Secession-Proofness in Large Heterogeneous Societies*

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March 2006

Abstract

This paper examines the model of multi-jurisdictional formation considered by Alesina and Spolaore (1997) and Le Breton and Weber (2003), where the distribution of individuals is given by Lebesgue measure over a bounded interval. Every jurisdiction chooses a location of a public good and shares its cost of production among its residents. In addition, each individual incurs a *transportation cost*. We consider a notion of *secession-proof allocation* where no group of individuals can make all its members better off by choosing both a location of the public good and a cost-sharing mechanism among its own members. We examine secession-proof allocations and show that they may fail to satisfy some desirable requirements, including the Rawlsian principle. We show however that secession-proofness can be reconciled with an approximate Rawlsian principle in large societies.

Keywords: Optimal jurisdictions, Secession-proofness, Rawlsian allocations, Efficiency.

JEL Classification Numbers: D70, H20, H73.

*The analysis conducted in this paper elaborates on Drèze, Le Breton and Weber (2006) who explore the limit case of an unbounded society.

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1 Introduction

In this paper we consider a large heterogeneous society whose members make selections from the given policy space. The selection may consist of a unique policy wherein the society remains undivided. However, if more than one policy is chosen, it creates a partition of the entire society into smaller subsocieties; called hereafter *jurisdictions*, each one associated with a chosen policy. The basic reason for a possible partition of the society into smaller groups is the conflict between increasing returns to scale on one hand, and heterogeneity of agents' preferences on the other. Indeed, in a large group the per capita contributions towards financing of a public project could be lower than in a smaller group. However, the policy choices made by a large group could be quite distant from the ideal choices of some of its members. Thus, the benefits of group size are not unlimited, and the increasing returns could be outweighed by costs of heterogeneity of agents' characteristics and tastes. It is possible, therefore, that a decentralized organization might be superior to the grand coalition that embraces the entire society.

The policy choices of the society and the composition of the formed jurisdictions do not provide a complete description of the problem we consider. Since the policies are costly in our framework, one has to identify a mechanism of sharing the costs within each jurisdiction. Thus, when a set of appropriate policies has been chosen and a partition of individuals has formed, the cost allocation among individuals must be determined. When all three elements of the collective choice problem (namely projects' choice, assignment to the projects, and cost allocation) are in place, one can turn to a stability test of the proposed arrangement. There could be a single individual or a group of individuals (not necessarily from the same jurisdiction) who object to the proposed arrangement. The rejection is credible if there is a policy with concomitant allocation of its costs among the members of the deviating group that would make each of them better off with respect to the original arrangement. This group therefore poses a threat of "secession." Hence, the set of cost allocations that do not permit credible rejections is called "secession-proof." Accordingly, the main focus of this paper is analysis of existence and characterization of the set of secession-proof cost allocations.

We examine a society of individuals who make policy choices from the unidimensional policy space. The cost of each policy is given by the positive parameter g. The preferences of each individual are single-peaked and symmetric with respect to her peak, which represents her favorite policy choice. The heterogeneity of individuals' preferences is described by the distribution of the peaks over the real line. Alesina and Spolaore (1997) study this problem in the case where the distribution λ is the Lebesgue measure over the unit interval. In their framework, the only cost allocation available for each jurisdiction is the *equal-share*, when all individuals within the same jurisdiction make an identical contribution towards the policy costs. Le Breton and Weber (2003) examine a class of absolutely continuous distributions, and demonstrate the existence of secession-proof cost allocations for high values of policy costs g that make it optimal to form a single jurisdiction. They establish a principle of partial equalization asserting that, in general, secession-proofness entails some, but not full, compensation of citizens disadvantaged by the assigned public policy. In particular, it could be the case that both equal-share allocation (no equalization) and Rawlsian allocation (full equalization) are not secession-proof. Haimanko, Le Breton and Weber (2004) consider an arbitrary probability measure with bounded support, and establish the existence of secession-proof cost allocations regardless of the value of policy costs g.

In this paper we consider distributions of individuals' ideal points by the Lebesgue measure on the large bounded interval $[-\theta, \theta]$. We examine the crucial role played under these circumstances by the Rawlsian principle under which the society maximizes the utility level of the most disadvantaged individual. In our framework, the Rawlsian principle implies the full equalization of all individuals' utilities, and, therefore, entails the full compensation to every individual for being assigned to the policy different from her favored one.

The implications of the Rawlsian principle in the society where the population is distributed over the entire real line, i.e. $\theta = \infty$, have been studied by Drèze, Le Breton and Weber (2006). Their main result asserts that the *unique* secession-proof allocation is, in fact, Rawlsian, and the only mechanism to guarantee stability in this setting societies is the full equalization of utilities of all individuals. In this paper, we show that this result cannot be extended to societies whose population distribution has a finite support, i.e., $\theta < \infty$, and the Rawlsian principle and secession-proofness are, in general, inconsistent. In an attempt to bridge the chasm between two requirements, we relax the Rawlsian principle by considering instead a mild "privilege-free" condition that rules out granting excessive "privileges" to some individuals, and, hence, imposes the additional cost burden of public policies on the rest of the society. However, even this mild requirement, which holds for all Rawlsian allocations, is shown to be incompatible with secession-proofness. We then introduce the notion of approximate Rawlsian allocation, where the proportion of individuals receiving a privilege of the given magnitude (relative to the Rawlsian allocation), is sufficiently small. We show that in large finite societies the approximate Rawlsian principle is consistent with secession-proofness.

We also examine the basic differences between the multi-jurisdictional and one-jurisdictional settings. In particular, we show that the single-jurisdictional version of the Le Breton and Weber (2003) result, which yields the existence of secession-proof cost allocations that satisfy the principle of partial equalization, does not always hold in the multi-jurisdictional framework.

The paper is organized as follows. In the next section we present the model and analyze the efficient partitions of the entire society. In Section 3 we introduce the notion of a cost allocation, the principle of partial equalization, and define intra- and inter-jurisdictional symmetric and Rawlsian allocations. In Section 4, we introduce the notion of secession-proofness and show that the Rawlsian principle and secession-proofness are, in general, incompatible. In Section 5 we demonstrate that even a relaxation of the Rawlsian principle does not reconcile it with secession-proofness. We show, however, that in a large finite society, secession-proofness yields cost allocations that satisfy the approximate Rawlsian principle. The proofs of all results are relegated to the Appendix.

2 The Model and Efficient Partitions

We consider a group of individuals and the set of feasible policies represented by the real line $\Re$. Every individual has an ideal policy in $\Re$ and her preferences over policies are single-peaked and symmetric with respect to the ideal point. This allows us to identify an individual with her ideal point and to represent the society by the distribution of individuals' ideal points, which is given by the Lebesgue measure λ over the interval $R_\theta = [-\theta, \theta]$, where θ is a positive real number. The group choice consists of three elements:

- A partition of R_θ into measurable subsets with positive measure, called *jurisdictions*;
- A policy in R_θ for each jurisdiction;
- A cost-sharing scheme among individuals, where the sum of individual contributions covers the total cost of public projects.

For illustrative purposes, we may interpret this problem as follows: an urban population located on the all real line is faced with a selection of a number and location of public projects (say, libraries), an assignment of each individual to one of the projects, and, finally, an allocation of projects costs to society members. We assume that the cost of every project is a positive number g. We therefore adopt a spatial interpretation of R_θ, where a policy choice is represented by the location of the public project. That allows us to introduce the disutility or transportation cost incurred by individuals. An individual located in t (labelled t henceforth) faces the transportation cost $d(t,p)$ between t and the location p of the public project she is assigned to. We assume that transportation cost is simply represented by the distance, i.e., $d(t,p) = |t-p|$.

For any bounded measurable subset $S \subset \Re$, denote by

$$D(S) = \inf_{p \in I} \int_S d(t,p)dt,$$

the minimal aggregate transportation cost of the individuals in S. It is useful to make the following observation:

Remark 2.1: Every jurisdiction S would minimize its total transportation cost by choosing the location of the public project at the ideal point of its "median voter". That is, the median of S, denoted $m(S)$, is the cost-minimizing location for S. Thus, if the group S is an interval of length s, its total transportation cost $D(S)$ is given by

$$\int_S \mid t - m(S) \mid dt = \frac{s^2}{4}. \qquad (1)$$

At this point we do not examine how the monetary contribution of each individual t towards the cost of the public projects is determined. The only condition we require is the *balanced budget* that requires the society to finance all chosen public projects.

Since the total cost incurred by jurisdiction S consists of policy and transportation components, the average cost of an individual t in S is

$$\frac{g + D(S)}{\lambda(S)}.$$

The following remark, which plays an important role in our analysis, determines the optimal size of a jurisdiction that minimizes the per capita cost of its members:

Remark 2.2: If a jurisdiction is represented by an interval of length s, by (1), the per capita total cost of its members is

$$f(s) \equiv \frac{s}{4} + \frac{g}{s}.$$

For a given value of $g > 0$, the function $\frac{s}{4} + \frac{g}{s}$ is convex and obtains its minimum at $s^* = 2\sqrt{g}$. Therefore, the value of s^* can be viewed as an *optimal size* of a jurisdiction, and an interval of length s^* as an *optimal jurisdiction.*

For every measurable subset S of R_θ denote by $\mathcal{P}_S$ the set of all finite partitions of the set S into measurable subsets with positive measure. If $S = R_\theta$, we will simply write $\mathcal{P}$.

Definition 2.4: The partition $P \in \mathcal{P}$ is *efficient* if for every partition P', we have

$$\sum_{S \in P'} [D(S + g] \leq \sum_{S \in P'} [D(S) + g].$$

An efficient partition simply minimizes the aggregate sum of transportation and policy costs over all partitions in $\mathcal{P}$. The main result of this section guarantees that, in general, there is a unique efficient jurisdictional structure which consists of jurisdictions of equal size.

Proposition 2.5: For every θ, every efficient partition consists of intervals of equal size. Moreover, the optimal number[1] of jurisdictions $N(R_\theta)$ is given by[2]

$$N(R_\theta) = \begin{cases} \lfloor \frac{2\theta}{s^*} \rfloor & if \quad \frac{2\theta}{s^*} \leq \sqrt{\lfloor \frac{2\theta}{s^*} \rfloor (\lfloor \frac{2\theta}{s^*} \rfloor + 1)} \\ \lfloor \frac{2\theta}{s^*} \rfloor + 1 & if \quad \frac{2\theta}{s^*} \geq \sqrt{\lfloor \frac{2\theta}{s^*} \rfloor (\lfloor \frac{2\theta}{s^*} \rfloor + 1)}. \end{cases}$$

The size of each jurisdiction in an efficient partition is $s(R_\theta) = \frac{2\theta}{N(R_\theta)}$.

Remark 2.6: It would be useful to expand the notion of an optimal number of jurisdictions to an interval of the arbitrary length L and to consider an efficient partition of such an interval into $N(L)$ equal size intervals. As in Proposition 2.5, we have:

$$N(L) = \begin{cases} \lfloor \frac{L}{s^*} \rfloor & if \quad \frac{L}{s^*} \leq \sqrt{\lfloor \frac{L}{s^*} \rfloor (\lfloor \frac{L}{s^*} \rfloor + 1)} \\ \lfloor \frac{L}{s^*} \rfloor + 1 & if \quad \frac{L}{s^*} \geq \sqrt{\lfloor \frac{L}{s^*} \rfloor (\lfloor \frac{L}{s^*} \rfloor + 1)}. \end{cases}$$

The optimal number of jurisdictions $N(R_\theta)$ is determined through the interplay of two opposite forces. The creation of a new jurisdiction reduces the aggregate transportation cost but adds an additional cost of public project g. Once we determine that all jurisdictions are intervals of the same length, one can formulate the problem in terms of minimization of the total aggregate cost with respect to the length s of a typical jurisdiction. In general, the optimal value s^* is such that $\frac{2\theta}{s^*}$ is not an integer and to find the minimum in this case, we take advantage of convexity of the total aggregate cost as a function of s. After deriving the unconstrained minimum s^* of the total aggregate cost with respect to s, it remains to examine the nearest value(s) of s on the right and on the left of s^* for which the value $\frac{2\theta}{s}$ is an integer. An evaluation of the total aggregate cost in these two points yields the optimal number of jurisdictions. In the (non generic) case where these two values of s yield the same cost, there are two optimal jurisdictional structures with a different number of jurisdictions.

The "connectedness" of all optimal jurisdictions is a general property that can be derived under more general assumptions on the individual transportation cost functions and on the distribution of individuals over R_θ. However, the fact that these intervals are of equal length, is driven by

[1]In the expression, $\lfloor \frac{2\theta}{s^*} \rfloor$ denotes the largest integer (called integer part) that does not exceed $\frac{2\theta}{s^*}$.
[2]Obviously, if $\lfloor \frac{2\theta}{s^*} \rfloor = 0$, then $N(R_\theta) = 1$.

our assumption of the uniform distribution of the population over R_θ. In the next proposition we evaluate the optimal length of jurisdictions $s(L)$ for an interval of size L and the average individual contribution in an optimal partition:

Proposition 2.7: For all positive values L we have:

$$(i) \quad 2\sqrt{g} - \tfrac{4g}{L+2\sqrt{g}} \le s(L) \le 2\sqrt{g} + \tfrac{4g}{L-2\sqrt{g}},$$
$$(ii) \quad f(s(L)) \le f(2\sqrt{g}) + \tfrac{27\sqrt{g^3}}{L^2}.$$

3 Cost Allocations

Let us introduce the notion of a *cost allocation* that determines a monetary contribution of each individual towards the cost of public projects in group S.

Definition 3.1: A measurable function x, defined on the bounded subset S is called an S-cost allocation[3] if it satisfies the budget constraint, i.e., the total contribution of all members of S, $x(S)$, is equal to the cost of the public project:

$$x(S) \equiv \int_S x(t)dt = g.$$

We allow for lump sum transfers and do not restrict the mechanism for reallocation of benefits within each potential jurisdiction S.

Definition 3.2: A measurable function x defined on R_θ is called a *P-cost allocation* if it satisfies the aggregate budget constraint:

$$\sum_{S \in P} (x(S) - g) = 0. \tag{2}$$

The results of the previous section show that efficiency leads to a well-defined jurisdictional structure, whose aggregate policy cost is to be shared among all individuals. In this section, we introduce a set of normative principles that will impose some constraints on the choice of cost allocations. These principles formulate various fairness requirements, including the Rawlsian principle and its modifications.

[3]We use the term cost allocation without qualification when $S = R_\theta$ represents the entire population.

We now define *intra-jurisdictional symmetry* and *the principle of partial equalization* of cost allocations (the latter is introduced in Le Breton and Weber (2003)). Let S be an arbitrary jurisdiction and x an S-cost allocation.

Definition 3.3 - Intra-Jurisdictional Symmetry: x satisfies intra-jurisdictional symmetry if $x(t) = x(s)$ whenever $d(t, m(S)) = d(s, m(S))$.

Definition 3.4 - Partial Equalization: x satisfies the principle of partial equalization if

$$\begin{aligned} (i) \quad 0 &\leq x(t) - x(s) \leq t - s \text{ for all } t, s \in S \text{ such that } s < t \leq m(S) \\ (ii) \quad 0 &\leq x(t) - x(s) \leq s - t \text{ for all } t, s \in S \text{ such that } s > t \geq m(S). \end{aligned}$$

Since a project supported by group S is located at its median $m(S)$, the intra-jurisdictional symmetry requires that individuals who are equidistant from the location of the public project should make an equal contribution. The principle of partial equalization suggests some form of compensation within every jurisdiction. Precisely, those closer to the project location contribute more than those who are further away. But these contribution differentials cannot exceed the transportation cost differentials and ultimately those closer to the project location would still have a (weakly) lower aggregate cost.

If the principle of partial equalization is replaced by full equalization, the appropriate cost allocation would equalize the total of transportation costs and contribution towards the cost of public project for all members of S. This allocation is called *Rawlsian* as it minimizes the highest total cost burden among all individuals in S:

Definition 3.5 - Full Equalization (Rawlsian allocation): An S-cost allocation x is called Rawlsian if it yields the *full equalization*, i.e.,

$$x(t) - x(s) = t - s \text{ for all } t, s \in S \text{ such that } s < t \leq m(S) \qquad \text{and}$$

$$x(t) - x(s) = s - t \text{ for all } t, s \in S \text{ such that } s > t \geq m(S).$$

The Rawlsian allocation will be denoted x_R^S.

The location of the public project in S at the jurisdiction's median $m(S)$ implies that the total individual contribution of citizen t in S is given by the term $d(t, m(S)) + x(t)$. Since the aggregate transportation and policy costs in S combine to $D(S) + g$, it follows that

$$x_R^S(t) = \frac{g + D(S)}{\lambda(S)} - d(t, m(S)).$$

Note that if $S \subset R_\theta$ is a connected interval $[a, b]$, the allocation x

satisfies intra-jurisdictional symmetry if $x(t) = x(b + a - t)$ for all $t \in S$;

satisfies the principle of partial equalization if two conditions hold

$$0 \leq x(t) - x(s) \leq t - s \text{ for all pairs } s, t \text{ such that } a \leq s \leq t \leq \frac{a+b}{2},$$
$$0 \leq x(t) - x(s) \leq s - t \text{ for all pairs } s, t \text{ such that } b \geq s \geq t \geq \frac{a+b}{2};$$

is Rawlsian if

$$x(t) = x_R^S(t) = \begin{cases} t + \frac{g}{b-a} - \frac{b-a}{4} & if \quad t \in \left[a, \frac{a+b}{2}\right] \\ -t + \frac{g}{b-a} + \frac{3(b-a)}{4} & if \quad t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

The notions of intra-jurisdictional symmetry, partial equalization and Rawlsian allocation are extended to a multi-jurisdictional set-up as follows. Consider a pair (P, x), where P is a partition of R_θ and x is a P-cost allocation. x is intra-jurisdictionally symmetric (respectively, Rawlsian or satisfies the principle of partial equalization) if its truncation to S is intra-jurisdictionally symmetric (respectively, Rawlsian or satisfies the principle of partial equalization) for every $S \in P$. We shall denote by x_R^P the Rawlsian allocation associated with the partition P. It should be noted that these principles do not impose any link between cost allocations in different jurisdictions and do not restrict the sharing of the total public projects costs Ng across N jurisdictions. Since x is a P-cost allocation, by (2), we have

$$\sum_{S \in P} x(S) = gN,$$

and if $x(S) \neq g$ for some S, then a cross subsidization takes place. If $x(S) < g$, then the jurisdiction S receives a subsidy to cover the cost of its public project, and if $x(S) > g$, the jurisdiction S is a donor that subsidizes other jurisdictions.

To introduce the last principle of this section, we limit our attention to the case where, as for efficient jurisdictional structures, the partition P consists of intervals of equal length. Since all jurisdictions are of the same size, from a normative point of view it makes sense to rule out a subsidization across jurisdictions and to require that cost allocations should be the same in all jurisdictions. Indeed, an external impartial observer contemplating that problem could argue that if we promote the use of x in a given jurisdiction, then x should be promoted in all other jurisdictions identical to that under consideration.

Definition 3.6: Let P be a partition of R_θ into n jurisdictions. A P-cost allocation x satisfies *inter-jurisdictional symmetry* if the equality

$$x(t + \frac{2n\theta}{N}) = x(t)$$

holds for all $t \in R_\theta$ for which the point $t + \frac{2n\theta}{N}$ also belongs to R_θ.

Intra and inter jurisdictional symmetry and partial equalization are general normative principles which are useful to guide a selection of the cost allocation. The set of cost allocations satisfying the three properties is nonempty, as it always contains the Rawlsian allocation. Given that the Rawlsian allocation meets several desirable fairness criteria, the natural question in our analysis of stability is whether the Rawlsian principle is compatible with the requirement of secession-proofness. The next section provides a negative answer to this question. We show, moreover, that even if the set of acceptable cost allocations is expanded to include all intra and inter symmetric cost allocations satisfying the principle of partial equalization, we still fall short of the reconciliation of our fairness requirements with the secession-proofness.

4 Secession-Proofness and the Rawlsian Principle

We now turn to the examination of "stable" partitions that are immune against a threat of deviation or secession by a group of individuals. We study a collective choice problem whose solution must be accepted by all individuals and all coalitions of individuals. That is, we require a voluntary participation of the individuals in potentially seceding groups. This necessitates an examination of mechanisms that allocate benefits among individuals in such a way that no coalition can generate a higher payoff to all its members. However, if a coalition S can make its members better off relative to the current arrangement, we will say that S is *prone to secession*. Formally,

Definition 4.1: Consider a pair (P, x), where P is a partition in $\mathcal{P}$ and x is a cost allocation. The jurisdiction S is prone to secession (given (P, x)) if

$$\int_S (d(t, m(S)) + x(t))dt > D(S) + g.$$

If no jurisdiction is prone to secession, then the pair (P, x) is called *secession-proof*; if there is no ambiguity we drop the first argument of the pair and simply refer to a secession-proof cost allocation.

The concept of secession-proofness introduced here is closely related to the notion of the core of a game with coalition structures (Aumann and Drèze (1974)), whose set of players is R_θ, and the set of feasible outcomes of a coalition S is given by all S-cost allocations. Since we do not use the game-theoretical analysis here, we chose to formulate our results without relying on it.

We would like to point out two immediate but nevertheless useful implications of secession-proofness:

Remark 4.2: (i) Every jurisdiction balances its budget. Indeed, no jurisdiction can be a net donor, since, otherwise it would secede and save the amount of the net transfer. No jurisdiction can be a net recipient either, as by (2), another jurisdiction would have to be a net donor.

(ii) Every individual makes a nonnegative contribution to the financing of the public project in her jurisdiction; otherwise, the coalition of other individuals within that jurisdiction who do make a positive contribution would be better off by breaking away from the jurisdiction.

As we argued above, the condition of secession-proofness is quite demanding as it requires that no coalition should be able to deviate and thereby improve the welfare of all its members. In this section we explore the extent to which this principle can coexist with the normative principles introduced in the previous section.

For a society R_θ denote by $P(\theta)$ an efficient partition of R_θ into $N(\theta)$ jurisdictions of equal size and by x_R^θ the Rawlsian allocation associated with partition P_θ. Given the cost of public project g, denote by $SP(\theta, g)$ the set of secession-proof cost allocations, the nonemptiness of which is guaranteed by Haimanko, Le Breton and Weber (2004). First, we describe the cases for which the Rawlsian principle is compatible with the stability requirement of secession-proofness and show that, in general, it is not the case:

Proposition 4.3: Let $\theta \geq \sqrt{g}$. The pair (P_θ, x_R^θ) is secession-proof if and only if the ratio $\frac{\theta}{\sqrt{g}}$ is an integer.[4]

The natural question then arises is whether the secession-proofness is compatible with a weaker set of requirements than the Rawlsian principle. Indeed, in the single-jurisdiction case Le Breton and Weber (2003) show that secession-proofness is compatible with intra-jurisdictional symmetry and the principle of partial equalization for a large class of the population distributions containing the Lebesgue measure. In the multi-jurisdictional set-up, the situation becomes more complicated. Unlike in the single-jurisdictional case, we have to face all additional constraints of preventing secession by groups containing individuals from different jurisdictions. Indeed we show that, even under inter-jurisdictional symmetry, the aforementioned compatibility does not hold any longer. Denote by $PES(\theta, g)$ the set of inter- and intra-symmetric cost allocations that satisfy the principle of partial equalization. Note that this set contains the Rawlsian allocation and is, therefore, nonempty. We have:

Proposition 4.4: If $N(R_\theta) > 1$, $SP(\theta, g) \cap PES(\theta, g) \neq \emptyset$ if and only if the ratio $\frac{\theta}{\sqrt{g}}$ is an integer.

In the next section we consider modified versions of the Rawlsian principle and examine their compatibility with the stability requirement of secession-proofness.

[4]This is reminiscent of the integer problem examined first by Pauly (1967), (1970) in the context of the theory of clubs. The Rawlsian allocation is secession-proof only for a non-generic subset of values θ.

5 Modifications of the Rawlsian Principle and Secession-Proofness

In the previous section we have shown that it is in general impossible to reconcile the standard Rawlsian principle with the secession-proofness principle. When the requirement of secession-proofness is maintained, it is therefore important to investigate possible modifications of the Rawlsian principle.

By Proposition 2.5, an optimal partition of the bounded society R_θ consists of $N(R_\theta)$ intervals of the equal size $s(R_\theta)$. Remark 2.2 implies that the lower bound on the average cost within each jurisdiction in the partition is $\sqrt{g}$. It can be attained only if every jurisdiction has the optimal size $2\sqrt{g}$. The latter, however, is possible only if θ is a multiple of $\sqrt{g}$. Thus, in general, the society will have to incur a per capita cost larger than $\sqrt{g}$. We impose a mild condition on cost allocations by requiring that no individual $t \in R_\theta$ is assigned a cost contribution, that, combined with her transportation cost, would be less than $\sqrt{g}$. This condition rules out cost allocation that grant privileges to some individuals and shift a higher cost burden to the rest of the society. Unless one wishes to provide an individual or a group with specific privileges, it is hard to offer a priori reason, immune to an ethical or normative appraisal, to grant an individual a cost contribution below the reservation payoff $\sqrt{g}$. Note the latter is attained only when the economies of scale generated by a large jurisdiction are perfectly balanced by the heterogeneity costs incurred by its size. The application of the 'privilege-free" condition to all subsets S of the society R_θ amounts to the following inequality:

$$\int_S (x(t) + d(t, m(S(t))))dt \geq \lambda((S)\sqrt{g}. \tag{3}$$

In fact, we will require (3) to hold only for intervals of the optimal size s^*:

Definition 5.1: A cost allocation x is "privilege-free" if the inequalities (3) are satisfied for all groups $S \subseteq R_\theta$ such that $S = [t, t + s^*]$.

Obviously every Rawlsian allocation is privilege-free whenever θ is a multiple of $\sqrt{g}$.

We shall show now that the privilege-free principle is, in general, incompatible with secession-proofness in the multi-jurisdictional framework (or when $\sqrt{g} < \theta$) if the continuity of cost allocations is imposed:

Proposition 5.2: Let $\sqrt{g} \leq \theta$. Then the set of continuous, secession-proof and privilege-free allocations on R_θ is nonempty if and only if θ is a multiple of $\sqrt{g}$.

Let $\theta \geq \sqrt{g}$. Since the inequality

$$\int_{z}^{z+2\sqrt{g}} (x(t) + d(t, m(S(t))))dt \leq 2g$$

is satisfied for every secession-proof allocation x and all $z \in \left[-\theta, \theta - 2\sqrt{g}\right]$, the inequality (3) implies that

$$\int_S (x(t) + d(t, m(S(t))))dt = 2g \tag{4}$$

for all optimal jurisdictions S.[5] The equations (4) guarantee that the total cost $x(t) + d(t, m(S(t)))$ is a periodic function with the period of $2\sqrt{g}$.

In order to reconcile a version of the Rawlsian principle with secession-proofness, we examine an "approximate" Rawlsian principle and demonstrate that for large populations with a bounded support, secession-proof cost allocations[6] are in some sense not "too far" from the Rawlsian solution. We formalize a notion of privilege granted to a group of individuals when each of its members contributes less than the amount prescribed by the Rawlsian allocation reduced by some given value δ.

Definition 5.3: Let $\delta > 0$ be given. Let P be an efficient partition of R_θ. We say that a P-cost allocation grants a *privilege of magnitude δ to the group S* (with respect to the Rawlsian allocation x_R^θ) if for all $t \in S$ the following inequality holds:

$$x(t) \leq x_R^\theta(t) - \delta.$$

[5]Note that if $\theta < \sqrt{g}$, the society R_θ does not contain optimal groups and the equations (4) are vacuous.

[6]The structure of the set of secession-proof cost allocations is rather intricate. From Proposition 5.2., we know that in general some individuals are going to incur a total cost smaller than $\sqrt{g}$. We have derived some very preliminary insights on this set but a full understanding of the implications of secession-proofness is currently out of reach.

We denote by $\mathcal{G}(x, \delta, \theta)$ the family of measurable sets S that are granted a privilege δ via allocation x.

The main result of this section shows that if a society is large enough then the size of a coalition receiving a privilege of the given magnitude via secession-proof allocation is relatively small:

Proposition 5.4: Let $\delta > 0$ and a bounded set $S \subset \Re$ be given. Then there exists $\bar{\theta}$ with $S \subset R_{\overline{\theta}}$ such that for every $\theta > \bar{\theta}$ and every secession-proof allocation $x \in SP(\theta, g)$,

$$S \notin \mathcal{G}(x, \delta, \theta).$$

Proposition 5.4 asserts that if x is secession-proof, then for any privilege level, the size of the privileged group must be small if the size of the entire population is large enough. In some sense, Proposition 5.4 is a formal statement of the claim that secession-proofness implies the approximate Rawlsian recommendation.

The assertion of Proposition 5.4 can actually be strengthened. Let us denote by $\mathcal{G}'(x, \delta, \theta)$ the collection of groups S such that

$$\int_S (x(t) + \mid t - m(S(t)) \mid) \, dt \leq \int_S (x_R^*(t) + \mid t - m(S(t)) \mid) \, dt - \delta\lambda(S),$$

where the privilege $\delta\lambda(S)$ is granted to the entire set S. The average benefit of members of S is δ which is a weaker requirement than insisting on the privilege level of δ for every member of S. Obviously,

$$\mathcal{G}(x, \delta) \subseteq \mathcal{G}'(x, \delta).$$

We have a stronger version of Proposition 5.4:

Proposition 5.5: Let $\delta > 0$ and a bounded set $S \in \Re$ be given. Then there exists $\bar{\theta}$ with $S \subset R_{\overline{\theta}}$ such that for every $\theta > \bar{\theta}$ and every secession-proof allocation $x \in SP(\theta, g)$,

$$S \notin \mathcal{G}'(x, \delta, \theta).$$

6 Appendix

We start by stating several claims.

Claim A.1: Let P be an efficient partition. Then $S_P(t) \in \arg\min_{S \in P} \mid t - m(S) \mid$. That is, every individual is assigned to the public project closest to her location.

Follows immediately from efficiency of P.

Claim A.2: Let P be an efficient partition. Then every element $S \in P$ is an interval.

Proof: Consider $S \in P$ and $a, b \in S$ with $m(S) \leq a < b$. We claim that every point $c \in [a, b]$ belongs to S. Indeed, suppose that such an c is assigned to another jurisdiction, say, S'. First note that $m(S) \neq m(S')$. Otherwise, the merger of jurisdictions S and S' would violate the efficiency of P. Claim A.1 implies that $m(S') > m(S)$, but the inequality $\mid c - m(S') \mid \leq \mid c - m(S) \mid$ yields $\mid b - m(S') \mid < \mid b - m(S) \mid$. Thus, b is closer to $m(S')$ than to $m(S)$, a contradiction to Claim A.1. $\square$

Claim A.3: Let P be an efficient partition. Then all intervals S in P have the same length.

Proof: Let S and S' be two adjacent intervals in P and let l and l' denote their respective lengths. Since P is efficient, $D(S) + D(S')$ is minimal among all possible partitions of $S \cup S'$ into two intervals. Thus, (1) implies that $l^2 + (l')^2 \leq x^2 + y^2$ for all nonnegative numbers x, y satisfying $x + y = l + l'$. Since under the constraint $x + y = l + l'$ the convex function $x^2 + y^2$ reaches its minimum at $x = y = \frac{l+l'}{2}$, it follows that l must be equal to l'. $\square$

Proof of Proposition 2.5: We prove here a version of proposition 2.5 by considering an interval S of the length L rather than the set R_θ. Let $N(L)$ be the efficient number of jurisdictions when the population is distributed over S. Let $L = 2m\sqrt{g} + r$, where $m = \lfloor \frac{L}{2\sqrt{g}} \rfloor$, is an integer and $0 \leq r < 2\sqrt{g}$. We consider the case where $m > 0$ (otherwise, $N(L)$ is trivially equal to 1).

When S is divided in N intervals of equal size, the total aggregate cost is

$$N\left(g + \frac{L^2}{4N^2}\right) = Ng + \frac{L^2}{4N}$$

The first order condition over the set of real numbers yields

$$g - \frac{L^2}{4N^2} = 0, or$$

$$N' = \frac{L}{2\sqrt{g}} = \frac{2m\sqrt{g} + r}{2\sqrt{g}} = m + \frac{r}{2\sqrt{g}}.$$

Since $Ng + \frac{L^2}{4N}$ is convex in N, and N' is not necessarily an integer, it follows that $N(L)$ is either m or $m+1$. The optimal choice is m if and only if

$$gm + \frac{L^2}{4m} \leq g(m+1) + \frac{L^2}{4(m+1)},$$

which, after simplifications, is equivalent to

$$\frac{L^2}{4g} \leq m(m+1), \quad \text{or} \quad \frac{L}{s^*} \leq \sqrt{m(m+1)}.$$

□

Proof of Proposition 2.7: (i) Let $L = 2m\sqrt{g} + r$, where $m = \lfloor \frac{L}{2\sqrt{g}} \rfloor$ is an integer and $0 < r < 2\sqrt{g}$. By Proposition 2.5, the optimal number of intervals is either m or $m+1$. This implies that $s(L)$ is either equal to $\frac{L}{m}$ or to $\frac{L}{m+1}$. Thus,

$$\frac{2m\sqrt{g} + r}{m+1} \leq s(L) \leq \frac{2m\sqrt{g} + r}{m},$$

and, therefore,

$$2\sqrt{g}\frac{r - 2\sqrt{g}}{L - r + 2\sqrt{g}} \leq \frac{r - 2\sqrt{g}}{m+1} \leq s(L) - 2\sqrt{g} \leq \frac{r}{m} = 2\sqrt{g}\frac{r}{L-r}.$$

Since $0 < r < 2\sqrt{g}$, this implies

$$2\sqrt{g} - \frac{4g}{L + 2\sqrt{g}} \leq s(L) \leq 2\sqrt{g} + \frac{4g}{L - 2\sqrt{g}}.$$

(ii) The per capita aggregate cost of an individual in an interval of length s:

$$f(s) = \frac{1}{s}(g + \frac{s^2}{4}).$$

We evaluate the values $f(s(L))$ when L tends to infinity. By assertion (i), the value of $s(L)$ for large L is close to $2\sqrt{g}$, and we examine the second order Taylor expansion of f at $2\sqrt{g}$. Assertion (i) implies that there is

$$z(L) \in \left[2\sqrt{g} - \frac{4g}{L+2\sqrt{g}}, 2\sqrt{g} + \frac{4g}{L-2\sqrt{g}},\right]$$

such that

$$f(s(L)) - f(2\sqrt{g}) = f''(z(L))\frac{(s(L)-2\sqrt{g})^2}{2} = \frac{g(s(L)-2\sqrt{g})^2}{z(L)^3}.$$

Since $f'''(s) = -\frac{6g}{s^4} < 0$, the last inequality yields

$$f(s(L)) - f(2\sqrt{g}) \leq \frac{16g^3(\frac{1}{L-2\sqrt{g}})^2}{8\sqrt{g^3}(\frac{L}{L+2\sqrt{g}})^3}. \tag{5}$$

If $L > 4\sqrt{g}$ then $\frac{L}{L+2\sqrt{g}} > \frac{2}{3}$ and $\frac{1}{L-2\sqrt{g}} < \frac{2}{L}$, yielding

$$f(s(L)) - f(2\sqrt{g}) \leq \frac{27\sqrt{g^3}}{L^2}.$$

If $L \leq 4\sqrt{g}$ then $s(L)$ is either 1 or 2, and $f(s(L)$ is either $\frac{g}{L} + \frac{L}{4}$ or $\frac{2g}{L} + \frac{L}{8}$. It is easy to verify that (5) holds in this case as well. □

Proof of Proposition 4.3: Assume first that $\frac{\theta}{s^*}$ is an integer. Then the size of each jurisdiction in an optimal partition is s^* and the total cost incurred by an individual $t \in R_\theta$ under allocation x^*_R is $\sqrt{g}$.

Consider a jurisdiction S. By Remark 2.1, the total cost incurred by S is at least $g + \frac{(\lambda(S))^2}{4}$ and each $t \in S$ on average contributes at least

$$f(\lambda(S)) = \frac{g}{\lambda(S)} + \frac{\lambda(S)}{4}.$$

Since the minimum of the function $\frac{g}{s} + \frac{s}{4}$ is attained at $s = 2\sqrt{g}$, it follows that $\frac{g}{\lambda(S)} + \frac{\lambda(S)}{4} \geq \sqrt{g}$, and the coalition S is not prone to secession. Thus, the sufficiency part is completed.

Assume now that the value $\frac{\theta}{\sqrt{g}}$ exceeds 1 and is not an integer. Suppose that the Rawlsian allocation x_R^θ is secession-proof. The efficiency implies that the total cost incurred by the entire society R_θ is

$$2\theta\left[\frac{gN(R_\theta)}{2\theta}+\frac{\theta}{2N(R_\theta)}\right].$$

Then

$$\lambda(S)\left[\frac{gN(R_\theta)}{2\theta}+\frac{\theta}{2N(R_\theta)}\right]\leq g+\frac{\lambda(S)^2}{4}$$

for all S. Since $f(\lambda(S))=\frac{g}{\lambda(S)}+\frac{\lambda(S)}{4}$ is minimal for at $2\sqrt{g}$, which does not exceed 2θ, the secession-proofness of x_R^θ implies that

$$\frac{gN(R_\theta)}{2\theta}+\frac{\theta}{2N(R_\theta)}\leq\sqrt{g}.$$

But this inequality is violated whenever $N(R_\theta)\neq\frac{\theta}{\sqrt{g}}$, a contradiction. Thus, the Rawlsian allocation is not secession-proof.□

Proof of Proposition 4.4: Let $\frac{\theta}{\sqrt{g}}$ be an integer. By Proposition 4.3, the set $SP(\theta,g)$ contains the Rawlsian allocation, which is, as we observed earlier, is an element of $PES(\theta,g)$. Thus, the intersection of the sets $SP(\theta,g)$ and $PES(\theta,g)$ is non-empty.

Assume now that $\frac{\theta}{\sqrt{g}}$ is not an integer and let x be a cost allocation in $PES(\theta,g)$. By Proposition 2.5, there are two possibilities.

If $N=\lfloor\frac{\theta}{\sqrt{g}}\rfloor$, then the size $s=s(R_\theta)$ of a jurisdiction in P is larger than $2\sqrt{g}$. Since there are at least two jurisdictions, consider the jurisdictions $S_1\equiv[-\theta,-\theta+s]$, $S_2\equiv[-\theta+s,-\theta+2s]$. Take an optimal group $S^*=[-\theta+s-\sqrt{g},-\theta+s+\sqrt{g}]$. Since x satisfies intra and interjurisdictional symmetry we have

$$\int_{S^*}(x(t)+d(t,m(S(t)))dt=2\int_{-\theta+s-\sqrt{g}}^{-\theta+s}(x(t)+d(t,m(S(t)))dt. \tag{6}$$

Moreover, the principle of partial equalization and the fact that $s>2\sqrt{g}$ imply

$$\frac{\int_{-\theta+s-\sqrt{g}}^{-\theta+s}(x(t)+d(t,m(S(t)))dt}{\sqrt{g}}\geq\frac{2}{s}\int_{-\theta+\frac{s}{2}}^{-\theta+s}(x(t)+\mid t-m(S_1)\mid)dt. \tag{7}$$

Since

$$\int_{-\theta+\frac{s}{2}}^{-\theta+s}(x(t)+\mid t-m(S_1)\mid)dt=\frac{g}{2}+\frac{s^2}{8}>\frac{s\sqrt{g}}{2}. \quad (8)$$

By combining (6)-(8), we obtain

$$\int_{S^*}(x(t)+d(t,m(S(t)))dt>2g.$$

However, the jurisdiction S^* is optimal and, therefore,

$$g+D(S^*)=\int_{-\theta+s-\sqrt{g}}^{-\theta+s+\sqrt{g}}(\mid t+\theta-s\mid)dt=2g.$$

Thus, x is not secession-proof.

Consider now the case where $N(R_\theta)=\lfloor\frac{\theta}{\sqrt{g}}\rfloor+1$. The optimal size $s=s(R_\theta)$ is smaller than $2\sqrt{g}$. However, it is still bounded from below, namely,

$$s>\frac{4}{3}\sqrt{g}. \quad (9)$$

Indeed, since

$$N(R_\theta)<\frac{\theta}{\sqrt{g}}+1,$$

it follows that

$$s>2\sqrt{g}\frac{\theta}{\theta+\sqrt{g}}.$$

If $\theta\geq 2\sqrt{g}$, (9) follows immediately. If $\theta<2\sqrt{g}$ then $N(R_\theta)=\lfloor\frac{\theta}{\sqrt{g}}\rfloor+1=2$. Since a two-jurisdictional structure yields a lower aggregate cost than the grand coalition, by Remark 2.1, we have

$$2(g+\frac{\theta^2}{4})\leq g+\frac{(2\theta)^2}{4},$$

or $\theta>\sqrt{2}g$. Since $s=\theta$ and $\sqrt{2}>\frac{4}{3}$ it follows that (9) holds in this case as well.

Consider the coalition $S'\equiv\left[-\theta,-\theta+2\sqrt{g}\right]$. Since $s\neq 2\sqrt{g}$, it follows

$$\int_{-\theta}^{-\theta+s}(x(t)+\mid t-m(S')\mid)dt>s\sqrt{g}.$$

Note that (9) implies $2\sqrt{g} < \frac{3s}{2}$, that is, the point $-\theta + 2\sqrt{g}$ is located to the left of the median of the adjacent jurisdiction $-\theta + \frac{3}{2}s$. The principle of partial equalization implies

$$\int_{-\theta+s}^{-\theta+2\sqrt{g}} (x(t)+ \mid t - m(S') \mid)dt > (2\sqrt{g} - s)\sqrt{g},$$

which leads to

$$\int_{-\theta}^{-\theta+2\sqrt{g}} (x(t)+ \mid t - m(S) \mid)dt > 2g.$$

However, the equality $g + D(S') = 2g$ implies that x is not secession-proof. □

Proof of Proposition 5.2: It remains to show that if $N(R_\theta) > 1$ and θ is not a multiple of $\sqrt{g}$, then any continuous secession proof allocation is not privilege-free. Suppose, in negation, that there is a continuous secession-proof privilege-free allocation x.

Since 2θ is not a multiple of $2\sqrt{g}$, Remark 2.2 implies that there exists $\bar{t} \in (-\theta, \theta)$ such that

$$x(\bar{t}) + d(\bar{t}, m(S(\bar{t}))) > \sqrt{g}.$$

Consider first the case where $\theta > 2\sqrt{g}$. Then there exists $\bar{\delta} > 0$ such that for every positive $\delta < \bar{\delta}$ either the coalition $\left[\bar{t} - \delta, \bar{t} + 2\sqrt{g}\right]$ or the coalition $\left[\bar{t} - 2\sqrt{g}, \bar{t} + \delta\right]$ is a subset of R_θ. Without loss of generality, assume that the latter holds and denote $S(\delta) = \left[\bar{t} - 2\sqrt{g}, \bar{t} + \delta\right]$. By (1) of Remark 2.1, we have

$$g + D(S(\delta)) = 2g + \delta\sqrt{g} + \frac{\delta^2}{4}. \tag{10}$$

Since x is continuous, there exists $0 < \delta' < \bar{\delta}$ and $\eta > 0$ small enough such that

$$x(t) + d(t, m(S(t))) > \sqrt{g} + \eta$$

for all $t \in [\bar{t}, \bar{t} + \delta']$. Furthermore, since x is privilege-free, for every $\delta, 0 < \delta < \delta'$, we have

$$\int_{S(\delta)} (x(t) + d(t, m(S(t))))dt > 2g + \delta\sqrt{g} + \eta\delta. \tag{11}$$

Then the secession proofness of x implies that the equation (10) and the inequality (11) are incompatible for sufficiently small values of δ, a contradiction.

Consider the case where $\theta < 2\sqrt{g}$. Since $N(R_\theta) > 1$, Proposition 2.5. yields that $N(R_\theta) = 2$ and $\theta > \sqrt{g}$. If there exists $\bar{t} \in \left[-\theta, \theta - 2\sqrt{g}\right) \cup \left(2\sqrt{g} - \theta, \theta\right]$ such that $x(\bar{t}) + d(\bar{t}, m(S(\bar{t}))) > \sqrt{g}$, we can proceed as in the previous case. Assume, therefore, that $x(\bar{t}) + d(\bar{t}, m(S(\bar{t}))) \leq \sqrt{g}$ for all $\bar{t} \in \left[-\theta, \theta - 2\sqrt{g}\right] \cup \left[2\sqrt{g} - \theta, \theta\right]$. Note that the budget balance condition implies

$$\int_{-\theta}^{0} (x(t) + d(t, \frac{\theta}{2}))dt = \int_{0}^{\theta} (x(t) + d(t, -\frac{\theta}{2}))dt = \frac{\theta^2}{4} + g > \theta\sqrt{g}. \tag{12}$$

Thus, in particular,

$$\int_{0}^{2\sqrt{g}-\theta} (x(t) + d(t, \frac{\theta}{2}))dt > (2\sqrt{g} - \theta)\sqrt{g}. \tag{13}$$

(12) and (13) imply that

$$\int_{-\theta}^{2\sqrt{g}-\theta} (x(t) + d(t, S(t)))dt = \int_{-\theta}^{0} (x(t) + d(t, -\frac{\theta}{2}))dt + \int_{0}^{2\sqrt{g}-\theta} (x(t) + d(t, \frac{\theta}{2}))dt > 2g.$$

Hence, a coalition $\left[-\theta, 2\sqrt{g} - \theta\right]$ is prone to secession, a contradiction to the assumed secession-proofness of the allocation x. $\square$

In the proof of our next propositions we use some results from the basic measure theory. The first two rely on the regularity of the Lebesgue measure (Billingsley (1986), Theorem 12.3):

Claim A.4: If S is a bounded and measurable subset of $\Re$ then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq S$ with $\lambda(S \setminus K_\varepsilon) \leq \varepsilon$.

Claim A.5: If S is a bounded and measurable subset of $\Re$ then for every $\varepsilon > 0$ there exists an open set $O_\varepsilon \supseteq S$ with $\lambda(O_\varepsilon \setminus S) \leq \varepsilon$.

Another is a well-known result (Billingsley (1986), page 231):

Claim A.6 - Lusin's theorem: Let A be a bounded measurable subset of $\Re$ and h is a measurable function on A. Then for every $\varepsilon > 0$ there exists a compact set $C_\varepsilon \subseteq A$ with $\lambda(\tilde{C}_\varepsilon) \geq \lambda(A) - \varepsilon$ and h is continuous on C_ε.

Finally, we will utilize the property of *essential boundedness* of secession-proof allocations in a multi-jurisdictional framework:

Claim A.7: Let $N(R_\theta) > 1$ and $x \in SP(\theta, g)$. Then there exists a constant $q > 0$ such that $x(t) \leq q$ almost everywhere on R_θ, or $\lambda(\{t \in R_\theta : x(t) > q\}) = 0$. In fact, q can be chosen as any number exceeding $2\sqrt{g}$.

Proof: Let $N(R_\theta) > 1$ and $x \in SP(\theta, g)$. We shall show that

$$\lambda(\{t \in R_\theta : x(t)+ \mid t - m(S(t)) \mid > 2s(R_\theta)\}) = 0,$$

where $s(R_\theta)$ is the optimal size in R_θ.

Suppose, to the contrary, that $\lambda(S) > 0$, where $S \equiv \{t \in \Re : x(t)+ \mid t - m(S(t)) \mid > 2s(R_\theta)\}$. Let S_1, S_2 be two adjacent jurisdictions in an optimal partition, with $\lambda(S') > 0$, where $S' = S_2 \cap S$. Denote $T \equiv S_1 \cup S'$ and define the T-cost allocation y as follows:

$$y(t) = \begin{cases} x(t) & if \quad t \in S_1 \\ 0 & if \quad t \in S'. \end{cases}$$

We have

$$D(T) + g \leq \int_T (y(t)+ \mid t - m(S_1) \mid)dt < \int_{S_1} x(t)dt \int_S \mid t - m((S_1) \mid dt.$$

Since $\mid t - m(S_1) \mid < x(t)+ \mid t - m(S_2) \mid$ for all $t \in S'$, it follows that

$$D(T) + g < \int_T (x(t)+ \mid t - m(S(t)) \mid)dt.$$

That is, T is prone to secession, contradicting our assumption that x is secession-proof.$\square$

Proof of Proposition 5.4: Let $\delta > 0$. By Proposition 2.7, the Rawlsian allocation x_R^θ is "close" to $\sqrt{g}$ and the optimal jurisdictional size is "close" to $2\sqrt{g}$ when θ is large enough. Thus, we can choose $\tilde{\theta}$ such that $\mid x_R^\theta - \sqrt{g} \mid < \frac{\delta}{4}$ and $s(R_\theta) < \frac{5}{2}\sqrt{g}$ for every $\theta > \tilde{\theta}$.

Let $\theta > \tilde{\theta}$. Suppose now that $y \in SP(\theta, g)$ and the set $S \in R_\theta$ with a positive measure is granted a a privilege of magnitude δ via allocation y. Let $\lambda(S) = \rho > 0$. We will use the following claim:

Claim A.8: There exists a finite family of pairwise disjoint intervals $\tilde{\mathcal{I}} = \{I_1, \ldots, I_m\}$ such that

(i)

$$\lambda(I) \geq \frac{\rho}{2}, \quad \text{where } I \equiv \bigcup_{I_i \in \tilde{\mathcal{I}}} I_i,$$

(ii)

$$\int_I z(t)dt < \int_I \left(x_\theta^R(t) + \mid t - m(S(t)) \mid\right) dt - \frac{\delta}{2}\lambda(I).$$

Proof: Let the number η be such that $0 < \eta < \frac{\delta}{10\sqrt{g}}$. Claim A.4 implies that there exists a compact subset K of S such that $\lambda(K) > \frac{3}{4}\rho$ and an open set O_η with $O_\eta \supseteq S$ such that

$$\lambda(O_\eta \backslash K) \leq \eta \lambda(K). \tag{14}$$

For every $t \in K$, let $I(t) \subset O_\eta$ be an interval that contains t. Since K is compact, the cover $\{I(t)\}_{t \in K}$ admits a subcover $\mathcal{I} = I_1, \ldots, I_N$. We may assume, without loss of generality, that all intervals in $\mathcal{I}$ are pairwise disjoint, and, moreover that O_η consists only of elements of $\mathcal{I}$.

Denote by $\tilde{I}$ the following subset of $\mathcal{I}$:

$$\tilde{\mathcal{I}} = \{I_i \in \mathcal{I} | \frac{\lambda(K^c \cap I_i)}{\lambda(I_i)} \leq 3\eta\}.$$

By (14), we have

$$\sum_{I_i \in \mathcal{I}} \lambda(K^c \cap I_i) = \lambda(O_\eta \backslash K) \leq \eta \lambda(K) < \frac{3}{4}\eta\rho.$$

But

$$\sum_{I_i \in \mathcal{I}} \lambda(K^c \cap I_i) = \sum_{I_i \in \tilde{\mathcal{I}}} \lambda(K^c \cap I_i) + \sum_{I_i \notin \tilde{\mathcal{I}}} \lambda(K^c \cap I_i) > 3\eta \sum_{I_i \notin \tilde{\mathcal{I}}} \lambda(I_i).$$

Thus,

$$\lambda(\bigcup_{I_i \notin \tilde{\mathcal{I}}} I_i) < \frac{1}{4}\lambda(K), \quad \text{and } \lambda(\tilde{I}) > \frac{3}{4}\lambda(K) > \frac{\rho}{2},$$

where $I = \bigcup_{I_i \in \tilde{\mathcal{I}}} I_i$. Moreover,

$$\int_I z(t)dt = \int_{K \cap I} z(t)dt + \int_{K^c \cap I} z(t)dt.$$

We have

$$\int_{K \cap I} z(t)dt < \lambda(I)(\sqrt{g} - \delta),$$

By Claim A.7, $z(t) < 5\sqrt{g}$ almost everywhere on R_θ, yielding

$$\int_{K^c \cap I} z(t)dt \leq 5\sqrt{g}\eta\lambda(I).$$

Since $\eta < \frac{\delta}{10\sqrt{g}}$, we have

$$\int_{\tilde{I}} z(t)dt < \lambda(I)(\sqrt{g} - \frac{\delta}{2}),$$

which completes the proof of the claim. □

Let $\mathcal{I}$ be a family of m intervals that satisfy conditions (i) and (ii) of Claim A.8. Then there exists an interval $I^* \in \mathcal{I}$ such that $\lambda(I^*) > \frac{\rho}{2m}$. Let $I^* = (a, b)$. We have

$$\int_I z(t)dt < \int_I (x_R^\theta(t)+ \mid t - m(S(t)) \mid)dt - \frac{\delta(b-a)}{2}$$

and

$$\int_{R_\theta \setminus I} z(t)dt > \int_{R_\theta \setminus I} (x_R^\theta(t)+ \mid t - m(S(t)) \mid)dt + \frac{\delta(b-a)}{2}. \tag{15}$$

Consider the coalition $T = T_1 \bigcup T_2$, where $T_1 = [-\theta, a]$ and $T_2 = [b, \theta]$. Assume that θ is large enough so that $\lambda(T_j) > \frac{\theta}{2}$, $j = 1, 2$.

By assertion (ii) of Proposition 2.7, the total contribution of individuals in T_1 and T_2 satisfies

$$f(s(T_j))\lambda(T_j) \leq \lambda(T_j)\sqrt{g} + \frac{27\sqrt{g^3}}{\lambda(T_j)} \leq \lambda(T_j)\sqrt{g} + \frac{54\sqrt{g^3}}{\theta}$$

for $j = 1, 2$. Thus, the total contribution of the individuals in T, $F(T) = f(s(T_1))\lambda(T_1) + f(s(T_2))\lambda(T_2)$ satisfies

$$F(T) \leq \lambda(T)\sqrt{g} + \frac{108\sqrt{g^3}}{\theta} \leq \int_T (x_R^\theta(t)+ \mid t - m(S(t)) \mid)dt + \frac{108\sqrt{g^3}}{\theta}.$$

But, by (15),

$$\int_T (y(t)+ \mid t - m(S(t)) \mid)dt > \int_T (x_R^\theta(t)+ \mid t - m(S(t)) \mid)dt + \frac{\delta(b-a)}{2}.$$

The secession-proofness of y implies

$$\int_T (y(t)+ \mid t - m(S(t)) \mid)dt \leq F(T)$$

and, therefore,

$$b - a \leq \frac{216\sqrt{g^3}}{\delta\theta}.$$

The inequality $b - a \geq \frac{\rho}{2m}$ immediately implies that

$$\rho \leq \frac{432m\sqrt{g^3}}{\delta\theta}.$$

Since m is independent of θ, it follows that the last inequality is violated when θ is large enough.

□

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