Legislative Lobbying under Political Uncertainty*

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Abstract

In this paper, we develop a duopolistic model of legislative lobbying. Two lobbies compete to influence the votes of a group of legislators who have a concern for both social welfare and campaign contributions. The type of a legislator is the relative weight he places on social welfare as compared to money. We study the equilibria of this lobbying game under political certainty and uncertainty and examine under which circumstances the policy is socially efficient, and how much money has been invested in the political process ?. A special attention is paid to three primitives of the environment : the intensity of the competition between the lobbies, the internal organization of the legislature and the proportion of bad and good legislators in the political arena.

*We would like to thank Ron Holzman for calling our attention on the least core and the nucleolus as a way to calculate the equilibrium of the lobbying game. We also thank seminar audiences in Leuven and Toulouse for their comments.

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1 Introduction

In all real polities, *special interest groups* or *lobbies*[1] participate actively in the policymaking process. An analytical apparatus has been developed aiming to provide a description of the channels through which the influence of these interest groups is exerted and a characterization of the main features of the equilibrium policies when this influence is accounted for. A common denominator of the research done in the last decade[2] on that topic has been to study structural models of the political process : economic and political actors behave rationally within well-specified economic and political institutions, where the policymaking process is formulated as an extensive form game. Methodologically, much progress has been made relatively to the traditional approaches, which were often based on inconsistent or irrational political and economic behavior, relying on non-derived influence functions, political support functions, or vote functions. While this new literature does not point out a single canonical model that would impose itself againsts its competitors, it is fair to say the the description of the competitive process among special interest groups as a *common agency game* (Bernheim and Whinston (1986)) has become a contender[3]. In this formulation, the principals are the lobbyists and the common agent is an incumbent politician depicted as having the power of selecting unilaterally the economic policy. The lobbyists move first : they, simultaneously or sequentially, offer to the common agent a menu of monetary payments conditional on the policy that will be ultimately selected. After contemplating the profile of offers, the politician decides which policy to select.

The anecdotical and empirical evidence is quite contrasted. Documenting that money affect policy outcomes is not an easy task. Indeed, as formulated by Grossman and Helpman (2001) "After all, it is difficult to know what a bill would have looked like absent the net effect of contributions. Even if we focus on roll-call votes, as many researchers have done, the effort is confounded by the counterfactual : how would a legislator have voted absent the contributions ? Perhaps a representative's vote on a bill was dictated by a concern for jobs in his district, which happens to be associated with the economic health of a contributor,

[1]It is not an easy task to define what is a special interest group (see Grossman and Helpman (2001) 's discussion on the matter). here we even use interchangeably the terms special interest groups and lobbies meaning that we ignore all the potential difficulties that a group may face to get some identity, and gives rise to a political organization/representation which is efficient. Not all groups are equal in that respect as suggested and investigated by Olson (1965). In this paper we skip this important aspect of the lobbying process known as the Olsonian program to focus on some other dimansions.

[2]See, for instance, Grossman and Helpman (2001) and Persson (1998).

[3]Laussel and Le Breton (2001) explore the structure of equilibrium payoffs. The common agency framework has been pionnered by Grossman and Helpman (1994)(2001) and followers to study trade policy, commodity taxation and other policies.

such as a large corporation. Or simply, the legislator was following the directives of party leaders". Variables should be introduced to control for these differents effects; for instance the legislator's ideological stance is reflected by his ratings with political organizations. Baldwin and Magee (2000) found that that the probability of a vote in favor of trade liberalization on the NAFTA and GATT Uruguay Round bills increases with the amount of contributions that a legislator receives from business interests and felt with the amount collected from labor unions. Stratmann (2003) studied the congressional votes on financial services legislation and conclude that contributions have changed voting behavior. These papers are just two examples of an antire genre of research[4] and several other authors have reached different, when not opposite conclusions. In the process of their analysis of the Tullock's (1972) puzzle about the little amount of money invested in U.S. politics, Ansolobehere, De Figueiredo and Snyder (2003) conclude that there is no econometric evidence that contributions have substantial effects on votes and legislative decisions and suggest an alternative explanation.

We depart from this literature in abandoning the assumption that policies are set by a single individual or by a cohesive, well-disciplined political party. In reality, most policy decisions, are made not by one person but by a group of elected representatives acting as a legislative body. Even when the *legislature*[5] is controlled by a single party (as it is necessarily the case in a two-party system if the legislature consists of a unique chamber[6]), the delegation members do not always follow the instructions of their party leaders. In situations with multiple independent legislators, special interest groups face a subtle problem in deciding how to allocate their resources to influence policy choices. For instance, should the lobby seek to solidify support among those legislators who would be inclined to support its positions anyway, or should it seek to win over those who might otherwise be hostile to its views ? The answer to this question and others like it depend on the rules of the legislative process i.e. the optimal strategy for wielding influence will vary with the institutional setting.

Many formal models of the *legislative process* have been developed by social scientists. The extensive game form describes the sequence of decision/information nodes of the legislators where a decision node typically consists in either the proposal of an alternative (there, the legislator acts as an agenda setter) or in expressing an opinion on a proposal (there, the

[4] Smith (1995) cites more than 35 studies poublished between 1980 and 1992 that attempted to explain roll-call votes in the U.S. Congress by campaign contributions from interested parties and by various indicatorss of a legislator's ideology.

[5] Like Diermeier and Myerson (1999), by legislators we mean here all individuals who have a constitutional role in the process of passing legislation. This may include individuals from what is usually refered to as being the executive branch like for instance the president or the vice-president.

[6] If instead, the legislature gives some power to actors from the "executive" branch then, this assertion does not necessarily holds true in case of divided government.

legislator acts as a voter). To each terminal node is attached a policy and the model of the legislative process is likely to depend upon the type of policy space which is considered. A classical model in that vein is the Baron-Ferejohn bargaining model (1989) describing the rules of the legislature to divide a fixed budget among the legislators. This legislative model has been paired with lobbying by Helpman and Persson (2001). Another very nice such model, due to Grossman and Helpman (2001), applies to any finite set of policies with one policy playing the role of the status quo. In their model, one legislator decides unilaterally upon an alternative (bill, amendement, motion, reform,...) that will confronted through a binary majority vote to the status quo. Lobbies have the opportunity to influence legislators in two occasions : first they will try to exert influence on the agenda setter and second they will also try to buy votes. In this paper, we focus on the binary setting i.e. we assume that the[7] policy space consists of two alternatives : the status quo (alternative 0) versus the change or reform (alternative 1). While simplistic, we think that many policy issues fit that formulation like for instance : to ratify or not a free-trade agreement, to forbid or not a free market for guns, to allow or not abortion. In such case, there is no room for agenda setting and the unique role of the legislature is to select one of the two options through voting. A legislature is then just described by a *simple game* $(N, \mathcal{W})$ where N is the set of legislators (or parties, if there is some strong party discipline) and $\mathcal{W}$ is the list of winning coalitions : the reform is adopted if and only if the coalition of legislators voting for the reform belongs to that list.

Lobby 0 (respectively lobby 1) preference is defined by the amount of dollars W_0 (respectively W_1) that would be lost (respectively gained) by its members if the reform was adopted and we asumme that both lobbies represent faithfully the two sides of the society on the issue uner scrutiny. Following Grosssman and Helpman (1994), we assume that each legislator seeks to maximize a weighted sum of social welfare and monetary contributions. Therefore, in this setting, each legislator i is simply described by a single parameter α_i denoting the weight that he puts on social welfare. This will be refered hereafter as being the type of the legislator. The lower is α_i, the cheapest is legislator i and therefore there is a sense in which we can qualify politicians with a low α as "bad" or corrupted politicians since they are more willing to depart from social welfare when deciding upon which policy to implement[8].

[7] The idea that α could be an adverse selection parameter is suggested in Grossman and Helpman (1992) and is the main motivation of Le Breton and Salanié (2003).

[8] Some empirical estimates of this parameter have been provided in the common agency setting. Interestingly, Golberg and Maggi (1999) find that the 1983 U.S. pattern of protection is consistent with the model of Grossman and Helpman and estimate the value of the parameter α to be between 50 and 88, a surprisingly high range of values. Gawande and Bandyopadhyay (2000) also conclude that the model of Grossman and Helpman is consistent with the data but estimate the value of α to be between 3 and 8. Bradford (2001)

The main purpose of our paper is to proceed to an equilibrium analysis of the lobbying game where the two lobbies make first offers to the legislators who then vote in favor or against the reform. Several variants of this game are examined in turn. In the first part, we assume that the types of the legislator are common knowledge, an environment that we call *political certainty* as all the relevant variables are known with certainty by all the players. In the second part, we assume instead that the types of the legislators are private informations. We refer to this environment as *political uncertainty* as the lobbies when buying votes and the legislators when voting do not know with certainty the consequences of their choices. In both cases, we assume that the two lobbies move simultaneously. The exogeneous ingredients of our strategic environment are :

· The economic stakes W_0 and W_1 whose respective magnitudes will define the intensity of the competition. We assume here that the reform is the socially efficient policy i.e. $W_1 \geq W_0$ and the ratio $\frac{W_1}{W_0} \geq 1$ will be called the *efficiency threshold*.

· The simple game $(N, \mathcal{W})$ which describes the legislative process.

· A probability distribution F over the positive real line which describes the respective frequencies of bad and good legislators.

We aim to examine the impact of each of these key parameters on the final equilibrium outcome of the political mechanism described by this influence game. The outcome has two dimensions :

· The policy which is ultimately selected by the legislators.

· The ex ante monetary offers of the lobbies to the legislators and their ex post implementation.

In the first part of the paper, we assume political certainty. In the case of pure strategies, we demonstrate that the equilibrium policy is efficient but we note also that existence is obtained only under very stringent conditions on $(N, \mathcal{W})$. We examine the equilibrium in mixed strategies in a specific majority setting and comment the relationship to the literature on Colonel Blotto Games

In the second part of the paper, we move to the case of political uncertainty and limit our attention to the majority legislative process. The lobbying game is much more intricate as solving the continuation voting subgames gives rise to reduced payoff functions for the lobbies which may display some irregularities. We first examine the case where only lobby 0 is active and explore its optimal lobbying strategy. Once again the efficiency threshold plays

proceeds to an empirical investigation of a variant of a model of Grossman and Helpman where politicians maximize votes and finds that politicians weight a dollar of campaign contributions about 15% more than a dollar of national income. This would lead to a value of α very close to 1.

a critical role in explaining the feature of this strategy and the probability of getting the efficient policy selected. One surprising feature of the optimal offer is that the larger is the stake W_0 of lobby 0, the smaller is the coalition of legislators who receive an offer. In the last part, we return to the game and offer some preliminary insights of the best responses when there are three legislators.

Related Literature

Many of the questions examined in this paper have been investigated by other authors. Some general positive models aiming to describe the lobbying process of a legislature have been proposed by Bennedsen and Feldmann (2002), Boylan (2002), Helpman and Persson (2001), Polborn (2002) and Snyder (1991) among many more. Some recent papers by Dal Bo (2002) and Felgenhauer and Gruner (2004) look at the impact of external influence on a legislature or committee from a mechanism design angle. In particular, they compare open and closed voting and reach interesting conclusions. Instead of us, they model the committee choice issue as a problem with common values as in Condorcet juries.

In contrast to Le Breton and Zaporozhets (2007), this paper assume that the lobbies make their offers simultaneously. Several important contributions have assumed instead that they move sequentially. This formulation removes the existence issue[9]. Groseclose and Snyder (1996) consider a sequential game where the two lobbies move in sequence and the bidding process stops when the second mover has reacted to the offer of the first mover. They examine the majority game with an heterogeneous legislature and look at the equilibrium size of the coalition of legislators who receive an offer from the lobby moving first. Banks (2000) pursues this line of investigation while Diermeier and Myerson (1999) assume instead a homogeneous legislature but an arbitrary simple game. They focus on the architecture of the legislative process that would maximize monetary offers. Dekel, Jackson and Wolinsky (2006a,b) examine an open-ended sequential game where lobbies alternate in increasing their offers to legislators. By allowing lobbies to keep responding to each other with counter-offers, their game eliminates the asymmetry and the resulting second mover advantage of the Groseclose and Snyder 's game. Several settings are considered depending upon the type of offers that lobbies can make to legislators (Up-front payments versus promises contingent upon the voting outcome) and upon the role played by budget constraints[10]. The difference

[9] Prat and Rustichini (2000) present a general abstract model extending the common agency framework as it has many principals (lobbies here) and many agents (legislators here). It does not cover our behavioral model . Further, in our case, a direct check is more efficient than an application of their balancedness condition.

[10] These considerations which are irrelevant in the case of our two-round sequential game are important in their game.

in the budgets of the lobbies plays a critical role in determining which lobby is successful when lobbies are budget constrained, and the difference in their wilingness to pay plays an important role when they are not budget constrained. When lobbies are budget constrained, their main result states that the winning lobby is the one whose budget plus half of the sum of the value that each legislator attaches to voting in favor of this lobby exceeds the corresponding magnitude calculated for the other lobby. In contrast, when lobbies are not budget constrained, what matter are the lobbies's valuations and the intensity of preferences of a particular "near-median" group of legislators. The lobby with a-priori minority support wins when its valuation exceeds the other lobby's valuation by more than a magnitude thant depends on the preferences of that near-median group.

Young (1978 a, b, c) and Shubik and Young (1978d) were the first point out the relevance of the least core and the nucleolus to predict some dimensions of the equilibrium strategies of the lobbyists.

This literature is also closely related to the popular Colonel Blotto with some differences however. If the competition among lobbies was modeled as a Colonel Blotto game, then they would not pay attention to the money spent in the process and focus exclusively on the probability of winning. Instead here, it is assumed that money is valuabl[11]. As noted by Young (1978b)[12] "In fact, the real, or net, payoffs to the lobbyists consist of their payoff in the above sense less their expenditures. However, we may imagine for present purposes that the payoff from winning is immeasurably gretater than the lobbyists's budgets". In that respect the game looks more to a multi-unit all-pay auctions. The value of a bundle of votes will depend upon the number and identities of the legislators and the money spent will be lost, irrespective of the outcome. On this last point, it should be noted however, as pointed out by Dekel, Jackson and Wolinski that several variants are conceivable depending upon the contingencies upon which the payment is made conditional. In spite of these differences, the analogy with Colonel Blotto games is instructive. Young (1978) demonstrates that if the simple game is dictatorial then both lobbies spend the totality of their budgets on the dictator. He also shows that if there are at least two vetoers, then lobby 0 wins for sure by playing a diffuse mixed strategy with support the set of vetoers. Surprisingly, he shows that if there is exactly one vetoer, then the game fails to possess an equilibrium in mixed strategies. Colonel Blotto Games are notoriously difficult to solve even in the simplest settings. As noted by Young, when the game admits an equilibriumin mixed strategies, we can calculate

[11] Shubik and Young (1978) examine a variant that they call the session lobbying game which admits an equilibrium in pure strategies.

[12] In (1978a), he writes "It is as if winning or losing is taken to be of incomparably greater value than the prices paid".

the expected amount of money received by each legislator. This leads to a price for the vote of each legislator. The discontinuities displayed by these games are not always covered by existing theorems. In the case, where they are three legislators deciding under the ordinary majority rule, equilibria in mixed strategies exist and have been computed explicitely in the symmetric case. In the asymmetric case, Weinstein (2005) offers insighful results on the probability of winning of the strongest player. When the monetary cost is included, as already mentionned, the nature of the game is changed. Kvasov (2007) and Szentes and Rosenthal (2003a,b) provide results on this class of games in some specific cases.

To the best of our knowledge, the case of political uncertainty has not been investigared before. Our model follows Le Breton and Salanié's model (2003) of common agency with adverse selection except that, in contrast to them, we ignore the free-riding dimension of the lobbying process and its impact on efficiency.

2 The Model

In this section, we describe the main ingredients of the problem as well as the lobbying game which will constitute our model of vote-buying by lobbyists.

The external forces that seek to influence the legislature are represented by two players, whom we will call lobby 0 and lobby 1. Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law[13]. Lobby 0 is opposed to this bill and wants to maintain the status quo. Lobby 0 is willing to spend up to W_0 dollars to prevent passage of the bill while lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we will refer to these two policies in competition as being policies 0 and 1. We will assume that $\Delta W \equiv W_1 - W_0 > 0$ i.e. that policy 1 is the socially efficient policy. The ratio $\frac{W_1}{W_0}$ which is (by assumption) larger than 1 will be called the *efficiency threshold*. It measures the intensity of the superiority of the reform as compared to the status quo and will be used repeatedly in the analysis.

The legislature is described by a *simple game*[14] i.e. a pair $(N, \mathcal{W})$ where $N = \{1, 2, ..., n\}$ is the set of legislators and $\mathcal{W}$ is the set of *winning* coalitions. The interpretation is the following. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions. A coalition C is *blocking* if $N \backslash C$

[13] The framework also covers the case of private bills as defined and analysed by Boylan (2002).

[14] In social sciences it is sometimes called a committee or a voting game. In computer science, it is called a quorum system (Holzman, Marcus and Peleg (1997)) while in mathematics, it is called a hypergraph (Berge (1989), Bollobas(1986)). An excellent reference is Taylor and Zwicker (1999).

is not winning : some legislators (at least one) are needed to form a winning coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions[15]; from the definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition. The simple game is called *strong* if $\mathcal{B} = \mathcal{W}$[16]. The set of minimal (with respect to inclusion) winning (blocking) coalitions will be denoted $\mathcal{W}_m$ ($\mathcal{B}_m$). A legislator is a *dummy* if he is not part of any minimal winning coalition while a legislator is a *vetoer* if he belongs to all blocking coalitions. A group of legislators forms an *oligarchy* if a coalition is winning iff it contains that group i.e. each member of the oligarchy is a vetoer and the oligarchy does not need any extra support to win[17] i.e. legislators outside the oligarchy are dummies. When the oligarchy consists of a single legislator, the game is called dictatorial.

In this paper, all legislators are assumed to act on behalf of social welfare i.e. they will all vote for policy 1 against policy 0 if no other event interfers with the voting process. In contrast to Banks (2000) and Groseclose and Snyder (1996), we rule out the existence of a horizontal heterogeneity across legislators. However, legislators also value money and we introduce instead some form of vertical heterogeneity. Precisely, we assume that legislators differ among themselves according to their willingness to depart from social welfare. The type of legislator i, denoted α_i, is the minimal amount of dollars that he needs to receive in order to sacrify one dollar of social welfare i.e. Therefore if the policy adopted generates a level of social welfare equal to W, the payoff of legislator i if he receives a transfer t_i is:

$$t_i + \alpha_i W$$

This payoff formulation is compatible with two behavioral assumptions that will be considered in this paper. Either, the component W appears as long as the legislator has voted for a policy generating a level of social welfare W regardless of the fact that this policy has been ultimately selected or not : we will refere to this model, as *behavioral model P*, where P stands for procedural. Or, th component W appears as long as the policy ultimately seleted generates a level of social welfare W regardless of the fact that the legislator has voter for or againsts this policy : we will refere to this model, as *behavioral model C*, where C stands for consequential. In this paper, we focus on the behavioral model C.

To promote passage of the bill, lobby 1 can promise to pay money to individual legislators conditional on their supporting the bill. Similarly, lobby 0 can promise to pay money to

[15] In game theory, $(N, \mathcal{W})$ is called the dual game.

[16] When the simple game is strong, the two competing alternatives are treated equally.

[17] The five countries of the security council of the United Nations are vetoers but still do not form an oligarchy as they need some extra support to make a decision.

individual legislators conditional on their opposing the bill. We denote by $t_{i0} \geq 0$ and $t_{i1} \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted respectively by t_0 and t_1.

The timing of actions and events that we consider to describe the lobbying game is the following[18].

1. Nature draws the type of each legislator.
2. Both lobby 0 and lobby 1 make contingent monetary offers to individual legislators.
3. Legislators vote.
4. Payments (if any) are implemented.

This game has $n + 2$ players. A strategy for a lobby is a vector in $\Re^n_+$. Each legislator can chose among two (pure) strategies : to oppose or to support the bill.

The game is not fully described as we have not precisely defined yet what are the informations held by the players when they act. In this paper, we will consider two distinct settings concerning the move of player nature but we assume otherwise that the votes of the legislators are observable i.e. we assume open voting[19]. The first setting to which we will refer as *political certainty* corresponds to the case where the vector of legislators's types is common knowledge. This informational specification has two implications : first, the lobbies know the types of the legislators when making their offers and second each legislator knows the type of any other legislator when voting. The second setting to which we will refer as *political uncertainty* corresponds instead to the case where the type of a legislator is private information. In such case, not only the lobbies ignore the types of the legislators but each potential continuation voting subgame is a Bayesian game. This means that there is an adverse selection feature in the strategic relationship between lobbies and legislators and a Bayesian feature in the strategic interaction among legislators.

We examine the *subgame perfect Nash equilibria*[20] of this lobbying game. In section 3, we investigate the case of political certainty. Then, in section 4, we move to the case of political uncertainty.

[18] Specific details and assumptions will be provided in due time.

[19] The comparative analysis of closed(secret) versus open voting is the subject of several contributions among which Dal Bo (2002) and Felgenhauer and Grüner (2004).

[20] In the case of political uncertainty, the ultimate subgame is truly a Bayesian game that we solve using Bayesian-Nash equilibria. We don't use the word Bayesian subgame perfect Nash equilibrium as there is no updating operation of beliefs in our game.

3 Political Certainty

In this section, we consider the case where the vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ of legislators's types is common knowledge and, without loss of generality, we assume that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.. We first examine the set of subgame perfect Nash equilibria in pure strategies that we call compactly (with a slight abuse in the terminology) Nash equilibria. We show that they are efficient but exist only under very stringent conditions. We conclude by looking specifically at the case of the majority game with three legislators and calculate the Nash equilibrium in mixed strategies under some specific conditions.

3.1 Nash Equilibria in Pure Strategies

Let $(t_0, t_1) \in \Re_+^n \times \Re_+^n$ be a profile of lobbying strategies. In the continuation voting subgame, each legislator's behavior strongly depends on whether he is pivotal or not. Consider a legislator who expects to be non-pivotal, i.e. who expects that the outcome does not change no matter which policy he votes for. Then, such a legislator votes in favor of the policy preferred by the lobby that offers the largest monetary payment. If legislator i believes that he is not decisive, he votes for policy 1 if and only if

$$t_{i1} \geq t_{i0} \tag{1}$$

and for policy 0 otherwise.

If instead, legislator i thinks that he is pivotal, he votes for 1 if and only if

$$\alpha_i \Delta W + t_{i1} \leq t_{i0} \tag{2}$$

and for policy 0 otherwise. Clearly, a legislator with no offers from lobby 0 votes for policy1.

The first result asserts that under complete information only the efficient policy is chosen at equilibrium[21].

Proposition 1 *All Nash equilibria in pure strategies are Pareto efficient.*

Proof : Suppose on the contrary that there is a Nash equilibrium (t_0^*, t_1^*) for which policy 0 is chosen. Let N_0 be the coalition of voters supporting policy 0. Then, $N_0 \in \mathcal{B}$.

[21] Some readers may be surprised by the fact that we dont need to refine the set of Nash equilibria to reach that conclusion. In the common agency setting, Bernheim and Whinston (1986) use the thruthful refinement to rule our inefficient Nash equilibria. But such a refinement is not needed here as we have only two possible decisions.

Case 1 : $N_0 \in \mathcal{B}_m$. Then, each agent in this set is pivotal. Therefore, for any $i \in N_0$ (2) are satisfied. Any agent $i \in N_1$ is not pivotal and (1) should be satisfied. The net payoff of lobby 0 is $W_0 - \sum_{i \in N_0} t_{i0}^*$ while the net payoff of lobby 1 is $-\sum_{i \in N_1} t_{i1}^*$. Lobby 1 could pay 0 to all legislators in N_1 and gets 0 instead of a negative payoff. Therefore in such equilibrium, $t_{i1}^* = 0$ for all $i \in N_1$, and from (1) $t_{i0}^* = 0$ for all i$\in N_1$.

Therefore, both lobbies 0 and 1 make offers only to the legislators in N_0.

Next, if the inequalities (2) are strict, for each i, $t_{i0}^* > 0$ since the left-hand side is non-negative. Lobby 0 could reduce his transfer slightly without changing the outcome. Thus, the equalities must hold. Summing up these equalities for all $i \in N_0$ we get

$$\sum_{i \in N_0} t_{i1}^* = \sum_{i \in N_0} t_{i0}^* - \Delta W \sum_{i \in N_0} \alpha_i \leq W_0 - \Delta W \sum_{i \in N_0} \alpha_i < W_1$$

Since $\sum_{i \in N_0} t_{i1}^* < W_1$ and $t_{i0}^* = 0$ for all $i \in N_1$, lobby 1 could slightly increase its offers to all $i \in N_0$ and change the outcome from 0 to 1 in contradiction with our assumption.

Case 2 : $N_0 \notin \mathcal{B}_m$. Since none of the legislators from N_0 is pivotal, the following holds true :

$$t_{i0}^* \geq t_{i1}^* \text{ for all } i \in N_0$$

Then, the arguments used in case 1 apply □

The next proposition exhibits several necessary conditions on such equilibria. While stringent, these conditions cover the traditional common agency setting.

Proposition 2 *Let (t_0^*, t_1^*) be a Nash equilibrium.*
(i) If $\Delta W \sum_{i \in S} \alpha_n \geq W^0$ for all $S \in \mathcal{B}_m$, then $t_1^* = 0$.
(ii) If $\widehat{S}$ is an oligarchy and $\Delta W \alpha_i < W_0$ for all $i \in \widehat{S}$, then $t_1^* = \#SW_0 - \Delta W . \sum_{i \in \widehat{S}} \alpha_i$

Proof : The proof of (i) follows immediately from the observation that lobby 0 gross benefit is not enough to compensate a minimal blocking coalition of legislators. the proof of (ii) is also very simple. By proposition 1 the equilibrium is efficient. This implies that each veto player and therefore each member i of the oligarchy $\widehat{S}$ must receive at least $W_0 - \Delta W \alpha_i$. There is no need to make offer to any other player as they are dummies and to pay more the vetoers as it does not add anything □

Unfortunately, these results are mitigated by the fact that the lobbying game typically does not possess Nash equilibria in pure strategies. In the case where $\mathcal{W}$ is the majority game, the lobbying game has the structure of an asymmetric Colonel Blotto game (Gross and Wagner (1950), Laslier and Picard (2002)) for which it is well known that Nash equilibria in pure strategies does not exist as soon as the asymmetry is too small. In this literature, the two competitors are constrained by their budgets while here there are no such financial constraints. Note however that as long as we consider pure strategies, none of the lobby will spend more that its gross benefit and will spend the totality of this gross benefit if this can prevent the other lobby to win. This equivalence does not hold in the case of mixed strategies. While discontinuous, this games admit equilibria in mixed strategies; some features of these equilibria are described in subsection 3.3.

If the asymmetry between the lobbyists is large enough, existence of an equilibrium in pure strategies follows. It is immediate to see that if $\Delta W \sum_{i \in S} \alpha_n \geq W^0$, $(t_0^*, t_1^*) = (0, 0)$ is a Nash equilibrium. The second part of proposition 2 generalizes Le Breton and Salanié (2003) who consider the common agency framework i.e. the case of a dictatorial simple game. In that case, a Nash equilibrium always exists as we have assumed $W_1 > W_0$: the unique dictator receives W_0 from lobby 1. When there are least two vetoers in the oligarchy, existence is not guaranted. Consider the case where $N = \widehat{S} = \{1, 2\}$, $\alpha_1 = \alpha_2 = 0$ and $W_1 < 2W_0$. From proposition 2 (ii), we deduce that $t_1^* = 2W_0$ which is not an equilibrium behavior as $W_1 < 2W_0$.

In fact, this logical argument against existence of a Nash equilibrium in pure strategies applies to any simple game which is not oligarchic. let (t_0^*, t_1^*) be a Nash equilibrium. From proposition 1, the reform is selected. This implies that a winning coalition S of legislators votes for the reform. Therefore lobby 0 does not implement any monetary offer. This means that $t_{i0}^* - t_{i1}^* < \alpha_i \Delta W$ for all $i \in N$. We deduce that there is at most a minimal winning coalition $T \subseteq S$ such that $t_{i1}^* > 0$. Since for all $i \notin T$, $t_{i1}^* = 0$, we deduce that $t_{i0}^* = 0$ for all such i as none of these legislators is pivotal. Further since the simple game is not oligarchic, there is a minimal winning coalition different from T. Let T' be any such coalition and $t_{i1}^{**} = 0$ for all $i \in (N \backslash T) \cap T'$ and must be minimal winning, as any additional offer would useless but accepted. we deduce from that observation and the previous claim that $t_{i1}^{**} = t_{i1}^* - \varepsilon$ for some small enough for all $i \in T \cap T'$. From the construction, legislators in T' vote for the reform. Since the cost of t_1^{**} is smaller than the cost attached to the strategy t_1^*, we contradict our assumption that (t_0^*, t_1^*) is a Nash equilibrium.

Given the difficulty to obtain existence of an equilibrium in pure strategies; we consider, as suggested by Grossman and Helpman (2001), the possibility for the lobbyists to randomize over their monetary transfers. In contrast to the sequential version which is an alternative way to remove the existence issue but which offers a strong adavantage to the second mover, randomization may help in maintaining some balance between the strategic powers of the two competitors. To verify if this assertion holds true in full generality turns to be an extremely demanding technical task . In the case where $\alpha_1 = \alpha_2 = \alpha_3 = 0$, as already pointed out, the game is a colonel Blotto game, a discontinuous game for which, however, equilbria in mixed strategies exist and have been calculated. In the simple proposition below, we illustrate the difference between the two settings under the simplifying assumption that the third legislator (the more costly from the perspective of lobby 0) is too expensive to be condidered. The competition is then limited to legislators 1 and 2.

Proposition 3 *Assume that $\Delta W(\alpha_1 + \alpha_2) \leq W^0 < \Delta W \alpha_3$. Then, a mixed-strategy equilibrium exists with the following features. Lobby 0 makes offers to legislators $i = 1, 2$ with the probability that the offer to legislator i does not exceed x given by :*

$$F_i^0(x) = \begin{cases} 1 - \frac{\widetilde{W_0}}{2W_1} \text{ for } x \in [0, \Delta W \alpha_i), \\ \frac{W_1 - \widetilde{W_0}/2 - \Delta \alpha_i}{W^1 - x} \text{ for } x \in \left[\Delta W \alpha_i, \Delta W \alpha_i + \widetilde{W_0}/2\right] \end{cases}.$$

Lobby 1 makes an offer just to one of the legislators $i = 1, 2$ with equal probability. The probability that the offer to legislator i is less or equal to x is

$$F_i^1(x) = \frac{x + \Delta W \alpha_{-i}}{W_0 - (x + \Delta \alpha_i)} \text{ for } x \in \left[0, \widetilde{W_0}/2\right], \text{ where } \widetilde{W_0} = W_0 - \sum_{i=1}^{2} \Delta W \alpha_i.$$

Proof. A priori any legislator i prefers to vote for policy 1, and to make him indifferent between two policies lobby 0 has to pay $\Delta W \alpha_i$. We show that strategies described above form an equilibrium.

First, consider the choice of lobby 1. Given that lobby 0 is bribing agents $i = 1, 2$, it is enough for lobby 1 to bribe just one of these agents to get policy 1. Since the highest amount that lobby 0 can offer to legislator i is $\Delta \alpha_i + \widetilde{W_0}/2$, lobby 1 will never offer more than $\widetilde{W_0}/2$. It remains to show that lobby 1 randomizes among the transfers on the interval $\left[0, \widetilde{W_0}/2\right]$, i.e. the group is indifferent among these alternatives.

Suppose that lobby 1 make positive offer x to legislator 1 and zero to legislator 2. Then expected payoff of lobby 1 given the behavior of lobby 0 is $E\left[U_1 \,|x, 0\right] = W_1 F_i^0(x + \Delta \alpha_1) - x$. Since it is equal to $W^1 - \widetilde{W_0}/2$, $E\left[U_1 \,|x, 0\right]$ does not depend on x. It is clear that $E\left[U_1 \,|x, 0\right] = E\left[U_1 \,|0, x\right]$. Therefore, lobby 1 achieves the same expected payoff for different contribution levels, and it is also indifferent between bribing legislator 1 or bribing legislator 2.

Now, lets consider the behavior of lobby 0. It needs to buy at least two votes (simple majority) to get its preferred outcome. The cheapest way is to bribe legislators 1 and 2.

For lobby 0, it is a waste of resources to offer to legislator $i = 1, 2$ a positive bribe less than $\Delta\alpha_i$, since in that case the legislator would prefer policy 1. If lobby 1 does not make an offer to legislator i, then by offering $\Delta\alpha_i$ lobby 0 makes legislator i indifferent between the two policies. Amount $W_0 - \sum_{n=1}^{K+1} \Delta\alpha_n$ can be divided equally between legislators $i = 1, 2$. Thus, the maximum possible offer is calculated as $\Delta\alpha_i + \widetilde{W}_0/2$. Then, it is necessary to show that lobby 0 is indifferent among the bribes (x, y) with $x \in \left[0, \Delta\alpha_1 + \widetilde{W}_0/2\right]$ and $y \in \left[0, \Delta\alpha_2 + \widetilde{W}_0/2\right]$.

Suppose that lobby 0 offers x and y respectively to legislator $i = 1, 2$. Given the equilibrium strategy of lobby 1 expected payoff of lobby 0 is calculated as

$$E\left[U_0\,|x, y\right] = \frac{1}{2}\left(W_0 - x - y\right) F_1^1(x - \Delta\alpha_1) - \frac{1}{2} y \left(1 - F_1^1(x - \Delta\alpha_1)\right) + \frac{1}{2}\left(W_0 - x - y\right) F_2^1(y - \Delta\alpha_2) - \frac{1}{2} x \left(1 - F_2^1(y - \Delta\alpha_2)\right) = 0.$$

So, $E\left[U_0\,|x, y\right]$ does not depend on x and y, i.e. lobby 0 achieves the same expected payoff for any pair of offers x and y, each of which is less or equal to $\Delta\alpha_i + \widetilde{W}_0/2$. Thus, lobby 0 is willing to randomize in the described manner as well as lobby 1. ■

Note that in this Nash equilibrium in mixed strategies, lobby 0 gets an expected payoff equal to 0, which is the same as if it had no opportunity to bid for votes or as in the case of the Stackelberg equilibrium. In contrast, lobby 1 earns a positive expected surplus equal to $W_1 - \frac{\widetilde{W}_0}{2}$ which is equal to $W_1 - \frac{W_0}{2}$ when $\alpha_1 = \alpha_2 = 0$. This payoff is larger than the payoff $W_1 - W_0$ that it gets in the Stackelberg equilibrium. This confirms that restoring more balance between the two lobbyists leads to an increase of the payoff of the stronger lobby.

On the other hand, in this equilibrium, the outcome is random. The efficient policy is chosen if the offer of lobby 0 to a contested legislator i does not exceed the offer of lobby 1 by an amount equal to $\Delta W\alpha_i$. Otherwise, the inefficient policy is chosen. The probability P of selecting the efficient outcome is given by :

$$\begin{aligned} P &\equiv \Pr\left(\text{Lobby 1 wins}\right) \\ &= \frac{W_1 + \frac{\widetilde{W}_0}{2}}{W_1} + \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_1}{W_1} \ln \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_1}{\widetilde{W}_0 + \Delta W\alpha_1} + \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_2}{W_1} \ln \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_2}{\widetilde{W}_0 + \Delta W\alpha_2} \end{aligned} \tag{3}$$

The computation of P proceeds stepwise. The probability of lobby 1's success, given the fact that it offers x to legislator 1 is $\Pr\left(M^1 \text{ wins}\,|x, 0\right) = F_1^0(x+\Delta\alpha_1)$. Using the distribution of x it is easy to show that :

$$\Pr(\text{lobby 1 wins}\,|\text{legislator 1 is bribed})$$
$$= \frac{\Delta\alpha_2}{\widetilde{W_0}+\Delta\alpha_2}\frac{W_1-\frac{\widetilde{W_0}}{2}}{W_1}+\int_0^{\frac{\widetilde{W_0}}{2}}\frac{W_1-\frac{\widetilde{W_0}}{2}+x}{W_1}\frac{\widetilde{W_0}+2\Delta\alpha_2}{\left(\widetilde{W_0}+\Delta\alpha_2-x\right)^2}dx$$

We compute similarly the probability that lobby 1 wins given that legislator 2 is bribed. Note further that :

$$\frac{\partial P}{\partial\alpha_1}=\frac{\Delta}{2W_1}\ln\frac{(W_0+\Delta\alpha_1-\Delta\alpha_2)(W_0-\Delta\alpha_1)}{(W_0-\Delta\alpha_1+\Delta\alpha_2)(W_0-\Delta\alpha_2)}+\frac{\Delta^2}{2W_1(W_0-\Delta\alpha_1)}$$

It follows that if $\alpha_1=\alpha_2$, then $\frac{\partial P}{\partial\alpha_1}>0$ since the first term is equal to zero. If $\alpha_1=0$, then $\frac{\partial P}{\partial\alpha_1}=\frac{\Delta}{2W_1}\left(\ln\frac{W_0}{W_0+\Delta\alpha_2}+\frac{\Delta}{W_0}\right)$.

4 Political Uncertainty

In this section, we analyse the lobbying game under political uncertainty in the case where the simple game is the majority game with an odd number $n=2k+1$ of legislators. We assume that the types α_i of the legislators are independently and identically distributed from a continuous cumulative distribution function[22] F with bounded support $[\underline{\alpha},\bar{\alpha}]$ where $0\leq\underline{\alpha}<\bar{\alpha}$. We denote by f the probability density function, which is assumed to be strictly positive on the whole interval $[\underline{\alpha},\bar{\alpha}]$. Finally, we assume that the hazard rate $\frac{F}{f}$ is increasing and that the hazard rate $\frac{1-F}{f}$ is decreasing.

4.1 The Optimal Strategy of Lobby 0 when Lobby 1 is Inactive

We first consider the case where lobby 1 is inactive[23] i.e. $T_1^*=0$. This an important benchmark to start our exploration of the competition between the two lobbies. In such context, the strategic interaction with the other lobby disappears and the game becomes merely an agency problem with adverse selection where the principal is lobby 0 and the agents are the n legislators. The conflict of interest arises from our assumption that, without compensation, legislators would vote for policy 1. The contractual problem faced by lobby 0 amounts to the selection of a vector $T_0\in\Re_+^n$ conditional on verifiable information. Given our

[22] Therefore, the probability that any legislator has a type less than or equal to some α is $F(\alpha)$.

[23] Some authors simply ignore the existence of several lobbies and the competitive aspect resulting from that. One possible justification in our context consists in saying that lobby 1 cannot overcome its own collective action difficulties and act efficiently with respect to its own global stake.

observability assumptions, this information consists of the n-dimensional vector of individual votes. In principle, lobby 0 could make the payment to legislator i contingent upon the votes of other legislators as well or a general statistic depending upon the all profile of votes. We assume here that the reward to legislator i is simply based on his own vote : legislator i receives t_{i0} if and only if he voted against the bill. This excludes, for instance, the ingenious contractual solution of Dal Bo (2002) where a given legislator is paid only in the event where his vote has been decisive.

The rest of this section is devoted to a complete analysis of this principal-agent(s) problem i.e. to a characterization of the main features of the optimal strategy T_0^*. We will denote by n_0^* the number of legislators who have been promised to receive bribes by lobby 0 in the optimal strategy i.e. :

$$n_0^* \equiv \# \{i \in N : t_{i0}^* > 0\}$$

This is an important feature of the strategy as it provides an answer to the question: how large is the supermajority bought by lobby 0 ? A second feature is the total amount of dollars paid by lobby 0. From its perspective, this is a risky prospect, as it does know since when the offers are made, it does not know for surewill be the behavioral response of the legislators. Therefore, the amount $M_0^* \equiv \sum_{i \in N} t_{i0}^*$ just represents the upper bound of the range of this random variable. Other parameters of interest are the first E_0^* and second V_0^* moments of this random variable. The expected rate of return of this "investment" is then given by :

$$\frac{W_0 - E_0^*}{E_0^*}$$

The third and last feature of the strategy that deserves to be investigated is the distribution of M_0^* across legislators. We have seen in section 3 that, when the simple game is not symmetric i.e. when some legislators are more powerful than others, i.e. when they are not perfect substitutes, we should expect some differentials in the way they will be treated by the lobbies. However, when the game is symmetric, they are all offered the same amount. Our assumption that the legislators are all identical ex ante together with the fact that the majority game is summetric suggest that it will happen here too. This is not straightforward and calls for a proof as the behavioral responses of the legislature following any possible history of offers is now more complicated. In case where uniformity across the bribed legislators is shown to be optimal, we can, without loss of generality, limit ourselves to strategies defined by two dimensions : an integer n_0^* and a real number t_0^*.

4.1.1 The Voting Subgame(s)

Given any profile of offers T_0, a Bayesian strategy for legislator i in the continuation voting subgame is a mapping σ_i from the set of types $[\underline{\alpha}, \bar{\alpha}]$ into $\{0,1\}$: $\sigma_i(T_0, \alpha_i) = 0$ means that legislator i votes for the status quo when T_0 is the vector of standing offers and his type is α_i.

A key determinant of legislator i strategic evaluation is the probability p_i of being pivotal. Legislator i of type α_i with an offer equal to t_{i0} votes for the status quo if and only if

$$t_{i0} + p_i \alpha_i W_0 \geq p_i \alpha_i W_1$$

The Bayesian decision rule is therefore described by a cut point $\widehat{\alpha}_i$: legislator i votes for the status quo if his type α_i is below the cutpoint and votes for the reform otherwise. The cut point $\widehat{\alpha}_i$ is defined as

$$\widehat{\alpha}_i = \max\left\{\underline{\alpha}, \min\left\{\frac{t_{i0}}{p_i \Delta W}, \bar{\alpha}\right\}\right\} \tag{4}$$

Let $N_0 \equiv \{i \in N : t_{i0} > 0\}$. Under the restriction that offers are uniform i.e. $t_{i0} \equiv t_0$ for all $i \in N_0$, all legislators in N_0 face the same decision problem. Hereafter, we will restrict our attention here to symmetric equilibria i.e. we assume that these legislators use the same decision rule. We will denote by $\widehat{\alpha}$ the cut point describing this strategy and by p the probability of being pivotal for any of them. For the legislators outside N_0, voting for the reform is a dominant strategy

For any legislator i in N_0 the probability p of being pivotal is simply the probability that exactly k other legislators vote for 0. Since the legislators in $N \setminus N_0$ always vote for the reform, this is the probability of the event that exactly k legislators from $N_0 \setminus \{i\}$ vote for the status quo. Given the cut point $\widehat{\alpha}$, it is possible to write down explicitly the formula for p :

$$p = p(t_0, n_0, \widehat{\alpha}) = B_k\left[n_0 - 1, F(\widehat{\alpha})\right] \tag{5}$$

where $B_k[n,q] = C_n^k q^k (1-q)^{N-k}$ denotes the probability of the event k for a binomial random variable with parameters n and q. The pivotal probability depends upon the voting strategies played by the other legislators. The equilibrium pivotal probability will be solution of ? when $\widehat{\alpha}$ is the equilibrium cut point. Since the equilibrium cut point is itself dependent upon the equilibrium pivotal probability, we are left with an existence issue which is covered by the following proposition[24].

[24] A game with similar features has been examined by Palfrey and Rosenthal (1985) as describing the

Proposition 4 *For any given $t_0 \geq 0$ and n_0, the continuation voting subgame has two interior symmetric equilibria $\underline{\alpha}< \widehat{\alpha}_L < \widehat{\alpha}_R < \overline{\alpha}$ and $\overline{\alpha}$ as a corner equilibrium. The low cut point equilibrium $\widehat{\alpha}_L$ Pareto dominates*[25] *the two other equilibria.*

Proof:

The proof of the firt assertion is divided into two cases.

(i) $n_0 = k+1$, i.e. the lobby offers positive transfers to a simple majority of voters. In this case, the unique cut-off level exists. Applying (5) one gets that $p = F^k(\widehat{\alpha})$. Substituting it into (4) it follows that for $t \in (\Delta\bar{\alpha}, \infty)$ $\widehat{\alpha} = \bar{\alpha}$, and for $t \in [0, \Delta\bar{\alpha}]$ the cut point $\widehat{\alpha}$ is defined by

$$\widehat{\alpha}F^k(\widehat{\alpha}) = t/\Delta \tag{6}$$

From assumptions on the distribution function it follows that the LHS of this equality is strictly increasing function of $\widehat{\alpha}$, therefore $\widehat{\alpha}$ is uniquely defined by (6). One can see that $\widehat{\alpha}$ is increasing function of t.

(ii) $n_0 > k+1$, i.e. the number of voters receiving positive offers from the lobby is more than a simple majority.

In this case there can be 3, 2 or 1 equilibrium cut-off levels. From (5) the probability of being pivotal is

$$C^k_{n_0-1}F^k(\widehat{\alpha})(1-F(\widehat{\alpha}))^{n_0-1-k}.$$

First, let us consider function $\widehat{\alpha}F^k(\widehat{\alpha})(1-F(\widehat{\alpha}))^{n_0-1-k}$. One can see that on the interval $[\underline{\alpha}, \bar{\alpha}]$ it is nonnegative: it is equal to zero at $\underline{\alpha}$ and $\bar{\alpha}$, and it is strictly positive elsewhere on the interval. It has exactly one maximum at $\alpha^*_{N_0} \in (\underline{\alpha}, \bar{\alpha})$, where $\alpha^*_{N_0}$ is defined from

$$\frac{\partial}{\partial\alpha}\left[\alpha F^k(\alpha)(1-F(\alpha))^{n_0-1-k}\right] = 0.$$

To see that $\alpha^*_{N_0}$ is uniquely defined on the interval $(\underline{\alpha}, \bar{\alpha})$ let's rewrite the derivative as

$$\alpha F(\alpha)(1-F(\alpha))\left[\frac{1}{\alpha} + k\frac{f}{F}(\alpha) - (n_0-1-k)\frac{f}{1-F}(\alpha)\right].$$

From the assumptions it follows that the function in the brackets is monotonically decreasing, and for $\alpha \to \underline{\alpha}$ it approaches to $+\infty$ and for $\alpha \to \bar{\alpha}$ it approaches to $-\infty$. Therefore, it can be equal to zero exactly at one point $\alpha^*_{N_0} \in (\underline{\alpha}, \bar{\alpha})$. For convenience let

$$t^*_0 = C^k_{n_0-1}\Delta\alpha^* F^k(\alpha^*)(1-F(\alpha^*))^{n_0-1-k}.$$

decision to vote in an election given that voters incur a private cost to do so. In their model voters compare this cost to the expected differential benefit. They also face the issue of multiplicity of equilibria.

[25] Some warning is needed about what we mean by Pareto dominance. Precisely, we refer to unanimity in restriction to the coalition N_0 of legislators. It represents a way to solve the coordination issue faced by this subset of players.

From (4), (5) if $t_0 = t_0^*$, $\widehat{\alpha} = \alpha_{N_0}^*$, if $t_0 \in [0, t_0^*)$ there are two solutions for $\widehat{\alpha}_L$ and $\widehat{\alpha}_R$ defined by

$$\alpha F^k(\alpha)(1 - F(\alpha))^{n_0-1-k} = \frac{t_0}{C_{n_0-1}^k \Delta} \tag{7}$$

and for all $t_0 \in (0, \infty)$ there is also solution $\widehat{\alpha} = \bar{\alpha}$.

Consider now the second assertion. The expected utility of agent n is

$$U_n(\alpha_n, \widehat{\alpha} = \begin{cases} P^1(\widehat{\alpha})\alpha_n W_1 + (1 - P^1(\widehat{\alpha}))\alpha_n W_0, \text{ for } \alpha_n \geq \widehat{\alpha} \\ P^0(\widehat{\alpha})\alpha_n W_0 + (1 - P^0(\widehat{\alpha}))\alpha_n W_1 + t_0, \text{ for } \alpha_n \leq \widehat{\alpha} \end{cases},$$

where $P^1 = \Pr(\text{at least } k \text{ from the other } n_0 - 1 \text{ agents choose } 1)$ and

$P^0 = \Pr(\text{at least } k \text{ from the other } n_0 - 1 \text{ agents choose } 0)$.

First, let us consider the case $\alpha_n \leq \widehat{\alpha}$. Expected utility can be written as $U_n(\alpha_n, \widehat{\alpha}) = \alpha_n W_1 - \Delta P^0(\widehat{\alpha})\alpha_n + t_0$.

Probability P^0 can be written as

$$P^0(\widehat{\alpha}) = \sum_{i=k}^{n_0-1} C_{n_0-1}^i F^i(\widehat{\alpha}) \left(1 - F(\widehat{\alpha})\right)^{n_{0.}-1-i}.$$

From lemma 1 in the appendix, it follows that

$$\frac{\partial P^0}{\partial \widehat{\alpha}} = f(\widehat{\alpha})(n_0 - k)C_{n_0-1}^{k-1} F^k(\widehat{\alpha})(1 - F(\widehat{\alpha}))^{n_0-1-k} \geq 0.$$

Thus, $P^0(\widehat{\alpha})$ is increasing and expected utility is decreasing in $\widehat{\alpha}$. Therefore $U_n(\alpha_n, \widehat{\alpha}_L) \geq U_n(\alpha_n, \widehat{\alpha}_R)$, i.e. in equilibrium $\widehat{\alpha}_L$ utility of each agent n is at least as high as in equilibrium $\widehat{\alpha}_R$. The case $\alpha_n \geq \widehat{\alpha}$ is similar.□

In solving backward the all game, we solve each terminal voting subgames following a pair (t_0, n_0) by considering the equilibrium $\widehat{\alpha}_L = \widehat{\alpha}_L(t_0, n_0)$ which will be denoted simply $\widehat{\alpha} = \widehat{\alpha}(t_0, n_0)$ without risk of confusion.

4.1.2 The Optimal Offer of Lobby 0

We are now in position to investigate the two dimensions of the optimal strategy of lobby 0. Given t_0 and N_0, the probability of accepting the bribe by any legislator in n_0 is simply $F(\widehat{\alpha})$ and the probability of success for the lobby 0 is $G(t_0, n_0) = \sum_{j=k+1}^{n_0} B_j \left[n_0, F(\widehat{\alpha})\right]$. Therefore, the expected payoff of the lobby 0 is $\Pi(t_0, n_0) = G(t_0, n_0)W_0 - n_0 F(\widehat{\alpha})t_0$. The following proposition describes the optimal amount of the offer t_0 when lobby 0 buy a minimal winning coalition.

Proposition 5 *When $n_0 = k+1$, the equilibrium offer t_0^* is uniquely defined:*

$t_0^* = 0$ *for* $W_0 \in [0, N_0\underline{\alpha}\Delta]$*;*

$t_0^* = aF^k(a)\Delta W < \Delta W$ *where* $\alpha \in (\underline{\alpha}, \bar{\alpha})$ *is the unique solution to the equation :* $W^0 - n_0 a\Delta W = \frac{F(a)}{f(a)}\Delta W$ *for* $W_0 \in \left(n_0\underline{\alpha}\Delta, n_0\bar{\alpha}\Delta + \frac{\Delta W}{f(\bar{\alpha})}\right)$,

$t^* = \Delta W\bar{\alpha}$ *for* $W_0 \in \left[n_0\bar{\alpha}\Delta W + \frac{\Delta W}{f(\bar{\alpha})}, +\infty\right)$

Proof. In this case the expected payoff of the lobby is defined by $\Pi(k+1, t_0) = F^{k+1}(\widehat{\alpha})\left(W_0 - (k+1)\Delta\widehat{\alpha}\right)$. Since the cut-off level $\widehat{\alpha}(t_0)$ is increasing function it is possible to substitute for t_0 from the second stage problem and to maximize with respect to $\widehat{\alpha}$.

$\frac{\partial\Pi}{\partial\widehat{\alpha}} = (k+1)F^k(\widehat{\alpha})\left[f(\widehat{\alpha})(W_0 - (K+1)\Delta\widehat{\alpha}) - \Delta F(\widehat{\alpha})\right] = 0$

First, consider $W_0 - (k+1)\widehat{\alpha}\Delta = \frac{F(\widehat{\alpha})}{f(\widehat{\alpha})}\Delta$. By assumption $\frac{F}{f}$ is increasing, therefore the RHS is increasing function of $\widehat{\alpha}$, and the LHS is decreasing. Therefore, these two functions can intersect at most once on the interval $(\underline{\alpha}, \bar{\alpha})$. It is easy to see that interior solution $a \in (\underline{\alpha}, \bar{\alpha})$ exists only if $W_0 - (k+1)\Delta\underline{\alpha} > 0$ and $W_0 - (k+1)\bar{\alpha}\Delta < \frac{\Delta}{f(\bar{\alpha})}$.

Summing up, there are three cases:

If $(k+1)\underline{\alpha}\Delta < W_0 < (k+1)\bar{\alpha}\Delta + \frac{\Delta}{f(\bar{\alpha})}$ the cut point is $a \in (\underline{\alpha}, \bar{\alpha})$ defined by $W_0 - (k+1)a\Delta = \frac{F(a)}{f(a)}\Delta$; $\frac{\partial\Pi}{\partial\widehat{\alpha}} > 0$ for $\widehat{\alpha} < a$ and $\frac{\partial\Pi}{\partial\widehat{\alpha}} < 0$ for $\widehat{\alpha} > a$.

In case $0 < W_0 < (k+1)\underline{\alpha}\Delta$ the cut point is $\underline{\alpha}$ since $\frac{\partial\Pi}{\partial\widehat{\alpha}} < 0$ on the whole interval $(\underline{\alpha}, \bar{\alpha})$.

If $W_0 - (k+1)\bar{\alpha}\Delta > \frac{\Delta}{f(\bar{\alpha})}$ the cut point is $\bar{\alpha}$ since $\frac{\partial\Pi}{\partial\widehat{\alpha}} > 0$ on $(\underline{\alpha}, \bar{\alpha})$. ■

Of course, it is not necessarily optimal for lobby 0 to buy a minimal winning coalition. It may prefer to buy a supermajority. Given the fact that the function Π is continuous with respect to t_0 and that n_0 takes a finite number of values, an optimal strategy is always well defined. It remains however that the derivation of general results concerning this policy are difficult to derive. The results that follow offer some preliminary insights in some more structured settings.

Proposition 6 *Assume that F is the uniform distribution on the interval* $[0,1]$ *and that* $n = 3$. *Then there exists* $\lambda \in \left]\frac{2}{3}, 1\right[$ *such that :*

(i) $n_0 = 2$ *iff* $W_0 \geq \frac{38}{7}\Delta W$

(ii) If $W_0 \in [0, \lambda\Delta W]$, *then* $M_0^* = 3t(\alpha_1)$ *where* $\alpha_1 \equiv \frac{3\Delta W + W_0 - \sqrt{9\Delta W^2 - 10\Delta W W_0 + (W_0)2}}{8\Delta W}$.

(iii) If $W_0 \in \left[\lambda\Delta, \frac{38}{7}\Delta\right]$, *then* $M_0^* = \frac{8}{9}\Delta W$.

(iv) If $W_0 \in \left]\frac{38}{7}\Delta, +\infty\right[$, *then* $M_0^* = 2\Delta W$.

Proof. (i) $n_0 = 2$. $\Pi(k+1, \alpha) = \alpha^2(W_0 - 2\Delta\alpha)$.

For $W_0 \in [0, 3\Delta]$ function $\Pi(k+1, \alpha)$ reaches its maximum at $a = W_0/(3\Delta)$. Corresponding transfer $t^* = t(a) = \frac{(W_0)^2}{9\Delta}$ and $\Pi(k+1, a) = \frac{(W_0)^3}{27\Delta^2} < \Delta$;

For $W_0 \in (3\Delta, +\infty)$ function $\Pi(k+1, \alpha)$ is increasing on the whole interval, therefore the maximum point is 1. Corresponding transfer $t^* = \Delta$ and $\Pi(k+1, 1) = W_0 - 2\Delta > \Delta$.

(ii) $n_0 = 3$.

In this case $\alpha^* = 2/3$.

$\Pi(k+2, \alpha) = \alpha^3 W_0 + 3(1-\alpha)\Pi(k+1, \alpha)$.

$\Pi(k+2, \alpha^*) = \frac{4}{9}\left(\frac{5}{3}W_0 - \frac{4}{3}\Delta\right)$.

The roots of the equation $\frac{\partial \Pi(k+2,\alpha)}{\partial \alpha} = 0$ are defined by

$$\alpha_{1,2} = \frac{3\Delta + W_0 \pm \sqrt{9\Delta^2 - 10\Delta W_0 + (W_0)^2}}{8\Delta}. \tag{8}$$

If the discriminant is non-negative $\Pi(k+2, \alpha)$ reaches its maximum at α_1 (the smallest root) and its minimum at α_2 (the largest root).

$\frac{\partial \Pi(k+2,\alpha)}{\partial \alpha} > 0$ on the whole interval $[0, \alpha^*]$ in the following two cases: for $W_0 \in [\Delta, 9\Delta]$ since the discriminant is non-positive, and for $W_0 \in (9\Delta, +\infty)$ since $\alpha_1 > 1$.

Therefore, for $W_0 \in (\Delta, +\infty)$ $\Pi(k+2, \alpha)$ reaches its maximum at α^*. Optimal transfer $t^* = \frac{8}{27}\Delta$ and $\Pi(k+2, \alpha^*) = \frac{4}{27}(5W_0 - 4\Delta)$.

It remains to consider three cases in turn.

1.
   - $W_0 \in (0, \Delta)$. Function $\Pi(k+2, \alpha)$ reaches its maximum on the interval $[0, \alpha^*]$ either at α_1 or α^* and $\Pi(k+1, \alpha)$ reaches its maximum on $[0, 1]$ at a. One can check that $\alpha_1 \geq a$ and $\Pi(k+1, \alpha) \leq \Pi(k+2, \alpha)$ on $[0, \alpha_1]$ (the proof is proved in more general case in the next section). Therefore, $n_0 = 3$ and maximum point is either α_1 or α^*. More precisely, for $W_0 \in (0, 2/3\Delta)$ maximum point is α_1 since $\alpha_2 > \alpha^*$. Since α_2 is decreasing in W^0 for $W^0 \geq 2/3\Delta$ minimum point $\alpha_2 < \alpha^*$.
   - $W_0 \in (\Delta, 3\Delta)$. It is always the case that $\Pi(k+1, a) \leq \Pi(k+2, \alpha^*)$.
   - $W_0 \in (3\Delta, +\infty)$. It is necessary to compare $\Pi(k+1, 1)$ and $\Pi(k+2, \alpha^*)$. It follows that $\Pi(k+1, 1) \geq \Pi(k+2, \alpha^*)$ for $W_0 \geq \frac{38}{7}\Delta$ and the opposite inequality is true otherwise.□

■

The strategy of lobby 0 described in proposition ? displays an interesting feature. Not surprisingly, the larger is the stake W_0, the more money the lobby spends to buy votes.

What is more intringing however, is that this money is on less legislators i.e. the size of the coalition to which offers are made becomes smaller. These are two equilibrium predictions in the above special setting. They suggest the following two general questions.

· Is it the case, that lobbying activities are normal goods i.e. exhibiting positive income effects ?

· Is it the case that the size of the coalition of legislators approached by the lobby decreases as the stake becomes larger ?

We strongly suspect that the answers to these two questions is yes when F is the uniform distribution on the interval $[0, 1]$. Precisely, we think that the following assertion is true but have not been able to prove it in full generality for the moment.

There exist thresholds $\overline{w}(m)$, $m = k+1, ..., n_0$, such that $\overline{w}(m)$ is decreasing in m and such that $n_0 = m$ for $W_0 \in [\overline{w}(m)\Delta, \overline{w}(m-1)\Delta]$. That is, starting with small values of W_0 lobbying group prefers to bribe all members of the committee ($n_0 = n$) and with the increase of W_0 it bribes less and less members bribing just simple majority for rather large values.

Figures 2, 3, 4 and 5 suggest the plausibility of such pattern. The following two propositions are additional pieces of evidence in defense of that conjecture.

Proposition 7 *Assume that F is the uniform distribution on the interval $[0, 1]$. Let $\alpha^*_{n_0}$ denotes the optimal cut point when lobby 0 restricts itself to a coalition of size n_0. The following statements hold true :*

(i) $\alpha^*_{n_0}$ is decreasing with respect to n_0.

(ii) If $W_0 \geq (k+2)\Delta W$, then for any $n_0 > k+1$, the function $\Pi(n_0, \alpha)$ is increasing in α on the whole interval $\left[0, \alpha^*_{n_0}\right]$.

(iii) $\Pi(n_0+1, \alpha) > \Pi(n_0, \alpha)$ for any $\alpha \in \left[0, \alpha^*_{n_0+1}\right]$ and such that $\Pi(n_0, \alpha) \geq 0$.

Proof.

1. For the case of uniform distribution $\alpha^*_{N_0} = \frac{K+1}{N_0}$.

2. The expected payoff can be written as

   $\Pi(K+1, \alpha) = \alpha^{K+1} \left(W^0 - (K+1)\,\Delta\alpha\right)$.

   For $N_0 > K+1$ $\Pi(N_0, \alpha) =$

   $C^{K+1}_{N_0} \alpha^{K+1} \left(1-\alpha\right)^{N_0-K-1} \left(W^0 - (K+1)\,\Delta\alpha\right) + W^0 \sum_{k=K+2}^{N_0} C^k_{N_0} \alpha^k \left(1-\alpha\right)^{N_0-k}$ or

$$\Pi(N_0, \alpha) = C^{K+1}_{N_0} \left(1-\alpha\right)^{N_0-K-1} \Pi(K+1, \alpha) + W^0 \sum_{k=K+2}^{N_0} C^k_{N_0} \alpha^k \left(1-\alpha\right)^{N_0-k}. \quad (9)$$

Taking derivative of (9) with respect to α we get

$\frac{\partial \Pi(N_0,\alpha)}{\partial \alpha} = C_{N_0}^{K+1}(1-\alpha)^{N_0-K-1}\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} -$

$-(N_0-K-1)\,C_{N_0}^{K+1}(1-\alpha)^{N_0-K-2}\,\Pi(K+1,\alpha)+$

$+W^0\frac{\partial}{\partial \alpha}\left(\sum_{k=K+2}^{N_0} C_{N_0}^{k}\alpha^k(1-\alpha)^{N_{0.}-k}\right).$

Substituting for $\Pi(K+1,\alpha)$ from (9) and using lemma 1 of appendix to simplify the last term one obtains

$$\frac{\partial \Pi(N_0,\alpha)}{\partial \alpha} = C_{N_0}^{K+1}(1-\alpha)^{N_0-K-2}\left[(1-\alpha)\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} + (N_0-K-1)(K+1)\,\Delta\alpha^{K+2}\right]. \tag{10}$$

It follows that $\frac{\partial \Pi(N_0,\alpha)}{\partial \alpha} \geq 0$ for $W^0 \geq (K+2)\Delta$ since $\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} \geq 0$. One can also notice from (10) that if $a(N_0)$ is a maximum point of $\Pi(N_0,\alpha)$ then necessarily $a(N_0) \geq a(K+1)$ for any $N_0 > K+1$.

3. $\Pi(N_0+1,\alpha)-\Pi(N_0,\alpha) = C_{N_0}^{K+1}(1-\alpha)^{N_0-K-1}\,\Pi(K+1,\alpha)\frac{(K+1)-(N_0+1)\alpha}{N_0-K}+W^0\sum_{k=K+2}^{N_0} C_{N_0}^{k}\alpha^k(1-\alpha)^{N_{0.}-}$ $W^0\alpha^{N_0+1}$.

   $k-(N_0+1)\,\alpha \geq 0$ for any $\alpha \leq \alpha^*(N_0+1)$ and $k \geq K+1$.

   From (9) it follows that $\Pi(N_0,\alpha) \geq 0$ is equivalent to

   $C_{N_0}^{K+1}(1-\alpha)^{N_0-K-1}\,\Pi(K+1,\alpha) \geq -W^0\sum_{k=K+2}^{N_0} C_{N_0}^{k}\alpha^k(1-\alpha)^{N_{0.}-k}$. Substituting this into the previous expression one gets

   $\Pi(N_0+1,\alpha)-\Pi(N_0,\alpha) \geq$

   $W^0\sum_{k=K+2}^{N_0} C_{N_0}^{k}\alpha^k(1-\alpha)^{N_{0.}-k-1}\left[\frac{(N_0+1)}{N_0+1-k}(1-\alpha)-\frac{(N_0+1)}{N_0+1-K}(1-\alpha)\right]$

   $+W^0\alpha^{N_0+1} > 0.$

■

Remark 1 *It can be shown that for* $W^0 < (K+2)\Delta$ *function* $\Pi(N_0,\alpha)$ *can have one interior maximum and one interior minimum. From (10)* $\frac{\partial \Pi(N_0,\alpha)}{\partial \alpha} = 0$ *is equivalent to* $\Delta(N_0+1)\,\alpha^2-(W^0+(K+2)\Delta)\,\alpha - W^0 = 0$. *It follows that maximum point* $a(N_0)$ *is increasing in* N_0 *and minimum point is decreasing.*

The next proposition shows that if lobby 0 buys a minimal winning coalition, then the stake must be larger than some minimal threshold. More precisely

Proposition 8 *Assume that F is the uniform distribution on the interval $[0,1]$. A necessary condition for $n_0^* = k+1$ is :*

$$W_0 \geq \frac{e-1}{e-2}(k+1)\Delta W$$

Proof. In order to get the result I compare $\max_{\alpha\in[0,1]} \Pi(K+1,\alpha)$ and $\max_{\alpha\in\left[0,\alpha^*_{N_0}\right]} \Pi(K+2,\alpha)$.

First, one can notice that $\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} = (K+1)\alpha^K \left(W^0 - (K+2)\Delta\alpha\right)$. Therefore, $\Pi(K+1,\alpha)$ is increasing on $[0,1]$ if $W^0 \geq (K+2)\Delta$ and otherwise it has maximum at $a(K+1) = \frac{W^0}{(K+2)\Delta}$. Second, if $\Pi(K+2,\alpha)$ has maximum at some $a(K+2) < \alpha^*_{N_0}$, then necessarily $a(K+2) > a(K+1)$. It follows from the fact that $\frac{\partial \Pi(K+2,\alpha)}{\partial \alpha} = (K+2)(1-\alpha)\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} + (K+2)(K+1)\Delta\alpha^{K+2}$. Therefore, $\frac{\partial \Pi(K+2,\alpha)}{\partial \alpha} = 0$ if and only if $(1-\alpha)\frac{\partial \Pi(K+1,\alpha)}{\partial \alpha} = -(K+1)\Delta\alpha^{K+2} < 0$.

- $W^0 \geq (K+2)\Delta$.

  According to the previous result function $\Pi(K+1,\alpha)$ reaches its maximum at $\alpha = 1$ and $\Pi(K+2,\alpha)$ - at $\alpha^*_{N_0}$.

  $\Pi(K+1,1) = W^0 - (K+1)\Delta$ and

  $\Pi(K+2,\alpha^*(K+2)) = W^0 \left(\left(\frac{K+1}{K+2}\right)^{K+1} + \left(\frac{K+1}{K+2}\right)^{K+2}\right) - (K+1)\,\Delta\left(\frac{K+1}{K+2}\right)^{K+2}$.

  $\Pi(K+1,1) - \Pi(K+2,\alpha^*(K+2)) =$

  $W^0\left[1 - \frac{2x-1}{x-1}\left(1-\frac{1}{x}\right)^x\right] - (K+1)\Delta\left[1 - \left(1-\frac{1}{x}\right)^x\right]$, where $x = K+2 \geq 3$.

  For $x \geq 3$ function $\phi(x) = \left(1-\frac{1}{x}\right)^x$ is positive and increasing with $\lim_{x\to\infty}\phi(x) = 1/e$. Function $g(x) = \frac{1-\phi(x)}{1-\frac{2x-1}{x-1}\phi(x)}$ is decreasing, $g(x) > 0$ for $x \geq 3$ and $\lim_{x\to\infty} g(x) = \frac{e-1}{e-2}$. Thus, $\Pi(K+1,1) - \Pi(K+2,\alpha^*(K+2)) > 0$ iff $W^0 > (K+1)\Delta g(K+2)$ and $\frac{e-1}{e-2} < g(x) < 19/7$.

- $W^0 \leq (K+2)\Delta$.

  $\Pi(K+1,\alpha)$ reaches its maximum at $a(K+1) < 1$. $\Pi(K+2,\alpha)$ is either increasing on the whole interval $\left[0,\alpha^*_{N_0}\right]$ or its maximum is at $a(K+2) < \alpha^*_{N_0}$.

  If $\Pi(K+2,\alpha)$ has maximum at $a(K+2)$ then $a(K+2) > a(K+1)$ and from proposition **??** $\Pi(K+2,a(K+2)) - \Pi(K+1,a(K+1)) \geq \Pi(K+2,a(K+1)) - \Pi(K+1,a(K+1)) > 0$.

  Let's $\Pi(K+2,\alpha)$ is increasing on the whole interval. Then the following is true $\Pi(K+2,\alpha^*(K+2)) - \Pi(K+1,a(K+1)) \leq \Pi(K+2,\alpha^*(K+2)) - \Pi(K+1,1) < 0$ if $W^0 > (K+1)\Delta g(K+2)$.

■

4.2 The Optimal Strategy of Lobby 0 when Lobby 1 is active

In this section, we return to the game theoretical framework i.e. we take into account the lobbying or counterlobbying strategy of lobby 1. As in the preceding section, we disregard the possibility for a lobby to offer different offers to those who receive offers and we denote by (t_0, n_0) and (t_1, n_1) the respective (pure) strategies of lobby 0 and lobby 1. It should observed that the two-player game describing this competition is quite unusual as the sets of pure strategies of the players are non convex subsets of the $(n-1)$-dimensional unitary simplex[26] as illustrated on figure 1. This implies that the equilibrium analysis will be rather intricate and requires a specific treatment. We first explore the nature of the best responses functions of the two lobbies.

Insert Figure 1 about here

Given a profile $((t_0, n_0), (t_1, n_1))$ of lobbying strategies, the legislators are partitioned into three groups:

· The group N_0 incudes the legislators who have received an offer exclusively from lobby 0.

· The group N_{01} incudes the legislators who have received an offer from both lobbies.

· the group $N \setminus (N_0 \cup N_{01})$ of legislators who did not receive any offer.

Note that at equilibrium, no legislator will receive a positive offer from lobby 1 exclusively as this would be from its perspective a wasteful investment.

4.2.1 The Voting Subgame(s)

As before, in solving continuation voting subgames, we restrict our attention to symmetric equilibria i.e. all legislators in similar position follow the same decision rule.The legislators in group N_0 will be dscribed by the cut point $\widehat{\alpha}$ while those in N_{01} will be described by the cut point $\widehat{\beta}$. Let us denote by p^0 and p^{01} the probability of being pivotal for a legislator in N_0 and N_{01} respectively. Then, the cut points are defined as

$$\widehat{\alpha} = \max\left\{\underline{\alpha}, \min\left\{\frac{t_0}{p^0 \Delta W}, \bar{\alpha}\right\}\right\}, \tag{11}$$

and

$$\widehat{\beta} = \max\left\{\underline{\alpha}, \min\left\{\frac{t_0 - t_1}{p^{01} \Delta W}, \bar{\alpha}\right\}\right\}. \tag{12}$$

The dominant strategy of legislators in $N \setminus (N_0 \cup N_{01})$ is to vote for the reform.

[26] This follows, of course from our uniformity assumption.

For any legislator i in $N_0 \cup N_{01}$ the probability of being pivotal is simply the probability that exactly k other agents vote for 0. It is equal the probability that exactly k agents from $(N_0 \cup N_{01}) \setminus \{i\}$ vote for 0. Given the cut points $\widehat{\alpha}$ and $\widehat{\beta}$, it is possible to write down explicitly the formula for the probabilities of being pivotal:

$$p^0(N_0, N_{01}, \widehat{\alpha}, \widehat{\beta}) = \sum_{r=\max(0,k-n_{01})}^{\min(k,n_0-1)} B_r\left[n_0 - 1, F(\widehat{\alpha})\right] B_{k-r}\left[n_{01}, F(\widehat{\beta})\right], \tag{13}$$

A similar expression is derived for $p^{01}(N_0, N_{01}, \widehat{\alpha}, \beta)$. The continuation voting subgames are more complicated to analyse than before as there are now a pair of equations and two variables $\widehat{\alpha}$ and $\widehat{\beta}$ to be determined.

4.2.2 The Optimal Strategies of Lobbies 0 and 1

We offer some preliminary but incomplete insights in the case where F is the uniform distribution on the interval $[0, 1]$ and $n = 3$.There are seven possible cases according to the number of agents belonging to each of three groups N_0, N_{01} and $N \setminus (N_0 \cup N_{01})$. They are denoted by 201, 300, 210, 120, 111, 021, 030. For example,case 201 describes the situation in which two legislators receive an offer from the lobbying group 0 and the third one does not receive offer at all. The first two regimes, namely 201 and 300 bring us back to the previous situation, in which only lobbying group M^0 is active.

We are primarily interested in the best response of lobby 0 to the strategy of lobby 1. The following table describes the possible regimes r for lobby 0 given (t_1, n_1).

$(\mathbf{1}, \mathbf{t_1})$	$(\mathbf{2}, \mathbf{t_1})$	$(\mathbf{3}, \mathbf{t_1})$
210	120	030
111	021	

For each r it is possible to calculate the best response $t_0(t_1)$ and also $\Pi_r(t_1) = \Pi_r^0(t_0(t_1), t_1)$. Then, given (n_1, t_1), the reaction $t_0(t_1)$ of lobby 0 maximizes $\Pi_r(t_1)$.

Case 1 : 210

- Pivotal probabilities are defined by

  $p^0 = \alpha\,(1 - \beta) + \beta\,(1 - \alpha)$

  $p^{01} = 2\alpha\,(1 - \alpha)$

  Then, the second stage solutions $\alpha\,(t^0, t^1)$ and $\beta\,(t^0, t^1)$ are defined from the following system

$$\begin{cases} \alpha = \max\left\{0, \min\left\{\frac{t^0}{\Delta[\alpha(1-\beta)+\beta(1-\alpha)]}, 1\right\}\right\} \\ \beta = \max\left\{0, \min\left\{\frac{t^0-t^1}{2\alpha(1-\alpha)\Delta}, 1\right\}\right\} \end{cases} \tag{14}$$

- Suppose that $\alpha, \beta \in (0, 1)$. Eliminating β from the system, one can express t^0 through α and t^1:

$$t^0 = 2\Delta\alpha^2 (1 - \alpha) - t^1 (1 - 2\alpha) \tag{15}$$

- Lobbying group 0 maximizes:

$$\Pi^0_{210} = \left(\alpha^2 + 2\alpha\beta (1 - \alpha)\right) W^0 - t^0 (2\alpha + \beta) \tag{16}$$

We take the grid for t^1 consisting from 100 points and for each value of t^1 calculate t^0, β and Π^0_{210} a functions of α. On the interval $[0, 1]$, equation (22) may define two solutions for $t^0 (\alpha)$ one of which is decreasing in α and the other is decreasing. It is more intuitive that a larger amount of bribe from lobby 0 corresponds to a higher probability for a legislator to accept the bribe and to vote against the reform. Therefore we consider the increasing solution.

We maximize the function Π^0_{210} with respect to α on the interval where $t^0 (\alpha)$ is positive and increasing. Below the graphs of the functions (22) and (23) are different values of t^1 and $W^0 = \Delta = 1$.As we can see the function $\Pi^0_{210} (\alpha)$ may have two local maxima on the considered interval. For small values of t^1, the function $\Pi^0_{210} (\alpha)$ reaches its maximum in the right bounadary of the interval (figure 10). With an increase of t^1, the point where the maximal is reached moves to the left : first to the higher and the then to the lower local maximum point (figures 11, 12). We can have a situation where the maximum is reached at both points.

Case 2 : 120 is symmetric with respect to α and β to the previous case. Therefore, the analysis is very similar.

Case 3 : 111 cannot appear at the equilibrium since the system for the second-stage solution is consistent if and only if $t^1 = 0$:

- The system for the second-stage solution is consistent if and only if $t^1 = 0$:

$$\begin{cases} \alpha = \max\left\{0, \min\left\{\frac{t^0}{\Delta\beta}, 1\right\}\right\} \\ \beta = \max\left\{0, \min\left\{\frac{t^0-t^1}{\Delta\alpha}, 1\right\}\right\} \end{cases}$$

Then, we are back to the case 201.

Case 4 : 021

- The second-stage solution $\beta(t^0, t^1)$ is defined by $\beta^2 = \frac{t^0-t^1}{\Delta}$ for $\frac{t^0-t^1}{\Delta} \in (0,1)$; $\beta = 0$ or 1 if $\frac{t^0-t^1}{\Delta} < 0$ or $\frac{t^0-t^1}{\Delta} > 1$ respectively.

  At the first stage each lobbying group maximizes its expected payoff:

$$\Pi^0_{021} = \beta^2 W^0 - 2\beta t^0$$

Case 5 : 030

- The second-stage solution is given by :

$$2\beta^2 (1-\beta) = \frac{t^0 - t^1}{\Delta}$$

LHS is increasing in t^0 and decreasing in t^1 on the interval $[0, 2/3]$. Similar to the previous case, we take the grid for t^1 and express t^0 as a function of β. The expected payoff of lobby 0 is given by :

$$\Pi^0_{030} = \left(2\beta^2 (1-\beta) + \beta^3\right) W^0 - 3\beta t^0$$

It is maximized for $\beta \in \left[0, \frac{2}{3}\right]$ since the function $t^0(\beta)$ is increasing on this interval and decreasing on $\left[\frac{2}{3}, 1\right]$.

5 Appendix

5.1 A Technical Lemma

Lemma 1 *The Function* $U(n_0, m, \alpha) = \sum_{k=m}^{n_0} C^k_{n_0} F(\alpha)^k (1 - F(\alpha))^{n_0 - k}$, *where* $m < n_0$ *is increasing with respect to* α *and* n_0. *More precisely,*

$$\frac{\partial U(n_0, m, \alpha)}{\partial \alpha} = (n_0 + 1 - m) f(\alpha) C^{m-1}_{n_0} F^{m-1}(\alpha) (1 - F(\alpha))^{n_0 - m} \tag{17}$$

and

$$U(n_0 + 1, m, \alpha) - U(n_0, m, \alpha) = C^{m-1}_{n_0} F^m(\alpha) (1 - F(\alpha))^{n_0 + 1 - m}. \tag{18}$$

Proof. $\frac{\partial U(N_0,m,\alpha)}{\partial\alpha} =$

$f(\alpha)\sum_{k=m}^{N_0-1} C_{N_0}^k \left[kF^{k-1}(\alpha)\left(1-F(\alpha)\right)^{N_0-k} - (N_0-k)F^k(\alpha)\left(1-F(\alpha)\right)^{N_0-k-1}\right] + N_0 f(\alpha)F^{N_0-1}(\alpha)$.

Substituting for $kC_{N_0}^k = (N_0-(k-1))\,C_{N_0}^{k-1}$ one gets that this is equivalent to

$f(\alpha)\sum_{k=m}^{N_0-1}(N_0-(k-1))\,C_{N_0}^{k-1}F^{k-1}(\alpha)\left(1-F(\alpha)\right)^{N_0-k} -$

$f(\alpha)\sum_{k=m}^{N_0-1} C_{N_0}^k (N_0-k)\,F^k(\alpha)\left(1-F(\alpha)\right)^{N_0-k-1} + N_0 f(\alpha)F^{N_0-1}(\alpha)$.

In the two sums all term except the first and the last ones are cancelled out. Thus, $\frac{\partial U(N_0,m,\alpha)}{\partial\alpha} = (N_0+1-m)f(\alpha)C_{N_0}^{m-1}F^{m-1}(\alpha)\left(1-F(\alpha)\right)^{N_0-m}$.

$U(N_0+1,m,\alpha) - U(N_0,m,\alpha) =$

$\sum_{k=m}^{N_0+1} C_{N_0+1}^k F^k(\alpha)\left(1-F(\alpha)\right)^{N_0+1-k} - \sum_{k=m}^{N_0} C_{N_0}^k F^k(\alpha)\left(1-F(\alpha)\right)^{N_0-k}$. After applying $C_{N_0+1}^k = C_{N_0}^k + C_{N_0}^{k-1}$ this difference is equal to

$F(\alpha)^{N_0+1} + \sum_{k=m}^{N_0} C_{N_0}^k F^k(\alpha)\left(1-F(\alpha)\right)^{N_0+1-k} + \sum_{k=m}^{N_0} C_{N_0}^{k-1}F^k(\alpha)\left(1-F(\alpha)\right)^{N_0+1-k} - \sum_{k=m}^{N_0} C_{N_0}^k F^k(\alpha)\left(1-F(\alpha)\right)^{N_0-k}$. Summing up the first and the third sums it is equivalent to

$F(\alpha)^{N_0+1} - \sum_{k=m}^{N_0} C_{N_0}^k F^{k+1}(\alpha)\left(1-F(\alpha)\right)^{N_0-k} + \sum_{k=m}^{N_0} C_{N_0}^{k-1}F^k(\alpha)\left(1-F(\alpha)\right)^{N_0+1-k}$. One can notice that in the two sums all the terms except the first and the last ones are cancelled out. Therefore, we get (18). ■

5.2 Figures

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