0.19 % Subsidy-Free Spatial Pricing

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Abstract

Consider a population of customers uniformly spread over the entire plane, that faces a problem of locating facilities to be used as delivery places by these customers. The cost of every facility is financed by its users only, who, in addition, face an idiosyncratic private access (say transporation) cost to the facility. We assume that the facilities' cost is independent of location and access costs are linear with respect to the Euclidean distance. We show that that an external intervention that covers 0.19% of the facility cost is sufficient to guarantee *the existence of subsidy free prices i.e. prices immuned to cross-subsidization* : no group of customers is charged more than its stand alone cost (the cost incurred if it acts on its own). Moreover, we demonstrate that under this minimal external intervention, only Rawlsian prices survive the cross subsidization test.

Keywords: secession-proofness, optimal jurisdictions, Rawlsian allocation, hexagonal partition, cross-subsidization.

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1 Introduction

When citizens using a public facility (e.g. a hospital, a library or a post-office) face, in addition to their contribution to the fixed costs of the facility, an idiosyncratic private access cost (e.g. transportation) two questions arise:

How does one characterize the optimal number and "locations" of the facilities?

Should the prices respond, ceteris paribus, to the differences in private cost?

In this paper, we consider a population that consists of a continuum of citizens uniformly distributed over the entire two dimensional Euclidean space $\Re^2$[1]. Facilities can be located anywhere in $\Re^2$ at a cost independent of location. All citizens face common private access costs proportional to the Euclidean distance to the chosen facility.

The characterization of efficient partitions in this geometric setting is a well documented problem in mathematics[2]. There is a uniquely determined optimal "density" of facilities, each serving a connected subset of citizens (called hereafter *jurisdiction*) of the "optimal" size. The qualitative result (re)discovered[3] by many authors states precisely that there is a unique "shape" of efficient partitions which consists of identical regular hexagons. In fact, the optimality of the hexagonal structure has been proved by Fejes Toth (1953) and independently by several authors, including Haimovich and Magnanti (1988), whereas in economic geography the optimality of hexagons has been known since, at least, the 1930's, due to the classic work of Christaller (1933) and Lösch (1954). Obviously, in our framework the size of regular hexagons in an efficient partition depends on the value of facility costs.

The pricing issue has not received as much attention. Different approaches can be considered depending upon the objectives pursued by the entreprise supporting the cost attached to the efficient solution that we have just described. If that firm is a private monopoly charging prices in order to

[1] In this paper we adopt the spatial interpretation of the horizontal differentiation setting.

[2] The regular hexagonal structure arises in others packing and partionning problems. The hexagonal honeycomb conjecture asserts that regular hexagons provide the least perimeter way to partition the all plane into unit areas. It has been proved recently by Hales (2001) (see also Hales (2000) and Morgan (2005) for beautiful accounts). Thue ((1892)(1910)) has proved that the densest packing of unit disks in the plane is obtained when we inscribe a disk in each hexagon in the regular heexagonal tiling of the plane

[3] A recent nice proof is due to Morgan and Bolton (2002).

maximize its benefit, then we obtain pricing functions which depend upon the quantity of the good consumed by the customer and his location with respect to the nearest facility. In the case where there is a single facility (the mill), the nature of the optimal pricing function has been investigated by several authors. Among other things, they have examined what circumstances lead the monopolist to a policy of F.O.B. mill selling (the shipping cost from the mill is paid by the customer) or instead a policy involving partial or total freight absoption (freight-absorption pricing) like for instance uniform delivery pricing. Smithies (1941) assumes for instance that "The monopolist will not charge delivered prices at any two points, such that the price charged at the point further from his mill is greater than that charged at the nearer point by more that the cost of transporting a unit of the commodity from one point to the other"[4]. Under this assumption and the linearity of the transportation cost, he demonstrates that the monopolist will sell F.O.B. mill if the logarithmic demand curve is linear or concave and will absorb freight if the logarithmic demand curve is convex. F.O.B. pricing is probably considered by most antitrust economists as the desired (truly competitive) spatial pricing technique. This point of view is challenged by Greenhut and Greenhut (1977) who demonstrate that the class of local demand functions generating linear delivered price schedules by spatial sellers is a narrow class. In location theory, discussion of spatial price policies chosen by a monopoly is often limited to three particular pricing methods : F.O.B. (uniform mill pricing), uniformed delived pricing, and discriminatory local pricing[5]. Spulber (1981) examines the structure of the optimal nonlinear outlay schedule at the mill through quantity discounts or premia and quantity bundling. He derives some features of this nonlinear policy under the specific assumption of constant marginal cost of production but general assumptions on the transportation technology and the population density otherwise. This nonlinear mill pricing is compared to local price discrimination. He shows that the relative levels of total output and profits and consumer welfare under these pricing policies depend crucially upon the spatial distribution of consumers : profits, output, and consumer welfare under nonlinear pricing are greater than, equal to, or less than profits, output, and consumer welfare under local price discrimination if consumer density decreases, remains constant, or increases with

[4]This importance of this condition was rediscovered by Le Breton and Weber (2003) where it is named principle of partial equalization.

[5]See for instance Beckmann (1968)(1976).

increasing distances from the mill. To the best of our knowledge, the optimal pricing policy when it is technologically optimal for the firm to open more than one mill has not been investigated.

When the firm is a public regulated firm or utility, the derivation of the pricing policy follows a different road as the objectives are typically different from profit maximization. One standard approach in the theory of regulation and public utility pricing is the derivation of Ramsey-Boiteux prices. These prices are derived from the maximization of a social objective (typically a weighted sum of the consumer surplus and the profit of the firm). Of course, when the utility is constrained to balance its budget, only the consumer surplus matters. Another approach, popular in the theory of regulation and public utility pricing[6], consists in imposing to the prices to be subsidy-free or equivalently immuned to cross-subsidization. On one hand, the total cost of the utility must be covered by the prices : no external subsidy should be considered to take care of a deficit and no profit should be made at the expense of consumers. On the other hand, the prices must be such that no group of customers is charged more than the cheapest cost of serving their demands when the other customers do not have any demand. This notion of subsidy-free prices which can be applied equally to group of commodities instead of group of customers has been pionnered by Faulhaber (1975)[7] and is now a cornerstone in the theory and practice of public enterprise pricing[8]. Given a population of consumers $N = \{1, 2, ..., n\}$, a list of products or services $M = \{1, 2, ..., m\}$, a multiproduct cost function $C(Q_1, Q_2, ..., Q_m)$ and a profile of demand functions $\left(Q^1(p), Q^2(p), ..., Q^n(p)\right)$, the price vector $p \in \Re^m_+$ is *commodity subsidy-free* if :

$$pQ(p) = C(Q(p)) \text{ and } p_S Q_S(p) \leq C(Q_S(p), 0_{N\setminus S}) \text{ for all } S \subseteq M$$

where p_S, Q_S denote the restriction of the price and quantity vectors to the set of commodities

[6]See for instance, Baumol, Panzar and Willig (1988), Brown and Sibley (1986), Ralph (1992), Rosenbaum (1997), Sharkey (1982b) and Zajac (1978).

[7]See also Faulhaber (1979)(2005).

[8]See Ralph (1992) for an historical account. He reports that Baumol was the very first one to propose informally in 1970 this notion of subsidy free prices as witness in a trial. The conditions known then under the name of the "burden test" were analysed by Zajac and others at Bell labs. In 1972, they discovered that the burden test could be impossible to satisfy alongside a zero profit constraint and that Ramsey prices need not be subsidy free. Many notable economists expressed strong surprise that these things could happen. This stimulated Faulhaber who began to think about these issues and realize the relevance of cooperative game theory and in particular the core solution. Faulhaber (1975) nonexistence example stimulated a vast body of literature culminating in Baumol, Panzar and Willig 's famous book (1988) on contestatble markets.

in S. Similarly, the price vector $p \in \Re^m_+$ is *consumer subsidy-free* if :

$$pQ(p) = C(Q(p)) \text{ and } pQ^S(p) \leq C(Q^S(p)) \text{ for all } S \subseteq N$$

where $Q^S(p) \equiv \sum_{i \in S} Q^i(p)$. In many environments like the spatial environment considered in this paper there is almost a one to one correspondence between the sets M and N : the same "physical" service provided to two customers located in different areas of the territory correspond to two different commodities. In such case, the vector of commodity prices accounts for the cost of production together with the cost of delivery. In general, the two notions are different and the relationship between the two set of prices depends critically upon the actual pattern of individual demands. A more restrictive approach that does not depend upon observing customer demands is to require that any conceivable set of consumer demands vectors generates no greater than its stand alone costs. Formally, the price vector p is *anonymously equitable* if :

$$pQ(p) = C(Q(p)) \text{ and } pq \leq C(q) \text{ for all } q \leq Q(p)$$

If p is anonymously equitable, then it is consumer and commodity subsidy free but the converse does not need to be true as illustrated by a counterexample developed in Faulhaber and Levinson (1981). They derive sufficient conditions on the cost function C for the equivalence to be true. There is also no guarantee at all that subsidy free prices and a fortiori anonymously equitable prices exist[9]. Existence is an intricate issue and several sufficient and (or) necessary conditions on the cost function. In a way or another, the cost function must exhibit some form of increasing returns to scale but subadditivity which is the weakest conceivable form of such property is not enough for existence[10]. If C is sustainable (Baumol, Bailey and Willig (1977) then anonymous equitable prices exist and if anonymous equitable prices exist, then C is supportable and subhomogeneous (Sharkey and Telser (1978)). Moreover, when the set of anonymous equitable prices is nonempty, it is not necessary the case that Boiteux-Ramsey prices are anonymously equitable. Baumol, Bailey and Willig (1977)

[9]The list of inequalities involves any possible group S of customers or products. When it is applied to a single individual i or a single product j, we obtain an upper bound on the price (the stand alone cost) and (using the fact that the budget is balanced) a lower bound (the incremental cost). Often, the assessment of cross subsidy is limited to this subset of inequalities.

[10]For a careful and detailed presentation of the properties of cost functions and their implications for public utility pricing, we refer the reader to Sharkey (1982a).

have shown that if C is subhomogeneous and transray convex, then Boiteux-Ramsey prices are anonymously equitable. Finally, using an early result of Panzar and Willig (1977), Faulhaber and Levinson (1981) have shown that if C satisfies general cost complementarity and if demands are independent, then anonymously equitable prices exist[11].

Why should we be interested in subsidy free prices ? This requirement acting as a constraint on the pricing policy may reflect a principle of customer equity (Willig (1979)) : no customer or group of customers should subsidy the consumption of other customer. The issues of cross subsidization play an important role in the modern regulatory practices as applied to sectors and industries including electricity and natural gas distribution, telecommunications[12], railways, water and sewerages services to cite a few. The methods which were used before the modern definition of Faulhaber, on top of which the fully distributed cost method (FDC), were exposed to many criticisms[13] and the method of Faulhaber is now commonly used[14]. Besides equity considerations, the concern for subsidy free prices arises as soon as we assume that production can be realized by outide firms. If all potential producers have access to the same technology as the multiproduct firm (utility), the problem of finding subsidy free prices becomes the problem of finding prices that will prevent entry by a competitive producer. This establishes the strong connection between this issue and the vast literature on contestable markets[15]. Finally, in the context of public facilities, where the service provided displays the features of a pure public good, the concern for subsidy free prices may simply describe the constraint of voluntary participation or secession-proofness : no coalition of citizens should receive incentives to cultivate secession feelings and decide to provide the good by itself if it is not socially efficient to do so. In any setting where citizens or groups of citizens cannot be forced to participate to a collective

[11]See also Calum (2003) and Jamison (1996).

[12]See Kaserman and Mayo (1994) and Kaserman, Mayo and Flynn (1990).

[13]See for instance Bonbright (1961) and Braueutigan (1980). These methods were also rejected by regulatory bodies. For instance, Bonbright quotes that "..the Illinois Commerce Commission refused to order the Commonwealth Edison Company of Chicago to make a fully distributed cost study in support of a proposed rate increase, because there were at least "twenty-nine rival formulas for the allocation of capacity costs alone", formulas each of which had received some profesional sponsorship".

[14]Among the many government authorities which use that approach we could cite the US Interstate Commerce Commission, the US Justice department, and the Australian Federal department of Transport and Communications. A recent study of the cost and tariff structures of Scottish Water (the utility incharge of water and sewerage services for the 2.3. millions customers of Scotland) for the Scottish executive illustrates remarquably this methodology. Household water customers receive a subsidy from non-household customers : it results in households paying £44m a year less for water supply services than it costs to provide them with these services.

[15]See for instance Mirman, Tauman and Zang (1985).

arrangement that they would not accept, such concern arises naturally[16].

Our paper is at the intersection of the two literatures described above. We examine the existence and the nature of subsidy free prices in an environment where customers are differentiated on a spatial basis. In contrast to the above literature, the cost function is not given a priori. Instead, it has to be constructed from some basic primitives : the transportation technology and the cost of opening and operating a facility in a given location. In this spatial framework, the multiproduct character follows from the fact that the product has to be delivered and therefore two different locations generate different idiosycratic costs. This results in our problem beeing far more complicated than the single mill situation assumed by spatial economists. On the other hand, we will here ignore the difficulties arising on the demand side by assuming that all the demands are inelastic and unitary. It is of course a simplifying assumption that allows us to focus on the supply side[17]. Under that assumption, the first challenge is therefore to determine $C(1, 1, ..., 1)$ and $C(1_S, 0_{N \setminus S})$ for any coalition S of customers. as already reported, this follows from the literature on the partioning in regular hexagons. The main contribution of the paper lies in the exploration of the set of subsidy free prices.

In the one dimensional setting (customers located on a line), Haimanko, Le Breton and Weber (2004) have shown that subsidy free prices always exist. In our two dimensional setting, the situation is quite different. The characterization of efficient partitions allows us to derive the first result of this paper that demonstrates that the set of subsidy free prices is empty. This simply means that it is impossible to allocate facilities' cost in an efficient partition in a way to prevent cross subsidization. The reason for this nonexistence result is based on the simple observation that the *optimal* shape of a jurisdiction that minimizes its per capita monetary burden is actually a disk and not a hexagon. Thus, for any efficient partition one can find a circle-shaped jurisdiction that guarantees its members a lower average contribution. Intuitively, as soon as a large enough fraction δ of the total cost is financed by an external source (in violation to the requirement of budget balancedness), we remove the difficulty. The severity of the violation depends of course on how large δ must be to rescue the

[16]See for instance Alesina and Spolaore (1997) and Le Breton and Weber (2003).

[17]Of course, when demands are responsive to prices a different (equilibrium) analysis must be conducted. In the context of cooperative games, we must move from cost games to welfare games (Sharkey (1982a)).

rest of the inequalities. Any set of subsidy free prices for which a fraction δ of the total cost has been financed extrnally is named hereafter δ-subsidy free prices. For which values of δ, do δ-subsidy free prices exist ? Ideally, we would like $\delta = 0$ but we just mention that it does not work. Surprisingly, we show here that this number can be made very small. To explain how small it can be[18], we define the gap between efficiency and optimality as the following expression $\delta^* = 1 - \frac{K(B)}{K(H)}$, where $K(B)$ is the average individual cost in an optimal ball and $K(H)$ is the average cost of individuals in an efficient hexagon. We show that in our framework the value of δ^* is very small, namely, it is $\approx 0.0019(!)$ i.e. a per capita subsidy of **less than** 0.2% is sufficient in guarantee the existence of subsidy-free prices. The proof of this result is quite involved and requires an application of Fubini's theorem.

The next important question we address is the characterization of the set of δ^*-subsidy free prices. For this end, we consider the Rawlsian principle, under which the society maximizes the utility level of the most disadvantaged individual. In our framework, the Rawlsian principle implies the full equalization of all individuals' total costs, and, therefore, entails the full compensation to every individual for being assigned to the policy different from her favored one; in the spatial literature, this corresponds to uniform delivery prices, or more accurately to basing point prices as, in general, it is efficient to have several mills. It turns out that the *Rawlsian prices* are the *unique* δ^*-subsidy free consistent prices[19].

The paper is organized as follows. The next section contains the model and introduces the definitions needed for our main results that are stated in section 3. The proofs of all results are relegated to the Appendix.

2 The Model

We consider a population with a continuum of customers together with a firm. The firm produce a single homogeneous good; from a technological perspective, it has to decide how many production units (called hereafter facilities) to open and where to locate these facilities in the two dimensional

[18]We could see δ^* as the minimal degree of outside intervention that can rectify the stability failure.

[19]In the one dimensional case , for which we know that $\delta^* = 0$, Drèze, Le Breton and Weber (2006) prove that when the population is uniformly spread over the entire real line, then the Rawlsian prices are the unique subsidy free prices. This result is not true for a bounded interval. Le Breton and Weber (2003) have shown that F.O.B. prices and uniform delivery prices are not subsidy free in general.

Euclidean space which is meant to model here the true geographical space. Each customer is itself located somewhere in the two dimensional Euclidean space and consumes a single unit of the commodity produced by the firm. We assume that their demand for that commodity is inelastic with respect to the price.

At that stage, besides the assumption that the space of facilities's feasible locations is the all two dimensional space (stated formally below as assumption 1), no assumption is made upon the technology of each facility. The setting is broad enough to accomodate several interpretations. the facility could be public facility (hospital, swimming-pool, post-office, library, ..) where most of the cost consists of a fixed part if there are no too much congestion, or a private facility (bank office, warehouse, factories,..) where variable costs are likely to be more important. The firm can be a private firm with private objectives, like profit maximization, or a firm under the supervision of a public regulator. Herafter, we will privilege that second situation as we will be interested in the determination of prices that meet a condition of sustainability introduced in the litterature on regulation : the prices are calculated in such a way to prevent entry on a market for which it is efficient to have a single firm i.e. a situation of natural monopoly. We will also introduce later some more specific assumptions on the technology.

Assumption A.1 — Multidimensionality: The space of facilities' locations is the two-dimensional Euclidean space $X = \Re^2$.

Customers have idiosyncratic preferences (or accession costs) over the facilities that they could visit to buy the product or service (to post a letter, to deposit a check on the bank account, to swim for some time or to consult a doctor at the hospital or to drive to a production center for delivery if the customer is in fact a retailer of the commodity). We assume that for every customer the accession cost is represented by the Euclidean distance from her bliss point to the facility in her jurisdiction:. In this paper we assume that the accession cost is incurred by the client but we could, with little changes, assume that the cost is incurred by the firm : for instance, in the case of a post office, we can either assume that the client goes to the post office to get her letter or instead that the firm paid first for that cost by hiring a postman delivering the letter to the residence of the client.

Assumption A.2 — Euclidean accession costs: For every individual located at $l = (l_1, l_2) \in X$, her accession cost to every $t = (t_1, t_2) \in X$ is given by

$$||t - l|| = \sqrt{|t_1 - l_1|^2 + |t_2 - l_2|^2}. \tag{1}$$

This formalization allows us to identify a customer with her location and to characterize the demand side by the distribution of individuals' locations. We assume that the customers are uniformly distributed over the entire space X:

Assumption A.3 — Uniform distribution: The customers' distribution is given by the two-dimensional Lebesgue measure[20] λ over $\Re^2$.

The area of a measurable[21] set S will be denoted by $\lambda(S)$, i.e., $\lambda(S) = \int\limits_S dt$. In what follows, the null-measured sets with $\lambda(S) = 0$ will be disregarded, so that the qualification "up to a null-set" should be added to almost all our results. Each customer will buy her unit from a given facility. Given the list of facilities opened by the firm; this leads to a partition of the population into subsets described by a one to one correspondence between the partition and the list of opened facilities. Hereafter, we will refer to any subset in that partition as a jurisdiction.

In our set-up, every jurisdiction is a measurable bounded subset of X with positive measure. The collection of such sets will be denoted by $\mathcal{M}(X)$. We assume that the cost of each facility is independent of its location and consists of a fixed cost, independent of the size of a jurisdiction, and a variable cost proportional to the jurisdiction size:

Assumption A.4 — Facility cost: For a facility assigned to a jurisdiction, the cost is given by

$$f(S) = g + \alpha\lambda(S), \tag{2}$$

where $g > 0, \alpha \geq 0$ are two constants.

We now formally introduce the notion of a partition of a measurable subset $S \subset X$:

[20] See Halmos (1950), p. 153.

[21] A subset of X is measurable if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

Definition 2.1: A partition P of (possibly infinite-measured) set S is a jurisdiction structure that consists of sets from $\mathcal{M}(X)$ which are "almost" pairwise disjoint, i.e., $\lambda(T \bigcap T') = 0$ for all $T \neq T'$ in P, and whose union constitutes the entire set S, i.e., $\bigcup_{T \in P} T = S$.

The set of partitions of S is denoted by $\mathcal{P}(S)$. Obviously, if the measure of S is infinite, then every $P \in \mathcal{P}(S)$ consists of an infinite number of jurisdictions.

Now let us turn to the determination of facility choices. For each $S \in \mathcal{M}(X)$ and a location $l \in S$ we denote by $D(S,l)$ the value of total accession cost in S (with respect to location l):

$$D(S,l) = \int\limits_{S} ||t - l||dt. \tag{3}$$

Suppose that jurisdiction S forms and chooses a location l. One can derive the total cost of members of S that combines the project cost $f(S)$ and the aggregate access cost, $D(S,l)$. Thus the *per capita* cost of members of S whose location is l is given by

$$K(S,l) = \frac{f(S) + D(S,l)}{\lambda(S)}. \tag{4}$$

Now, consider any measurable set $S \subset X$, finite or infinite-measured. This set can be partitioned into several jurisdictions each choosing a facility location. We define its stand alone per capita cost[22] as the minimum over all possible partitions P of S and sets of location choices $L = \{l(T)\}_{T \in P}$:

$$\tilde{K}(S) = \inf_{P,L} \frac{\sum_{T \in P} \left[D\left(T, l(T)\right) + f\left(T\right) \right]}{\lambda\left(S\right)}. \tag{5}$$

Any partition solving program (5) is called $S-$efficient. It is easy to see that efficiency implies that to every jurisdiction T will correspond a facility whose location is selected in order to minimize the total accession costs of its members. That is, the location l correponding to jurisdiction T will satisfy the following program:

$$D(T,l) \to \min_{l \in X}. \tag{6}$$

[22]Since the total cost could be infinite, an operational definition is needed. It consists of taking limits when the area uniformly approaches S.

The value of this problem is called "$MAT(T)$" (Minimal Aggregate Transportation cost of the set T).[23] Any solution to (6) is called a *central location* of T.[24] We use the following lemma:

Lemma 2.2: For every jurisdiction $T \in \mathcal{M}(X)$, the central location, denoted by $m(T)$, is unique.

Since Lemma 2.2 resolves the locational choice for every jurisdiction, it allows us to reduce the examination of efficiency to partitions only. For this end, denote

$$D(T) = D(T, m(T)),\ K(T) = K(T, m(T)). \tag{7}$$

and for any measurable set $S \subset X$, finite or infinite-measured,

$$\tilde{K}(S) = \inf_{P \in \mathcal{P}(S)} \frac{\sum_{T \in P} [D(T) + f(T)]}{\lambda(S)}. \tag{8}$$

We have

Definition 2.3: A partition P is S-efficient if it is a solution to (8). An X-efficient partition will be simply called an efficient partition.

In what follows, we will focus our analysis on efficient partitions. The characterization of efficient partitions in our geometric setting is a well documented problem in mathematics. The qualitative result (re)discovered by many authors states that there is a unique "shape" of efficient partitions which consists of identical regular hexagons.[25] We have:

Result 2.4: Partition P is efficient if and only if it is comprised of identical regular hexagons, whose stand-alone value is minimal among all regular hexagons.

The size of hexagons in efficient partitions obviously depends upon the value of the fixed component of facility costs: the smaller the cost, the smaller are jurisdictions in an efficient partition. The size of "efficient" hexagons is explicitly derived in the Appendix.

[23] A solution to this problem exists. Indeed, the integral in (6) is continuous in l, and for $l \to \infty$ the value of a program goes to $+\infty$.

[24] Note that in the unidimensional setting, for every bounded set T, a location is central if and only if it is a median of T.

[25] See Fejes Toth (1953), Haimovich and Magnanti (1988) as well as Christaller (1933), Lösch (1954), Bollobas and Stern (1972), and Stern (1972) in the economic geography context.

Before turning to the main results of the paper, let us examine pricing rules. In every potential jurisdiction $S \in \mathcal{M}(X)$, a pricing rule y is a measurable function on S that specifies individual contributions of members of S towards the cost of a facility, $f(S)$, if this jurisdiction forms. We impose the following budget-balancedness condition.

Assumption A.5 — Budget balancedness: The total contribution of members of S covers the cost of the facility:

$$\int_S y(t)dt = f(S). \tag{9}$$

When we turn to examination of efficient partitions, it would be useful to consider the notion of *consistent pricing rule*. Since the whole plane is partitioned into identical (hexagonal) jurisdictions, it makes sense to demand that the individuals in identical locations within different jurisdictions face the same prices. We impose a weak form of consistency that requires that any two individuals in any two different jurisdictions, whose location is identical with respect to their corresponding central points, make the same monetary contribution towards their facilities cost.[26]

Assumption A.6 — Consistent sharing rule: For every efficient partition P^*, every two different (hexagonal) jurisdictions $H, H' \in P^*$ and every two individuals $t \in H, t' \in H'$ satisfying $t - m(S) = t' - m(S')$, we have $y(t) = y(t')$.

The pricing rule y associated with partition P^* determines the following cost allocation c for any individual $t \in X$

$$c(t) = y(t) + ||t - m(H^t)||, \tag{10}$$

where $H^t \in P^*$ is the hexagon in P^* that contains t, and $m(H^t)$ is its center.

From now on, we fix one of the (fully equivalent to each other) efficient partitions, name it P^*. We now impose to the pricing rule to be chosen in such a way that no other firm (having access to the same technologies that the firm that we consider) can profitably enter into that market and captures a segment of customers by offering a pricing rule that this group of customers would find more attractive than the one offered by the firm. To prevent this, it must be the case that for any

[26] This assumption simplifies calculus of the proof, while it is not essential for the main result.

possible subset of customers, the aggregate cost imposed to that group through the pricing rule of the firm should not exceed the cost incurred by that group if it had acted on its own. This is exactly the Faulhaber's definition of a subsidy-free pricing rule adjusted to our setting as we had first to construct the cost function of the firm from its technology while he was considering directly the cost function. Formally,

Definition 2.5: Let a cost allocation c be given. A set $S \in \mathcal{M}(X)$ is *prone to entry capture* if

$$c(S) = \frac{1}{\lambda(S)} \int_S c(t)dt > K(S). \tag{11}$$

A cost allocation c is *subsidy-free* if no set $S \in \mathcal{M}(X)$ is prone to entry capture. The set of subsidy-free cost allocations on X will be denoted by $\mathcal{A}$.

Next definition introduces the allocations that satisfy the *Rawlsian principle* by minimizing the total cost of the most disadvantaged customer in each jurisdiction. It implies the *cost equalization* across the entire population:

Definition 2.6: A cost allocation r is called *Rawlsian* if the value $r(t)$ is constant within each $H \in P^*$, and, hence, on X. That is, for every $t, t' \in X$ we have $r(t) = r(t')$.

3 Results

We are now in position to state the main results of the paper. First, we demonstrate that under our assumptions, a subsidy-free allocation fails to exist.

Proposition 3.1: Suppose that assumptions A.1-A.6 hold. Then the set of subsidy-free allocations $\mathcal{A}$ is empty.

In absence of subsidy-free allocations, we will turn to the search for a solution which is the "closest" to be subsidy-free. For instance, we may assume that there is a fixed per capita cost of entry by any potential entrant; alternatively, in the case where the firm is public, one can consider government intervention which subsidizes a certain fraction of a total cost to every customer to prevent any entry. Both approaches are essentially equivalent and yield the following definition of δ-*subsidy-freeness*:

Definition 3.2: Let $\delta > 0$ be given. A cost allocation c is δ-*subsidy-free* if for all $S \in \mathcal{M}(X)$ the following inequality holds:

$$(1-\delta)c(S) \leq K(S). \tag{12}$$

The set of δ-subsidy-free allocations on X will be denoted by $\mathcal{A}(\delta)$.

In other words, if individuals follow the prescribed agreement, then the δ-part of their total cost is covered "from outside". If, however, a jurisdiction decides to go for another firm making alernative offers, then this deal is no longer valid.

This definition relaxes the constraints which determine subsidy-free allocations and, obviously, if δ is large enough then the set $\mathcal{A}(\delta)$ is nonempty. Moreover, if $\mathcal{A}(\delta)$ is nonempty for some δ, it is also the case for all $\delta' > \delta$. This allows us to derive the threshold value δ^* defined by

$$\delta^* = \inf\{\delta > 0 |\ \mathcal{A}(\delta) \neq \emptyset\}. \tag{13}$$

It will be shown that the set $\mathcal{A}(\delta^*)$ is itself nonempty. The value δ^* therefore can represent the *cost of stability*, which is the minimal per-capita subsidy which sustains subsidy-freeness. We can now state our main result:

Proposition 3.3: Under Assumptions A.1-A.6,

(i) $\delta^* \approx 0.0019$;

(ii) The set $\mathcal{A}(\delta^*)$ is a singleton, containing only the Rawlsian allocation.

That is, the cost of stability δ^* is very small. Moreover, the only δ^*-subsidy-free allocation is Rawlsian.

The statement of this proposition requires an explanation. Consider a hexagon H, which is an element of an efficient partition. Obviously, this hexagon is not optimal in terms of per capita cost of its members and the value $K(H)$ exceeds

$$\min_{S \in \mathcal{M}(X)} K(S). \tag{14}$$

In fact, no hexagon represents a solution for (14). Unsurprisingly, jurisdictions with the minimal per capita total cost are balls. Denote by $K(B)$ the value of the problem in (14). We then show that the cost of stability δ^* is given by

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \tag{15}$$

which, since $K(B) < K(H)$, is obviously positive. Thus, the cost gap between an efficient hexagon and an optimal ball necessitates the government intervention and subsidization of efficient partitions. It is important to point out that this feature does not appear in the uni-dimensional setting where efficient and optimal jurisdictions are intervals of the same size and the cost of stability is equal to zero (see Drèze, Le Breton and Weber (2005)).

4 Appendix

Proof of Lemma 2.2: Let $S \in \mathcal{M}(X)$ be given and assume that S has two different central points, m and m'. Let L be the straight line connecting m and m' and denote $S' = S \setminus L$ and $\bar{m} = \dfrac{m+m'}{2}$. Obviously then m and m' are central points of S' as well and $D(S) = D(S')$. Then for every $t \in S'$ we have

$$\frac{1}{2}\left(||t-m|| + ||t-m'||\right) > ||t-\bar{m}|| \tag{16}$$

and, since $\lambda(S) = \lambda(S') > 0$, this implies that

$$\int_{S'} ||t-\bar{m}||dt < \frac{1}{2}\left(\int_{S'} ||t-m||dt + \int_{S'} ||t-m'||dt\right). \tag{17}$$

However, by (6), the right-hand side of (17) is equal to $D(S) = D(S')$, a contradiction to m and m' being central points of S'. $\square$

Before proceeding with the proof of Propositions 3.1 and 3.3, we need some notation and preliminary results. From now on we shall assume, without loss of generality, that the variable component of facility costs α in Assumption A.4 is equal to zero.

Denote by B_a^l the ball with the center at $a \in X$ and the radius $l > 0$.

Lemma A.1: A set S is a solution of (14) if and only if $S = B_a^{l^*}$, where $a \in X$ and the value of l^* is given by

$$l^* = \left(\frac{3g}{\pi}\right)^{\frac{1}{3}} \approx 0.985 g^{\frac{1}{3}} \tag{18}$$

Moreover, the per capita cost in such a ball, $K(B)$, is equal to l^*.

From now on, the ball with the optimal size l^* and the center a will be referred to as simply B_a. Sometimes, if the center of the ball is not important, we will use the notation B^l for $l > 0$.

Proof: Take a set S that solves (14). Denote the ball of radius l with the center at $m(S)$ by B^l. There trivially exist l_1, l_2 with $0 \leq l_1 \leq l_2 < \infty$ such that, both $B^{l_1} \setminus S$ and $S \setminus B^{l_2}$ are null-sets, and two sets, $B^l \setminus S$ and $S \setminus B^l$ have a positive measure for all $l \in (l_1, l_2)$. We claim that l_1 and l_2 coincide, i.e., $S = B^{l_1} = B^{l_2}$.

Indeed, if not, take $l_3 = (2l_1 + l_2)/3$ and $l_4 = (l_1 + 2l_2)/3$. Then both $\lambda(S \setminus B^{l_4})$ and $\lambda(B^{l_3} \setminus S)$ are positive numbers. We can shift a positive mass of individuals from $S \setminus B^{l_4}$ to $B^{l_3} \setminus S$ so that the newly created set $\tilde{S}$ has the same measure as S. However,

$$D(\tilde{S}) = \int\limits_{\tilde{S}} ||p - m(\tilde{S})||dp \leq \int\limits_{\tilde{S}} ||p - m(S)||dp < D(S), \tag{19}$$

a contradiction to S being a solution of (14).

It is left to derive l^* and $K(B)$. Notice that for every ball B^l, the total access cost $D(B^l) = \dfrac{2\pi l^3}{3}$. Since the area of B^l is πl^2, the average cost within B^l is $K(B^l) = \dfrac{g}{\pi l^2} + \dfrac{2l}{3}$. It is straightforward to verify that the last expression attains its minimum at $l^* = \left(\dfrac{3g}{\pi}\right)^{\frac{1}{3}}$, yielding the minimal average cost $K(B) = l^*$.$\square$

We will also utilize the lemma that evaluates the average cost of jurisdictions that are "close" to optimal balls:

Lemma A.2: Let $\gamma > 0$ and set S is located between two balls with the same center, $B_a^{l^*-\gamma}$ and $B_a^{l^*}$, i.e. $B_a^{l^*-\gamma} \subset S \subset B_a^{l^*}$. Then $K(S)$, the aggregate average cost over S, differs from the aggregate average cost over optimal ball $K(B)$ only in a second order term:

$$K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{20}$$

Proof: Let $\tilde{S} \subset B_a^{l^*} \setminus B_a^{l^*-\gamma}$. In our estimations below we take into account three observations: the optimal l^* satisfies $g = \frac{\pi l^{*3}}{3}$, the total transportation cost within S increases(at least, do not decrease) if we replace the $m(S)$ by a, and that the distance between any point in $\tilde{S}$ to a is bounded from above by l^*. Denote $z = \frac{3}{\pi}\lambda(\tilde{S})$. We have:

$$K(S) = \frac{g + D(S)}{\lambda(S)} \leq \frac{g + \int\limits_S ||a-t||dt}{\lambda(S)} \leq \frac{g + D(B_a^{l^*-\gamma}) + l^*\lambda(\tilde{S})}{\lambda(B_a^{l^*-\gamma}) + \lambda(\tilde{S})} = \frac{(l^*)^3 + 2(l^*-\gamma)^3 + zl^*}{3(l^*-\gamma)^2 + z} <$$

$$\frac{3(l^*)^3 - 6(l^*)^2\gamma + 6l^*\gamma^2 + zl^*}{3(l^*-\gamma)^2 + z} = \frac{3l^*(l^*-\gamma)^2 + zl^*}{3(l^*-\gamma)^2 + z} + \frac{3l^*\gamma^2}{3(l^*-\gamma)^2 + z} =$$

$$l^* + \frac{3l^*\gamma^2}{(l^*-\gamma)^2 + z/3} \leq l^* + \frac{3l^*\gamma^2}{\left(\sqrt{3}l^*\right)^2/2^2} = l^* + \frac{4}{l^*}\gamma^2, \tag{21}$$

as for γ small enough we have $l^* - \gamma > \frac{\sqrt{3}}{2}l^*$.$\square$

Lemma A.3: Let H be a hexagon in an efficient partition. Then the per capita cost in H is given by

$$K(H) = \frac{\sqrt{3}}{2}\left(\frac{2}{3} + \ln\sqrt{3}\right)^{\frac{2}{3}} g^{\frac{1}{3}} \approx g^{\frac{1}{3}}. \tag{22}$$

Proof: Consider a regular hexagon H_l, where l denotes the distance between the center $m(H_l)$ and a midpoint of its side. The total access cost in H_l is

$$\begin{aligned} D(H_l) &= 12\int_0^l\int_0^{\frac{x}{\sqrt{3}}}\sqrt{x^2+y^2}dxdy = 6\int_0^l\left[y\sqrt{x^2+y^2} + x^2\ln\left(y+\sqrt{x^2+y^2}\right)\right]_0^{\frac{x}{\sqrt{3}}}dx \\ &= 6\int_0^l\left[\tfrac{x}{\sqrt{3}}\sqrt{x^2+\tfrac{x^2}{3}} + x^2\ln\left(\tfrac{x}{\sqrt{3}} + \sqrt{x^2+\tfrac{x^2}{3}}\right) - x^2\ln x\right]dx \\ &= 6\int_0^l x^2\left[\tfrac{2}{3} + \ln\sqrt{3}\right]dx = 2l^3\left[\tfrac{2}{3} + \ln\sqrt{3}\right]. \end{aligned} \tag{23}$$

Since the area of H_l is $2\sqrt{3}l^2$, the average cost per citizen in jurisdiction H_l is given by

$$K(H_l) = \frac{g}{2\sqrt{3}l^2} + \frac{l}{\sqrt{3}}\left[\frac{2}{3} + \ln\sqrt{3}\right], \tag{24}$$

which attains its minimum at the efficient hexagon H, i.e., when

$$l = \tilde{l} = \left(\frac{2}{3} + \ln\sqrt{3}\right)^{-\frac{1}{3}} g^{\frac{1}{3}}. \tag{25}$$

It is easy to verify that then the per capita average cost $K(H) = K(H_{\tilde{l}})$ is given by (22) which at the same time represents the average cost of the whole plane X under an efficient partition. □

Take the efficient partition P^* of X. For every positive integer N, consider a subset G_N of P^* that consists of N^2 adjacent hexagons (see Figure 1). Let the sequence $\{G_N\}_{N=1,\ldots,\infty}$ be nested, i.e., each G_N is imbedded into G_{N+2} "symmetrically", such that the set $G_{N+2} \setminus G_N$ is a "hexagonal ring" comprised of $4N+4$ regular hexagons. We have the following result:

Lemma A.4: For every $a \in G_N$, the ball B_a is contained in G_{N+2}.

Proof: Denote by $\bar{l}$ the side of a hexagon in partition P^*. Since the minimal width of the hexagonal ring F_N is equal to $\bar{l}$, it suffice to demonstrate that $\bar{l} > l^*$. Note that $\bar{l} = \frac{2}{\sqrt{3}}\tilde{l}$, where $\tilde{l}$ is the distance between the center of the efficient hexagon and the middle point of one of its sides, which has been derived in (25). Thus,

$$\bar{l} = \frac{2}{\sqrt{3}} \left(\frac{2}{3} + \ln\sqrt{3}\right)^{-\frac{1}{3}} g^{\frac{1}{3}}, \tag{26}$$

which, by (18), exceeds the value l^*. □

Let the efficient partition P^* be endowed with the sharing rule y, that generates cost allocation c, and H is a (hexagonal) jurisdiction in P^*. Denote by λ^H the Lebesgue measure of H and by λ^B the Lebesgue measure of an optimal ball.

For every $a \in X$ denote by the value $\varphi(a)$ the aggregated cost incurred by the members of the ball B_a:

$$\varphi(a) = c(B_a) = \int_{B_a} c(t)dt \tag{27}$$

Define $\bar{\varphi}$ as the aggregated cost incurred by the allocation c on all balls of optimal size whose centers belong to the hexagon H:

$$\bar{\varphi} := \int_H \varphi(a)da. \tag{28}$$

Note that, due to the consistency assumption A.6, the value $\bar{\varphi}$ is invariant to a choice of a hexagon in P^*. We need the following result:

Lemma A.5:

$$\bar{\varphi} = I, \quad \text{where} \;\; I := \lambda^B \int_H c(t)dt. \tag{29}$$

Proof: Define the function $\Psi(a,t)$ on $G_N \times G_{N+2} \subset \Re^4$ by

$$\Psi(a,t) = \begin{cases} c(t), & \text{if} \quad t \in B_a; \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

We will integrate the function $\Psi(a,t)$ over the set $G_N \times G_{N+2}$. According to Fubini's theorem (Halmos (1950), p.148), two different orders of integration yield the same result. First, we integrate with respect to t and then to a. By (27) and (28) we have

$$\int_{G_N} \left[\int_{G_{N+2}} \Psi(a,t)dt \right] da = \int_{G_N} \left[\int_{B_a} c(t)dt \right] da = \int_{G_N} \varphi(a)da = N^2 \int_H \varphi(a)da = N^2 \bar{\varphi}. \tag{31}$$

Before integrating in the reverse order, note that the following duality property

$$\{a | t \in B_a\} \equiv B_t \tag{32}$$

holds for every $t \in X$. This is a simple consequence of the symmetry of the distance $||t-p||$ as a function of two arguments, and the circle B_t being the set of points p for which $||p-t|| = ||t-p|| \leq l^*$. Take a point $t \in G_{N-2}$. By Lemma A.4, $B_t \subset G_N$, and

$$\int_{G_N} \Psi(a,t)da = \int_{B_t} c(t)da = c(t) \int_{B_t} da = \lambda^B c(t). \tag{33}$$

We have:

$$\int_{G_{N+2}} \left[\int_{G_N} \Phi(a,t)da \right] dt = \int_{G_{N-2}} \left[\int_{G_N} \Phi(a,t)da \right] dt + L_N, \tag{34}$$

where

$$L_N := \int\limits_{G_{N+2}\setminus G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right]. \tag{35}$$

By using (32), (33) and Lemma A.4, the first term in (34) can be presented as:

$$\int\limits_{G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt = \int\limits_{G_{N-2}} \lambda^B c(t)dt = (N-2)^2 I. \tag{36}$$

Fubini's theorem allows us to rewrite (34) as

$$N^2\bar{\varphi} = (N-2)^2 I + L_N = N^2 I + L_N - 4(N-1)I. \tag{37}$$

Let us estimate the absolute value of the last two terms. Since for any $t \in G_{N+2}$, hence, for any $t \in G_{N+2} \setminus G_{N-2}$, we have that $\int\limits_{G_N} \Phi(a,t)da = \int\limits_{G_N \bigcap B_t} c(t)da \leq \int\limits_{B_t} c(t)da = \lambda^B c(t)$, it follows that

$$|L_N - 4(N-1)I| \leq |L_N| + 4(N-1)I \leq 4(N-1)I + \int\limits_{G_{N+2}\setminus G_{N-2}} \lambda^B c(t)dt = (12N-4)I < 12NI. \tag{38}$$

Thus,

$$|N^2\bar{\varphi} - N^2 I| \leq 12NI, \quad \text{or} \quad |\bar{\varphi} - I| \leq \frac{12I}{N}. \tag{39}$$

Since N can be made arbitrarily large, it immediately yields the desired equality $\bar{\varphi} = I$.□

Proof of Proposition 3.1: It is a corollary of Proposition 3.3.

Proof of Proposition 3.3: Let us show first that

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \tag{40}$$

which, by Lemmas A.1 and A.3, can be calculated as

$$\delta^* = 1 - \frac{2}{\pi^{\frac{1}{3}} 3^{\frac{1}{6}} (\frac{2}{3} + \ln\sqrt{3})^{\frac{2}{3}}} \approx 0.0019. \tag{41}$$

We will demonstrate that the set of δ-subsidy-free allocations is empty if and only if $\delta < \delta^*$.

Consider a δ-subsidy-free allocation c. The budget balancedness assumption A.5 implies that the value of I, determined by (29), is equal to $\lambda^B\lambda^H K(H)$, and by Lemma A.4. so is the value of $\bar{\varphi}$.

Hence, there exists $a \in H$ such that $\varphi(a) \geq \lambda^B K(H)$. On the other hand, the stand alone aggregate cost in B_a is $\lambda^B K(B)$. Since c is δ-subsidy-free, Definition 3.2 implies that $(1-\delta)\lambda^B K(H) \leq \lambda^B K(B)$, or $\delta \geq 1 - \frac{K(B)}{K(H)}$.

Let us show that Rawlsian allocation is δ-subsidy-free whenever $\delta \geq \delta^*$. Indeed, since $r(t) = K(H)$ for every $t \in X$, then for $S = B^{l^*}$ — an optimal ball we observe that $(1-\delta)K(H) \leq K(B)$. Now, for any $S \in \mathcal{M}(X)$ we have $K(S) \geq K(B)$ and therefore $(1-\delta)K(H) \leq K(B) \leq K(S)$.

To complete the proof of the proposition, it remains to demonstrate that the Rawlsian allocation is the only one to be δ^*-subsidy-free. That is, the allocation that assigns every individual in X a contribution $K(H)$ is the only δ^*-subsidy-free. For this end, consider an arbitrary δ^*-subsidy-free allocation c and estimate the number of individuals whose contribution is "substantially" below the level $K(H)$.

Take a positive number $\varepsilon > 0$. Consider first an arbitrary ring $B_a \setminus B_a^{l^*-\gamma}$ and evaluate the measure of individuals t whose cost contribution $c(t)$ satisfies $c(t) < K(H) - \varepsilon$. Denote this set by U, and consider the set $S = B_a \setminus U$, for which, by Lemma A.2, we have $K(S) < l^* + \frac{4}{l^*}\gamma^2$. On the other hand,

$$c(S) = c(B^{l^*}) - c(U) \geq \lambda^B K(H) - \lambda(U)K(H) + \lambda(U)\varepsilon = \lambda(S)K(H) + \lambda(U)\varepsilon. \tag{42}$$

The δ^*-subsidy-free of c implies that the average per capita contribution in group S, adjusted by $1-\delta^*$, does not exceed its stand-alone value, $K(S)$:

$$(1-\delta^*)\frac{c(S)}{\lambda(S)} = (1-\delta^*)(K(H) + \frac{\lambda(U)}{\lambda(S)}\varepsilon) \leq K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{43}$$

Since $K(B) = l^* = (1-\delta^*)K(H)$, we have:

$$\lambda(U) \leq \frac{4\lambda(S)}{l^*(1-\delta^*)\varepsilon}\gamma^2 < \frac{4\pi(l^*)^2}{l^*(1-\delta^*)\varepsilon}\gamma^2 = W\gamma^2, \tag{44}$$

where W is a constant which is independent of γ.

Now consider the square Q with a side of $2\pi l^*$ with the center at the origin. For any small positive number γ, denote by $R[i,\gamma]$ the ring $B_{p_i} \setminus B_{p_i}^{l^*-\gamma}$ centered at the point $p_i = (i\gamma, 0)$, where i is any (positive or negative) integer. For large enough positive integer N we have the following inclusion:

$$Q \subset \bigcup_{i=-N}^{N} R\left[i, \frac{\pi l^*}{N}\right]. \tag{45}$$

Denote by U and $U_i, i = -N, \ldots, -1, 0, 1, \ldots, N$, the sets of individuals in Q and $R\left[i, \frac{\pi l^*}{N}\right]$, respectively, who contribute less than $K(H) - \varepsilon$ under the allocation c. By utilizing (44), we have $\lambda(U_i) \leq W\frac{(\pi l^*)^2}{N^2}$. Thus, since $U \subset \bigcup_{i=-N}^{N} U_i$, we have

$$\lambda(U) \leq (2N+1)W\frac{(\pi l^*)^2}{N^2} < \frac{3}{N}W(\pi l^*)^2. \tag{46}$$

Since N can be chosen arbitrarily large, (46) implies that $\lambda(U) = 0$. Note that this argument actually implies that for any square with the side of $2\pi l^*$, the Lebesgue measure of the set of individuals who contribute less than $l^* - \varepsilon$ under the allocation c has the zero measure.

Finally, the set of all individuals who contribute less than $K(H)$ under c is a countable union of its subsets, indexed by (n, i, j), each of which is the set of individuals contributing less than $K(H) - 1/n$ in the square with the side of $2\pi l^*$ and the center at $(2\pi i, 2\pi j)$. Since each of these subsets has zero measure, so does their union as well, hence, every individual contributes at least $K(H)$. Finally, due to the budget balancedness condition A.5, the set of those who contribute more than $K(H)$ has the zero measure as well. Thus, every $t \in X$ contributes $K(H)$, implying that the only δ^*-subsidy-free allocation is Rawlsian.□

5 References

Alesina, A. and E. Spolaore 1997 "On the number and size of nations", *Quarterly Journal of Economics*, 113, 1027-1056.

Analysis of whether there are significant cross-subsidies between the different customer groups served by Scottish water, a final report for the Scottish Executive, Scotthish Executive, 2005.

Baumol, W.J., Bailey, E.E. and R.D. Willig 1977 "Weak invisible hands theorems on the sustainability of multiproduct monopoly" *American Economic Review*, 67, 350-365.

Baumol, W.J., Panzar, J.C. and R.D. Willig 1988 *Contestable Markets and the Theory of Industry Structure*, Harcourt Brace Jovanovich Publishers, San Diego.

Beckmann, M. 1968 *Location Theory*, Random House, New York

Beckmann, M. 1976 "Spatial price policies revisited", *Bell Journal of Economics*, 7, 619-630.

Bollobas, B. and N. Stern 1972 "The optimal structure of market areas", *Journal of Economic Theory* 4, 174-179.

Bonbright, J.C. 1961 "Fully distributed costs in utility rate making", *American Economic Review, Papers and Proceedings*, 51, 305-312.

Braueutigam, R.R. 1980 "An analysis of fully distributed cost pricing in regulated industries", *Bell Journal of Economics*, 11, 182-196.

Brown, S. and D. Sibley 1986, *The Theory of Public Utility Pricing*, Cambridge University Press, Cambridge.

Calum, G. 2003 "On intertemporal subsidy-free prices and economic depreciaition : constrained market pricing revisited", *Journal of Regulatory Economics*, 24, 135-159.

Drèze, J. , Le Breton, M., and S. Weber 2006 "Rawlsian pricing of access to public facilities: a unidimensional illustration", *Journal of Economic Theory*, forthcoming.

Faulhaber, G.R. 1975 "Cross-subsidization pricing in public enterprise pricing" in *Pricing in Regulated Industries, Theory and Applications II*, J.T. Wenders (Ed), Denver.

Faulhaber, G.R. 1979 "Cross-subsidization in public enterprise", *American Economic Review*, 65, 966-977.

Faulhaber, G.R. and S.B. Levinson 1981 "Subsidy-free prices and anonymous equity", *American Economic Review*, 71, 1083-1091.

Faulhaber, G.R. 2005 "Cross-subsidy analysis with more than two services", *Journal of Competition, Law and Economics*, 1, 441-448.

Greenhut, J. and M.L. Greenhut 1977 "Nonlinearity of delivered price schedules and predatory pricing", *Econometrica*, 45,1871-1875.

Haimovich, M. and T. L. Magnanti 1988 "Extremum properties of hexagonal partitions and the uniform distribution in Euclidean location", *Siam Journal of Discrete Mathematics* 1(1), 50-64.

Hales, T.C. 2000 "Canonballs and honeycombs", *Notices of the American Mathematical Society*, 47, 440-449.

Hales, T.C. 2001 "The honeycomb conjecture", *Discrete and Computational Geometry*, 25, 1-22.

Halmos, P. 1950 *Measure Theory*, Van Nostrand.

Jamison, M.A. 1996 "General conditions for subsidy-free prices", *Journal of Economics and Business*, 48, 371-385.

Kaserman, D.L. and J.W. Mayo 1994 "Cross-subsidies in telecommunications : roadblocks on the road to more intelligent telephone pricing", *Yale Journal of Regulation*, 11, 120-147.

Kaserman, D.L., J.W. Mayo and J.E. Flynn 1990 "Cross-subsidization in telecommunications : beyond the universal service fairy tale pricing", *Journal of Regulatory Economics*, 2, 231-250.

Haimanko, O., Le Breton, M. and S. Weber 2003 "Voluntary formation of communities for the provision of public projects", *Journal of Economic Theory*, 115, 1-34.

Le Breton, M. and S. Weber 2003 "The art of making everybody happy : how to prevent a secession ?", *IMF Staff Papers*, 50, 403-435.

Mirman, J.J., Tauman, Y. and I. Zang 1985 "Supportability, sustainability and subsidy-free prices", *Rand Journal of Economics*, 16, 114-126.

Morgan, F. 1999 "The hexagonal honeycomb conjecture", *Transactions of the American Mathematical Society*, 351, 1753-1763.

Morgan, F. and R. Bolton 2002 "Hexagonal economic regions solve the location problem", *American Mathematical Monthly*, 109, 165-172.

Morgan, F. 2005 "Kepler's conjecture and Hales's proof", *Notices of the American Mathematical Society*, 52, 44-47.

Ralph, E. 1992 "Cross-subsidy : a novice's guide to the arcane", Duke University, Mimeo.

Panzar, J.C. and R.D. Willig 1977 "Free entry and the sustainability of natural monopoly", *Bell Journal of Economics*, 8, 1-22.

Rosenbaum, D. 1997 *Cross-Subsidy-Free Pricing : Upper and Lower Boundaries for Utility Pricing*, American Public Power Association.

Sharkey, W.W. 1982a "Suggestions for a game-theoretic approach to public pricing and cost allocation", *Bell Journal of Economics* 13, 57-68.

Sharkey, W.W. 1982b *The Theory of Natural Monopoly*, Cambridge University Press, Cambridge.

Sharkey, W.W. and L.G. Telser 1978 "Supportable cost functions for the multiproduct firm", *Journal of Economic Theory*, 18, 23-37.

Smithies, A. 1941 "Monopolistic price policy in a spatial market", *Econometrica*, 9, 63-73.

Spulber, D.F. 1981 "Spatial nonlinear pricing", *American Economic Review*, 71, 923-933.

Thue, A. 1892 "Om nogle geometrisk taltheorietiske theoremer", *Forandlingerneved de Skandinaviske Naturforskeres,* 14, 352-353.

Thue, A. 1910 "Über die dichteste zusammenstellung von kongruenten kreisen in der ebene", *Christina Vid Selsk Skr*, 1, 1-9.

Willig, R.D. 1979 "Customer equity and local measured service" in *Perspectives on Local Measured Service*, J.A. Baude (Ed), Kansas City.

Zajac, E.E. 1978 *Fairness and Efficiency : An Introduction to Public Utility Pricing*, Ballinger Publishing Company, Cambridge.