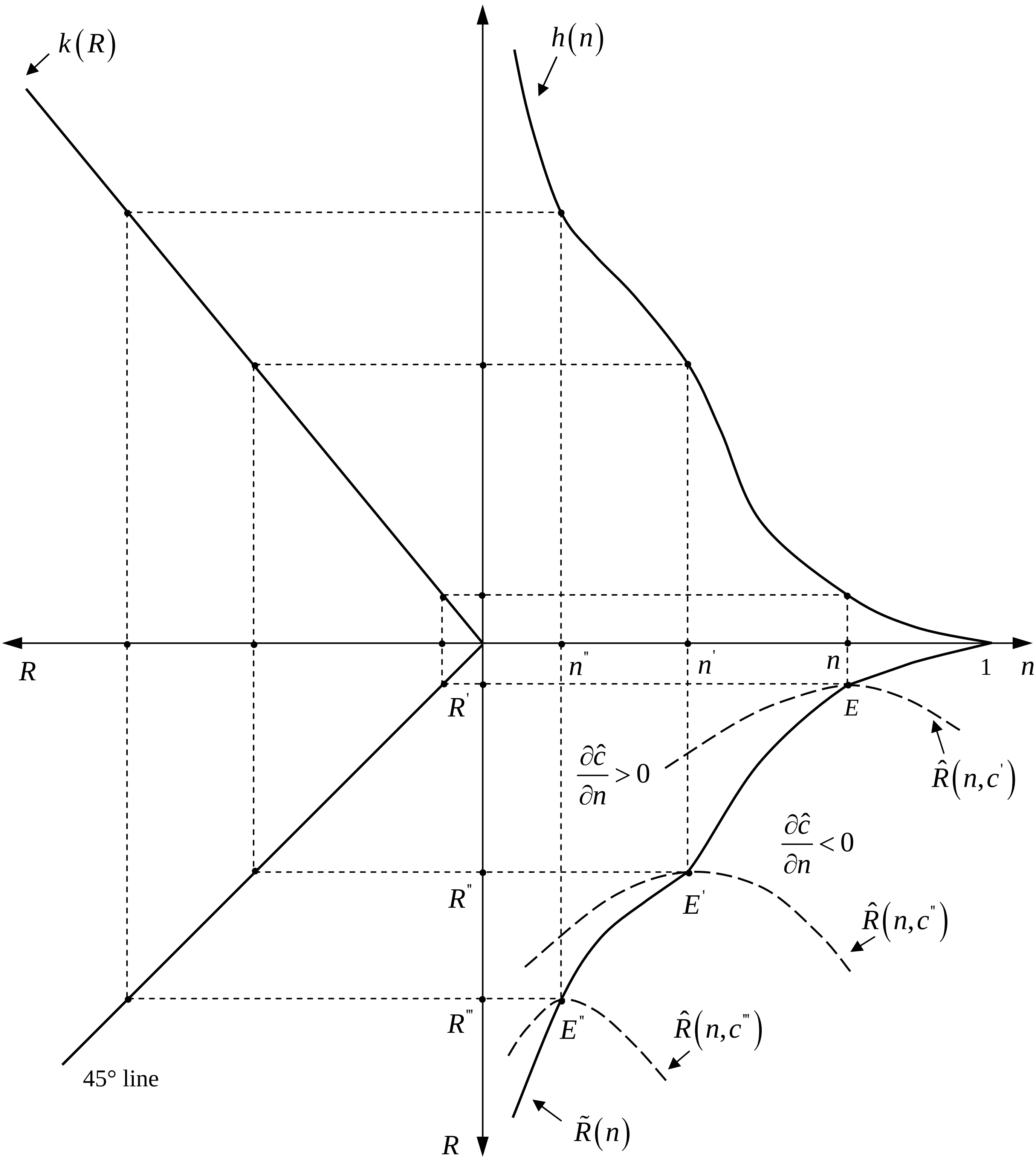

Figure 1: **Determination of the loci of feasible F.E regular paths and efficient regular paths-CES case.**

N.B: $c' < c'' < c'''$; **N.W quadrant drawn assuming** $\eta = 1$;
$R' = \tilde{R}(n'), R'' = \tilde{R}(n'')$ and $R''' = \tilde{R}(n''')$.

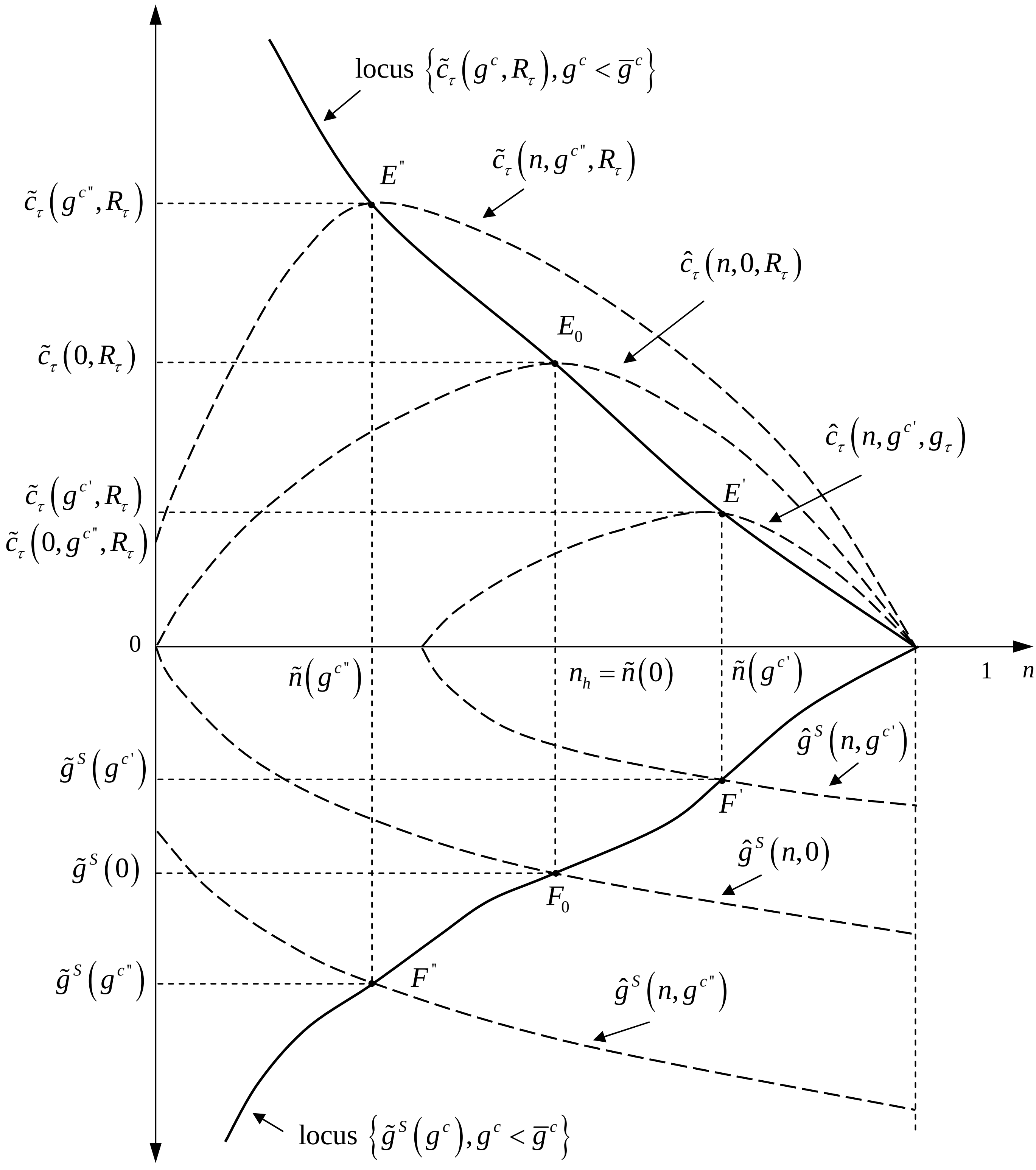

Figure 2: **Loci of F.E regular paths and efficient regular paths. Case Cobb-Douglas and** $\beta < 1$

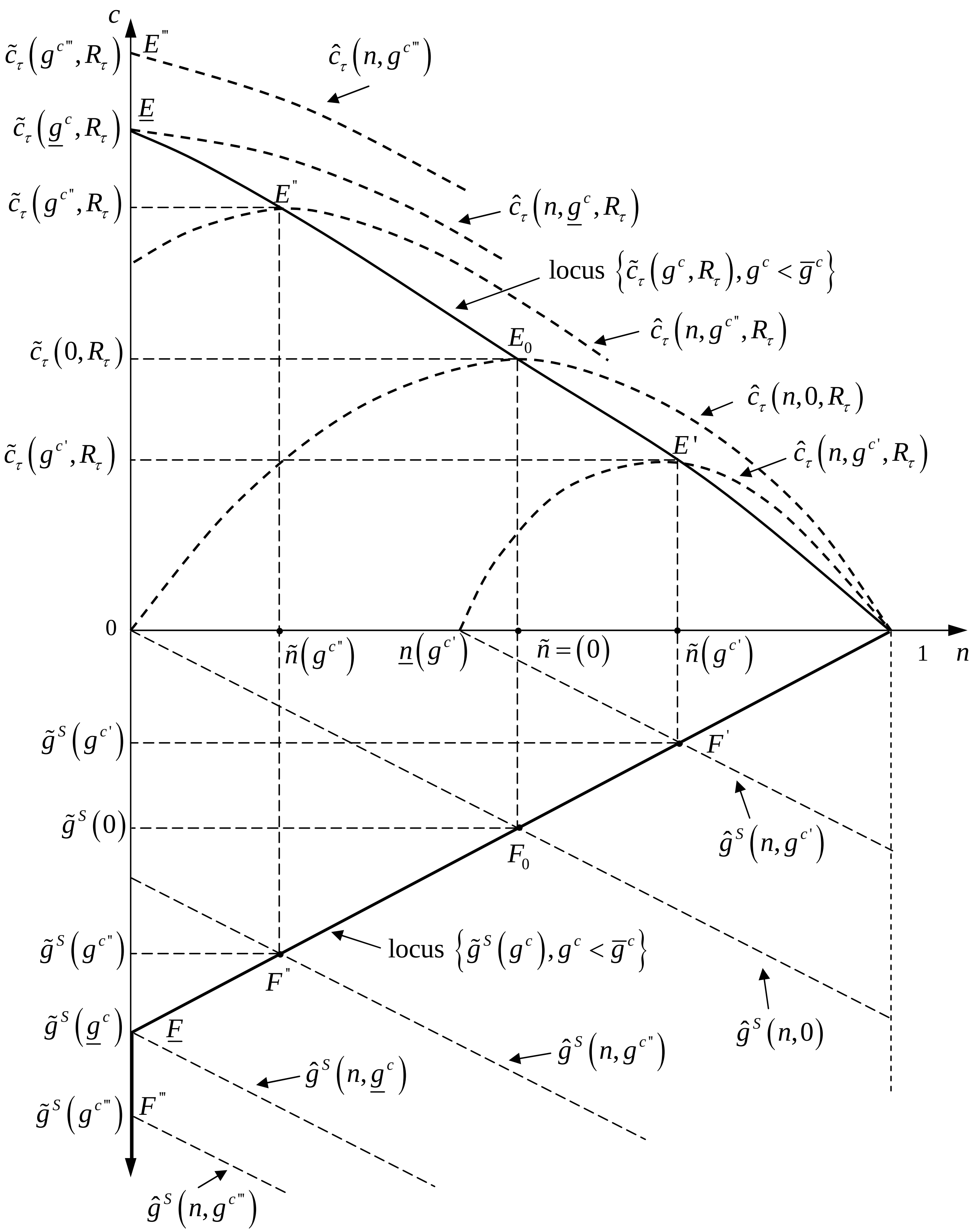

Figure 3: **Loci of regular paths and efficient $\beta = 1$ regular paths. Case Cobb-Douglas case**

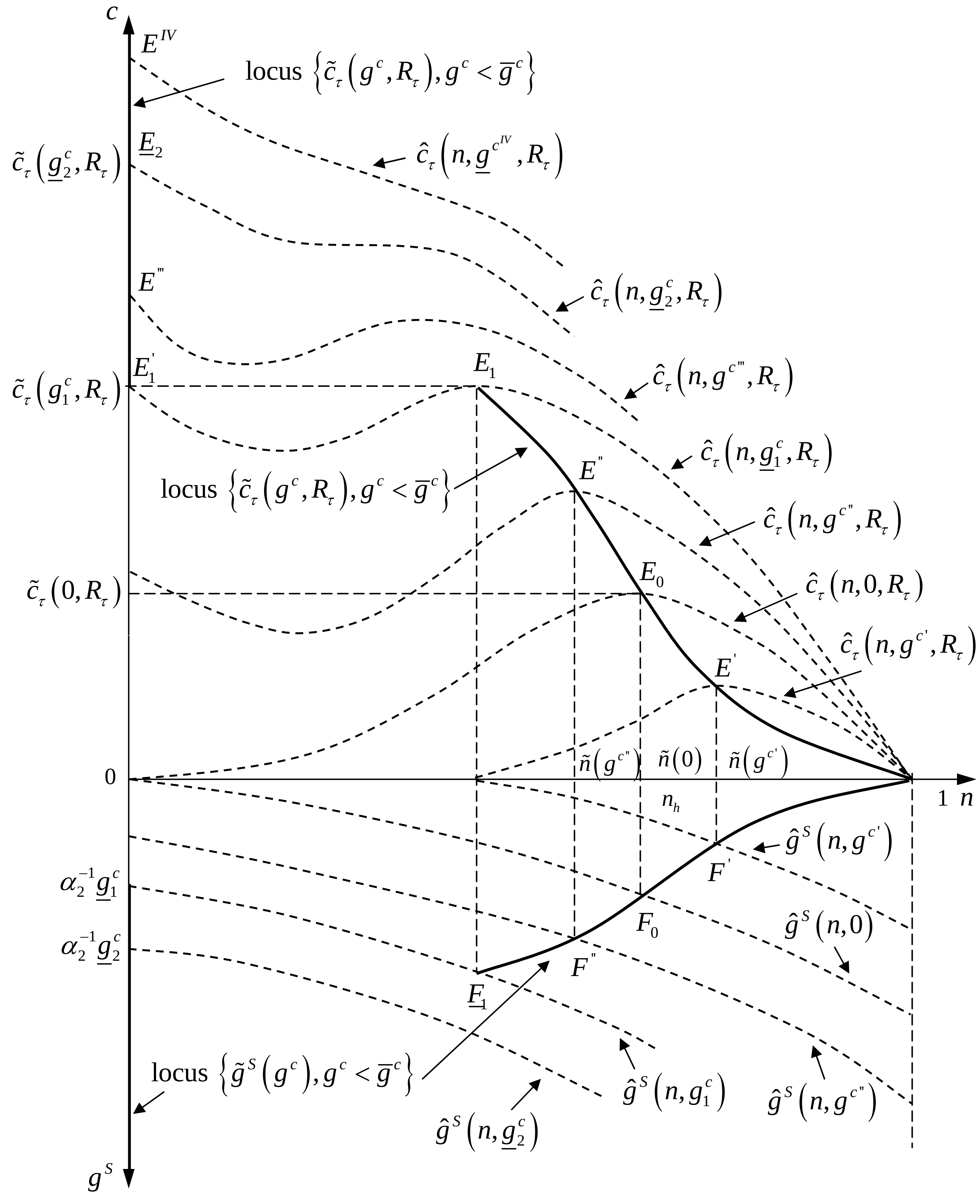

Figure 4: **Loci of F.E regular paths and efficient regular paths. Case Cobb-Douglas case and** $\beta = 1$
(NB: $\partial \hat{g} / \partial n \big|_{n=0} = 0$ if $\beta = 1$ **)**

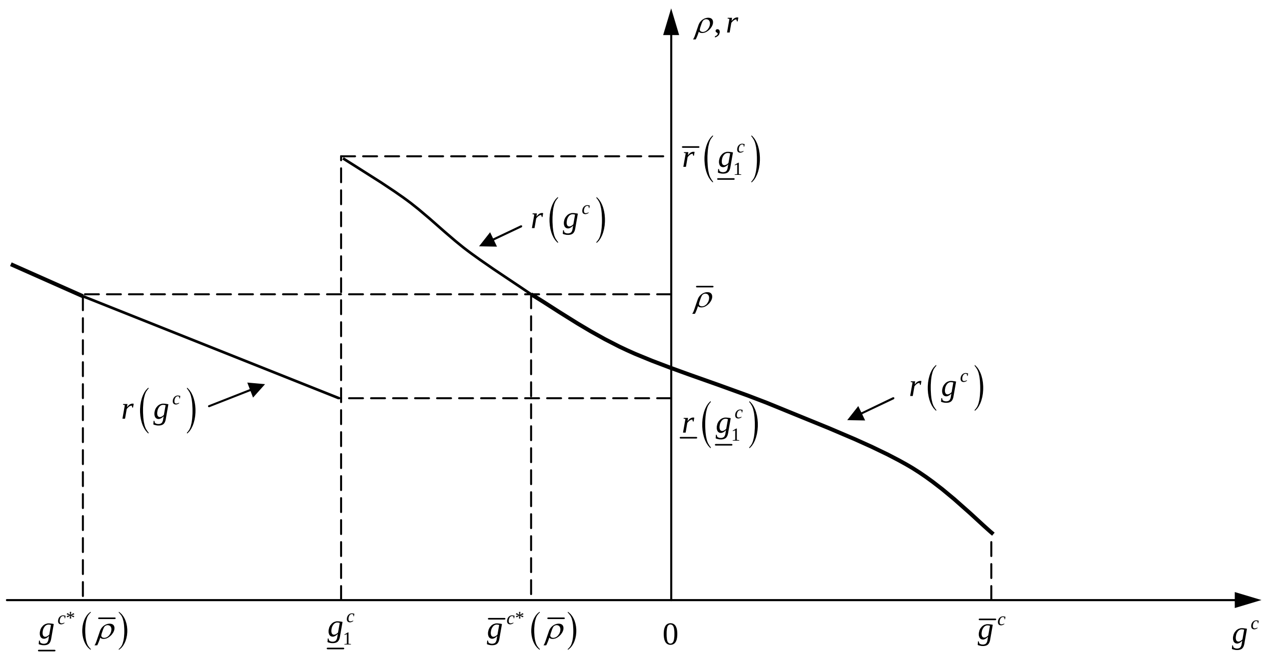

Figure 5: **Function** $r\left(g^c\right)$ **. Case** $\beta > 1$ **and** $\breve{g} < \underline{g}_1^c$

NB: $\underline{g}^{c*}(\bar{\rho})$ **and** $\bar{g}^{c*}(\bar{\rho})$, **optimal policies for** $\rho = \bar{\rho}$

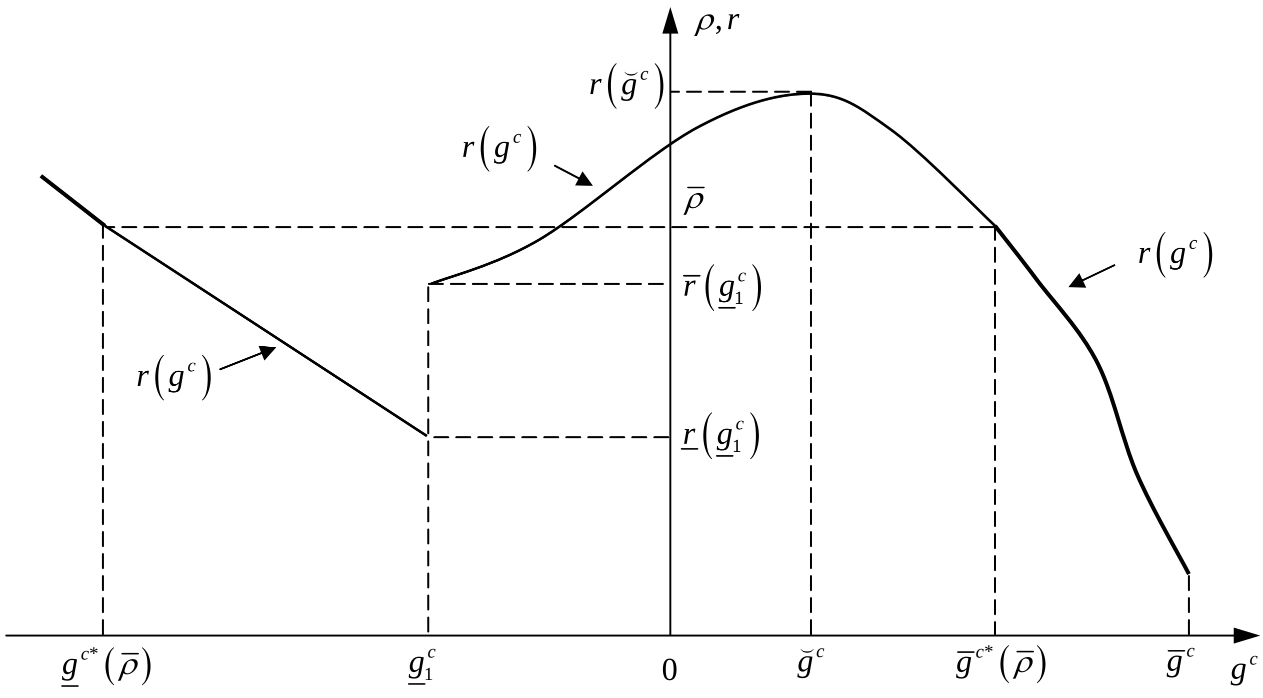

Figure 6: **Function** $r\left(g^c\right)$ **. Case** $\beta > 1$ **and** $\breve{g} > \underline{g}_1^c$

NB: $\underline{g}^{c*}(\bar{\rho})$ **and** $\bar{g}^{c*}(\bar{\rho})$, **optimal policies for** $\rho = \bar{\rho}$

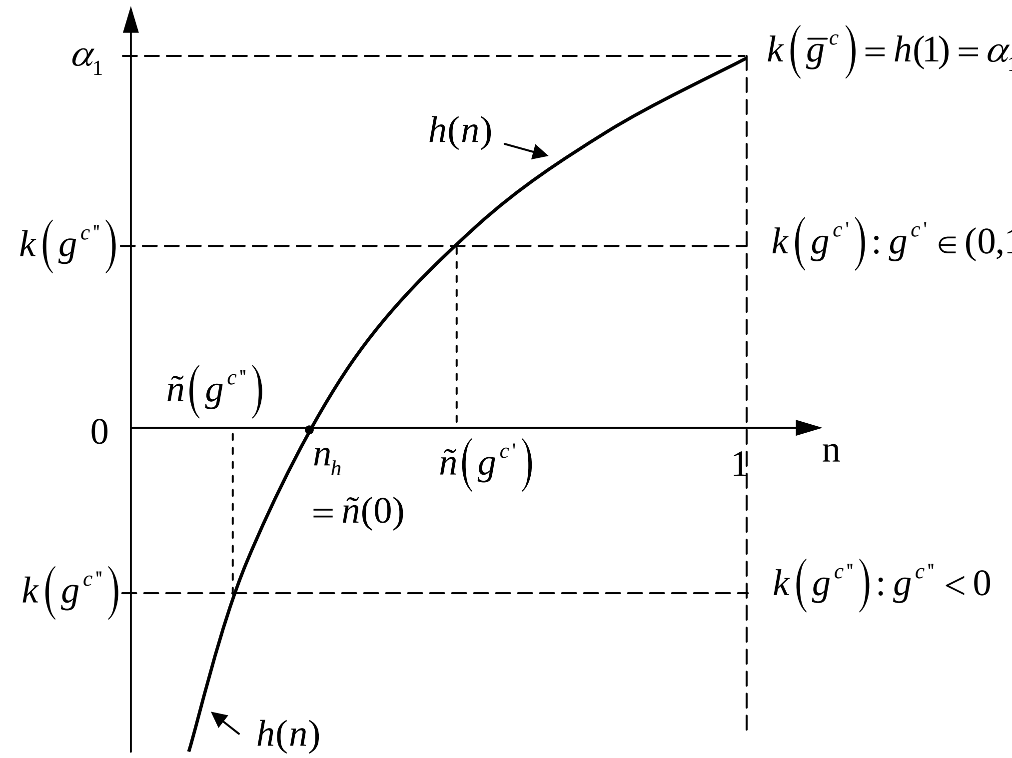

Figure A.2.1: **Case** $\beta < 1$

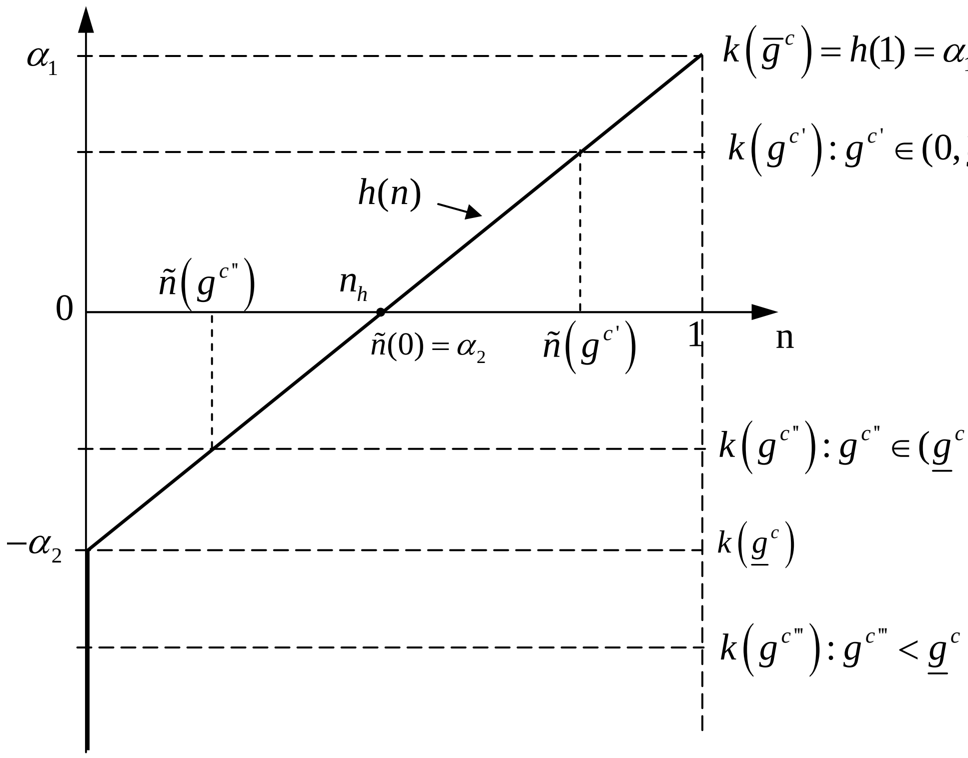

Figure A.2.2: **Case** $\beta = 1$

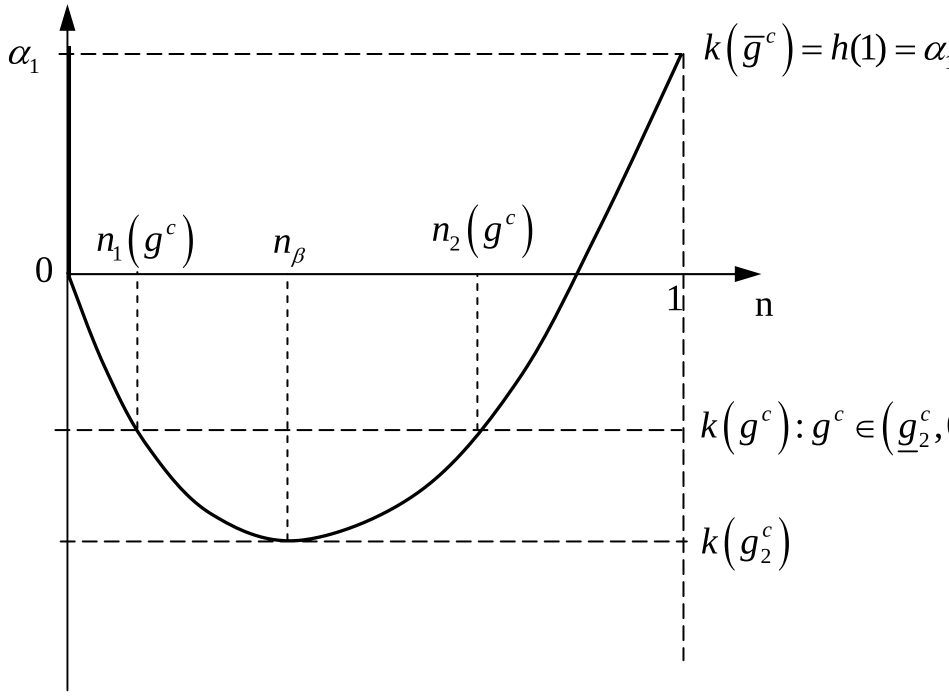

Figure A.2.3: **Case** $\beta > 1$