Research Cycles

Yann Bramoullé   Gilles Saint-Paul[1]

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Abstract: This paper studies the dynamics of fundamental research. We develop a simple model where researchers allocate their effort between improving existing fields and inventing new ones. A key assumption is that scientists derive utility from recognition from other scientists. We show that the economy can be either in a regime where new fields are constantly invented, and then converges to a steady state, or in a cyclical regime where periods of innovation alternate with periods of exploitation. We characterize the cyclical dynamics of the economy, show that indeterminacy may appear, and establish some comparative statics and welfare implications.

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"my love of natural science (...) has been much aided by the ambition to be esteemed by my fellow naturalists." Darwin (1958).

1   Introduction

This paper studies the dynamics of fundamental research. We observe that periods of intense innovations are followed by periods of exploitation of existing fields. We want to understand these dynamics and be able to study whether they are efficient from the point of view of social welfare.

A key aspect we are interested in is the credential one. Scientists derive utility from recognition from other scientists, which often takes the form of citations. In our model, the value derived by a scientist from a paper he has written is the sum of an "intrinsic" value of the paper, which depends on the field in which it is written and its order of appearance in that field, and a "citation premium" which depends on the number of subsequent papers written in that field.

We show that the economy can be either in a regime where new fields are constantly invented, and then converges to a steady state, or in a cyclical regime where periods of innovation alternate with periods where one only exploits existing fields. Furthermore, these cycles are very irregular and the duration of a cycle is "unpredictable" from the duration of the previous cycle – i.e. related to it by a very nonlinear function.

We are able to perform comparative statics in the convergence regime and show that a (i) higher citation premium raises the equilibrium rate of innovation, (ii) a mean-preserving spread in the distribution of the value of new fields reduces the equilibrium rate of innovation and (iii) a larger citation premium makes researchers less risk-averse, in that it alleviates (and potentially reverses) that effect.

We introduce the distinction between extensive and intensive search to the study of scientific progress. Studies of technological changes have long stressed the difference between improvements of known processes and innovations leading to new products (e.g. Rosenberg (1972)). Similarly, it seems that some scientific contributions are pioneering and open up new avenues for future research, while others mainly refine or extend previous work. This distinction lies at the core of Kuhn (1970)'s influential theory of scientific evolution. In his view, science alternates between periods of normal science and scientific revolutions. Under normal science, progress is gradual and builds up on past achievements. In contrast, scientific revolutions correspond to paradigm shifts during which scientists qualitatively change their focus and assumptions. Without necessarily adhering to Kuhn's view, other observers have noted the importance of fads and fashions in science. Bronfenbrenner (1966) gives an early account of fads in economics. Stephan and Levin (1991) discuss how scientific fashions might affect a scientist's career. Sunstein (2001) relates academic fads to informational cascades. We develop the first formal model of the evolution of science that gives rise to innovation cycles and scientific fashions.

Our paper advances the economics of science. Economic analysis of scientific activities is still rather undeveloped.[2] We study the first dynamic model of scientific progress. Our model explicitly accounts for the unique reward structure of science, by assuming that scientists care for recognition by their peers through citations of their work.[3]

In the literature on growth, several papers look at innovation cycles. Jovanovic and Rob (1990) and Jovanovic and Nyarko (1996) build learning models. In Jovanovic and Rob (1990), an agent can explore a known or an

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[2]Exceptions include Levin and Stephan (1991), Brock and Durlauf (1999), Goyal et al. (2006), see also Stephan (1996) and the references therein.

[3]Scientists could also care for citations because of the financial gains they generate, see e.g. Diamond (1986).

unknown dimension of the technological space. Innovation cycles emerge for an intermediate range of the parameters. In Jovanovic and Nyarko (1996), the agent chooses between a known and a better, but unknown, technology. Permanent upgrading and growth can coexist with technological lock-in and stagnation. Matsuyama (1999) shows how innovation cycles can emerge in an endogenous growth model. Phases where investment is concentrated on old technologies alternate with phases with innovation. Innovation cycles emerge because of the temporary monopoly power enjoyed by innovators. A main novelty of our approach is the citation premium.[4] Through citations, payoffs from a scientist's current choice depend on the future evolution of science. This introduces a non-trivial dynamic linkage between current and future actions. We show that innovation cycles still exist, although their characteristics are affected. Especially, the initial improvement phase is shorter when the citation premium is positive.

The rest of the paper is organized as follows. We introduce the model in section 2. We present our main result in section 3 and interpret it in section 4. We derive comparative statics in section 5 and comparative dynamics in section 6. We show that indeterminacy may appear in section 7, and establish some welfare results in section 8.

2　The model

We consider an infinite horizon model with discrete time. At each date t there is a continuum of existing fields of research, which we index by i. Each field is characterized by a stock of contributions (or 'papers') $n_t(i)$ at the end of period t.

Papers are produced by researchers. Researchers live for two periods,

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[4]Another difference is that the agent can make one search per period in Jovanovic and Rob (1990) and Jovanovic and Nyarko (1996), while in our model research effort is continuous and allocated among different alternatives in every period.

hence we have an overlapping generation structure. In the first period of their life, researchers produce contributions. In the second period of their life, they enjoy the returns from their scientific "reputation", which defines their utility function. A researcher's scientific reputation is the sum of the contribution of each individual paper he or she has written. An individual paper written at date t in existing field i yields the following contribution to its author's reputation:

$$v_t(i) = \omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)).$$

This reputation is the sum of two terms. The first term, $\omega(i) - \beta(\ln n_t(i) - \ln \bar{n})$, defines the *intrinsic* value of the paper. $\omega(i)$ is a field-specific constant which represents the field's value (or initial research potential) as a whole. The term $\beta(\ln n_t(i) - \ln \bar{n})$, where β and $\bar{n}$ are positive parameters, captures the fact that there are decreasing returns to research: the larger the stock of knowledge in field i, the smaller the intrinsic value of additional contributions. The second term, $\theta(\ln n_{t+1}(i) - \ln n_t(i))$, is the *citation premium.* It tells us that the reputation obtained from papers written at t is greater, the greater the flow of further advances in the relevant field at $t + 1$. Underlying this formulation is the idea that papers come in a given order, and that new papers cite previous papers, thus enhancing their author's reputation. Note that contemporaneous papers do not cite each other, so that what matters for citations is the log difference between the stock of papers written at the end of $t + 1$ and that at the end of t.

The total mass of researchers per generation is normalized to 1. Each researcher is endowed with ν units of time. He allocates his time optimally between writing papers in different fields. In addition to that, one may create new fields.

When one writes the first paper in a new field, its potential $\omega(i)$ is drawn from some distribution, with pdf $f(.)$, such that all moments exist. The

realization of $\omega(i)$ is unknown when one decides to write the paper. At the end of the period when the new field is created, its advancement level is set at the minimal value $\bar{n}$. Therefore, one must wait one period before making further contributions to a new field.

We assume that one unit of time produces either 1 paper in an existing field or γ papers in a new field.

We make two technical assumptions that we need to be able to solve the model:

Assumption A1 – If at date t, there is a strictly positive measure of new fields invented, then all fields invented before date t can no longer be researched from date $t + 1$ on.

This assumption is a useful simplification that avoids having to keep track of all the fields ever invented at any date t.[5] Only the fields invented in the last wave of innovation can be exploited at a given date.[6]

Assumption A2 – $\gamma < 1$.

This assumption states that inventing a new field requires more labor than writing a paper in an existingf field. It is a plausible, but merely technical assumption, required to prove the existence of an equilibrium for $\theta > 0$.[7]

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[5]This assumption is consistent with Kuhn's theory: "When it repudiates a paradigm, a scientific community simultaneously renounces most of the books and articles in which that paradigm had been embodied", Kuhn (1970). See also theorem 2 in Jovanonvic and Rob (1990) and the "no-recall" assumption of Jovanovic and Nyarko (1996).

[6]It is not necessary to make this assumption in the special case where $\theta = 0$. In such a case the value of inventing a new field is $V_N = \gamma\bar{\omega} = \gamma E(\omega)$, which is also the lower bound of the value of working on an existing field, since one could always produce new fields instead. Consequently, when new fields are invented, all previous fields reach their maximum advancement level, such that the value of the marginal paper is equal to V_N; they will not be exploited thereafter.

[7]It is again not needed for $\theta = 0$.

3 Equilibrium

In this section, we show the existence of an equilibrium, and the conditions under which it is cyclical as opposed to converging to a steady state. We provide a result for uniqueness in the case where $\theta = 0$. We first discuss the equilibrium conditions of the model in the two regimes of interest. We then state the paper's main result, whose proof is relegated to the Appendix. In the next section, we discuss its economic interpretation using a graphical illustration, confining ourselves to the $\theta = 0$ case. We then work out numerical examples. Finally, we give a sketch of the proof when the citation premium is positive.

3.1 Equilibrium conditions

At any point in time, the economy may be in one of two regimes:

In Regime I, all the research input is allocated to improving existing fields. There exists a shadow value of time λ_t; a field is exploited if and only if the first paper written in the current period has a value greater than λ_t, that is:

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}(i)) > \lambda_t.$$

The number of papers written in such a field, at t, must satisfy

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

The equilibrium value of λ_t must adjust so that the total mass of papers being written is equal to υ. Call s the last period where invention took place, and μ_s the mass of new fields invented at s. The evolution of one of these fields will clearly be a sole function of its intrinsic value ω. Thus n_t can be written as a sole function of ω rather than the field's specific index i. The preceding conditions are equivalent to

$$\omega > (\beta + \theta) \ln n_{t-1}(\omega) - \beta \ln \bar{n} - \theta \ln n_{t+1}(\omega) + \lambda_t, \tag{1}$$

and

$$n_t(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega-\lambda_t}{\beta+\theta}}. \tag{2}$$

The full employment condition can therefore be written as

$$\mu_s \int_{\omega>(\beta+\theta)\ln n_{t-1}(\omega)-\beta\ln\bar{n}-\theta\ln n_{t+1}(\omega)+\lambda_t} (n_t(\omega) - n_{t-1}(\omega))f(\omega)d\omega = v. \tag{3}$$

Finally, the value of writing a new paper, denoted by V_t, must be lower than that of working on an existing field:

$$V_t < \lambda_t.$$

In Regime II, people exploit existing fields, and work on new fields as well. They must be indifferent between the two activities, so that one must have $\lambda_t = V_t$. An existing field is exploited if and only if

$$\omega > (\beta + \theta)\ln n_{t-1}(\omega) - \beta\ln\bar{n} - \theta\ln n_{t+1}(\omega) + V_t. \tag{4}$$

Its advancement level then proceeds to

$$n_t(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega-V_t}{\beta+\theta}}.$$

Because of Assumption (A1), the existing fields will disappear at $t+1$ and be replaced by the mass μ_t of new fields, which will start with advancement level $\bar{n}$ at $t+1$. Therefore, $n_{t+1}(\omega) = n_t(\omega)$, since existing fields at t are no longer exploited at $t+1$. Substituting, the final advancement level is:

$$n_t(\omega) = \bar{n}e^{\frac{\omega-V_t}{\beta}}, \tag{5}$$

while (4) can be rewritten

$$\omega > V_t + \beta(\ln n_{t-1}(\omega) - \ln\bar{n}).$$

Note that this condition collapses to

$$\omega > V_t \tag{6}$$

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if the economy was also in regime II at date $t - 1$, since the field must then have been invented at that date.

In regime II, the resource constraint states that total time devoted to existing fields cannot exceed υ :

$$\mu_s \int_{\omega > \beta(\ln n_{t-1}(\omega) - \ln \bar{n}) + V_t} (n_t(\omega) - n_{t-1}(\omega)) f(\omega) d\omega \leq \upsilon.$$

The remaining time endowment must be devoted to new fields; this determines the mass of new fields invented at t :

$$\mu_t = \gamma \left[\upsilon - \mu_s \int_{\omega > \beta(\ln n_{t-1}(\omega) - \ln \bar{n}) + V_t} (n_t(\omega) - n_{t-1}(\omega)) f(\omega) d\omega \right].$$

Finally, in both regimes, the value of working in a new field V_t is determined as follows. Consider a researcher writing a paper in a new field with value ω. Then, $n_t(\omega) = \bar{n}$. If (1) holds at $t + 1$, which is equivalent to

$$\omega > \theta \ln \bar{n} - \theta \ln n_{t+2}(\omega) + \lambda_{t+1},$$

then the field will be active, and the inventor will benefit from citations. The value to the inventor is then given by

$$V_t(\omega) = \omega + \theta(\ln n_{t+1}(\omega) - \bar{n})$$

where $n_{t+1}(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+2}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega - \lambda_{t+1}}{\beta+\theta}}$.

Otherwise, the field will not be active at $t + 1$, and the inventor just gets the intrinsic value of the first paper:

$$V_t(\omega) = \omega.$$

Thus, the value of working on a new field at t is given by:

$$V_t = \gamma \left[\bar{\omega} + \frac{\theta}{\beta + \theta} \int_{\omega > \lambda_{t+1} - \theta(\ln n_{t+2}(\omega) - \ln \bar{n})} (\omega - \lambda_{t+1} + \theta(\ln n_{t+2}(\omega) - \ln \bar{n})) f(\omega) d\omega \right].$$

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3.2 Existence, uniqueness, and cycles

We now state the central results of the paper. To do so, we need to introduce the following two functions:

$$\Phi(y) = \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_y^{+\infty} (\omega - y) f(\omega) d\omega \right], \tag{7}$$

and

$$I^*(y) = \int_y^{+\infty} (e^{\frac{\omega - y}{\beta}} - 1) f(\omega) d\omega.$$

Most of the equilibrium conditions involving the value of a new field V can be stated in terms of the Φ function; for example, it can be easily shown that if the economy is in regime II in two consecutive periods t and $t + 1$, then $V_t = \Phi(V_{t+1})$. As for I^*, it shows up whenever one is adding progresses across exploited fields to get the associated total labor input. For example, if $\theta = 0$ and fields are exploited for the first time, then the LHS of (3) is equal to $\mu_s \bar{n} I^*(\lambda_t)$.

Both functions are continuous and decreasing. Since $\Phi(0) \geq \gamma \bar{\omega}$ and $\Phi(+\infty) = \gamma \bar{\omega}$, Φ has a fixed point $\bar{V}$:

$$\Phi(\bar{V}) = \bar{V}. \tag{8}$$

The paper's main result can be stated as follows:

PROPOSITION 1 – Assume that the economy starts at $t = 0$ with an initial mass of fields μ_{-1}, whose intrinsic value is distributed with $f()$, and whose initial advancement level is given by $\bar{n}$. Then:

(i) There exists an equilibrium path.

(ii) If

$$I^*(\bar{V}) > \frac{1}{\gamma \bar{n}}, \tag{9}$$

then any equilibrium is cyclical, i.e. periods in regime I alternate with periods in regime II. During periods in regime II, the mass of invented fields follows

explosive oscillations, until the economy reverts to regime I. During periods in regime I, the set of exploited fields grows. The duration of a period in regime I cannot exceed $\gamma \bar{n} I^*(\gamma \bar{\omega})$.

(iii) If

$$I^*(\bar{V}) < \frac{1}{\gamma \bar{n}},$$

then there exists an equilibrium such that

-the economy is in regime II from $t = 0$ on.

-the value of working in a new field is equal to $\bar{V}$ at all dates.

-the mass of invented fields converges to its steady state value, given by

$$\bar{\mu} = \frac{\gamma \nu}{1 + \gamma \bar{n} I^*(\bar{V})},$$

by dampened oscillations.

(iv) If $\theta = 0$, equilibrium is unique.[8]

PROOF – See Appendix.

4   Interpretation

4.1   Without citation premium

To analyze the reason behind cycles, let us focus on the simpler case where $\theta = 0$. In the absence of a citation premium, inventors of new fields just get the intrinsic value of the field, ω, as a reward. Consequently, the value of a new field is pinned down and equal to $V = \gamma \bar{\omega}$ in any period.

Figure 1 plots the value of working in an existing field at any date t, λ_t, as a function of the total input in existing fields; that defines the LL schedule. This curve is downward-sloping, because of decreasing returns, captured by

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[8]We conjecture that the equilibrium is unique for θ small enough, but cannot prove it.

the $-\beta(\ln n_t(i) - \ln \bar{n})$ term in the utility function. For the same reason, its position is lower, the higher the initial advancement level of those fields, $n_{t-1}(i)$. Finally, given that level, its position is higher, the greater the mass of available fields μ_s, since the same total research input is now associated with a lower advancement level $n_t(i)$ in each field.

If, as is the case in Figure 1, that schedule intersects the horizontal line VV at $\lambda = V$, then the economy is in regime II. The horizontal distance AB determines the labor input into new fields, and hence the mass of fields being invented.

If that is not the case, then the economy must be in regime I, and equilibrium determination is illustrated in Figure 2. At date t, all researchers work in existing fields. Advancement in these fields generate a downward shift in LL, and the intercept of the LL schedule for the next period must be equal to λ_t – which simply means that the value of the first marginal paper at $t+1$ in a given field is equal to the value of the last paper written in that field at t. The process continues until the LL schedule cuts the VV schedule, in which case one is back to regime II (at $t+2$ in the case of Figure 2). This must happen in finite time, otherwise decreasing returns would eventually drive VV below the $\lambda = 0$ line. Note that the λ_ts fall during the regime I period. That is the reason why the set of fields being exploited grows during that phase.[9]

What happens, next, in regime II? At each date, a given mass of fields is invented. The greater that mass, the greater the value of exploiting these fields next period (i.e. LL shifts up). On average, one field invented at date t, with a quality distribution $f(\omega)$, triggers an amount $\bar{n}I^*(V)$ of research

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[9]Because of assumption A1, a field exploited during that phase must have been invented in the last period in regime II before the regime I phase. It enters regime I with an initial advancement level equal to $\bar{n}$. Using (1) with $\theta = 0$, it will therefore be exploited as soon as $\omega > \lambda_t$. Because the λs are falling, it will continue to be exploited until the economy reverts to regime II, when it becomes obsolete.

input devoted to exploiting that field at date $t+1$. That reduces the amount of time devoted to innovation: the greater the mass of fields invented today, the lower the mass of fields invented tomorrow. The evolution of μ_t, the mass of fields invented at t, evolves according to

$$\mu_t = \gamma(\upsilon - \mu_{t-1}\bar{n}I^*(V)). \tag{10}$$

If these dynamics are stable ($\bar{n}I^*(V) < 1$, Figure 3), then the economy converges to a steady state. Otherwise, ($\bar{n}I^*(V) > 1$, Figure 4), the economy cannot remain in regime II forever: it will revert to regime I. As regime I itself cannot last forever, the two regimes must prevail alternatively.

The instability condition $\bar{n}I^*(V) > 1$ simply means that a unit of labor employed in inventing a new field today attracts more than one unit of labor into exploiting that field tomorrow. That in turn reduces the amount of labor inventing new fields tomorrow more than one for one, thus generating the explosive oscillatory dynamics and the subsequent exit from regime II. The greater the quantity $I^*(V)$, the more invented fields are attractive, and the more likely it is that cycles arise.

4.2 Constructing an equilibrium

Constructing an equilibrium turns out to be easy in the $\theta = 0$ case. Consider Regime I. At any date, there are two kind of fields: those who are exploited, and those who have never been exploited since their invention. The latter are such that $\omega \leq \lambda_t$ have an advancement level equal to $\bar{n}$; the former, such that $\omega > \lambda_t$, reach an advancement level equal to $n_t(\omega) = \bar{n}e^{\frac{\omega-\lambda_t}{\beta}}$. At $t+1$, all these fields are exploited, plus a mass of extra fields such that $\lambda_t > \omega > \lambda_{t+1}$.

The research input into existing fields at date $t+1$ is therefore equal to

$$v_{At+1} = \mu_s \int_{\omega > \lambda_{t+1}} (n_{t+1}(\omega) - n_t(\omega)) f(\omega) d\omega$$

$$= \mu_s \int_{\lambda_t > \omega > \lambda_{t+1}} (\bar{n} e^{\frac{\omega - \lambda_t}{\beta}} - \bar{n}) f(\omega) d\omega + \mu_s \int_{\omega > \lambda_t} (\bar{n} e^{\frac{\omega - \lambda_{t+1}}{\beta}} - \bar{n} e^{\frac{\omega - \lambda_t}{\beta}}) f(\omega) d\omega$$

$$= \bar{n} \mu_s (I^*(\lambda_{t+1}) - I^*(\lambda_t)).$$

In the first period of regime I, $t = s + 1$, no field has been exploited and the equation boils down to

$$v_{As+1} = \bar{n} \mu_s I^*(\lambda_{s+1}).$$

Regime I proceeds as long as $v_{At} = v$. In the first regime II period, T, after regime I, one must have $\lambda_T = V = \gamma \bar{\omega}$, and $0 \le v_{AT} < v$. Therefore, it must be that

(i) $\lambda_t = I^{*-1}((t - s)\frac{v}{\bar{n} \mu_s})$.

(ii) The duration of the regime I phase is entirely pinned down by the condition $I^{*-1}((T - s)\frac{v}{\bar{n} \mu_s}) \le V < I^{*-1}((T - 1 - s)\frac{v}{\bar{n} \mu_s})$, i.e.

$$T - s - 1 = INT(I^*(V)\frac{\bar{n} \mu_s}{v}). \tag{11}$$

Note that one may have $T = s + 1$, in which case there is no regime I phase. This happens if and only if

$$I^{*-1}(\frac{v}{\bar{n} \mu_s}) \le V. \tag{12}$$

(iii) The economy enters regime II with an initial measure of new fields given by

$$\mu_T = \gamma(v - v_{At})$$
$$= \gamma(v - \bar{n} \mu_s(I^*(V) - I^*(\lambda_{T-1})))$$
$$= \gamma v(1 - DEC(I^*(V)\frac{\bar{n} \mu_s}{v})). \tag{13}$$

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Therefore, given an initial measure of invented fields at s, we can construct a unique, possibly empty, phase in regime I, and compute its length and the initial measure of invented fields in the first period T of the subsequent regime II phase.

The regime II phase is then constructed by mechanically applying (10). The economy asymptotically converges to the steady state if $\bar{n}I^*(V) < 1$; otherwise, it remains in regime II until date s' such that

$$\mu_{s'}\bar{n}I^*(V) > \upsilon \tag{14}$$

This gives the measure of invented fields available for the next regime I phase. One can then iterate the procedure. Note that s' is determined uniquely: on the one hand, the economy cannot remain in regime II after the first date which satisfies (14), as $\mu_{s'+1}$ would then be negative. On the other hand, if one picks up a date prior to s' for the transition to regime I, it would satisfy (12), thus leading to an empty regime I phase, which would not change the equilibrium.

It should be noted that equation (13), which related the mass of invented fields after a phase in regime I to the mass of invented fields before that phase, is discontinuous and highly non linear. Thus we expect cycles to be irregular: the characteristics of a cycle, such as the time spent in each regime and the mass of invented fields, are almost "unpredictable" from the characteristics of the preceding cycle.[10]

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[10] A similar nonlinearity applies to the mass of invented fields at the end of phase II, and the duration of phase II, as a function of the initial mass of invented fields μ_T. Solving for (10) yields

$$\mu_{t+1} = \frac{\gamma\nu}{1 + \gamma\bar{n}I^*} + \left(\mu_T - \frac{\gamma\nu}{1 + \gamma\bar{n}I^*}\right)(-\gamma\bar{n}I^*)^{(t-T)}.$$

One has to distinguish between two cases.

If $\mu_T > \frac{\gamma\nu}{1+\gamma\bar{n}I^*}$, then $s' - T$ must be even, and one must have

4.3 Numerical illustration

In this section we provide some simulations in order to get a better idea of the irregular nature of the innovation cycles. We assume that the quality of a field ω is drawn from a uniform distribution over $[0, \omega_u]$, implying $\bar{\omega} = \omega_u/2$.

Figures 5 to 10 report the simulation results for the following set of parameters: $\bar{n} = 2$; $\omega_u = 1$; $\beta = 0.3$; $\gamma = 0.7$; $\nu = 1$. The initial measure of existing fields was taken as $\mu_c = 1$.

It is easy to show that (9) holds in this case, so that the equilibrium must be cyclical. The simulation shows that the economy follows cycles that are irregular, both in the duration spent in regime I and the duration spent in regime II. The time spent in regime I oscillates between 1 and 2 periods (Fig. 5), while time spent in regime II oscillates between 1 and up to 6 periods (Fig. 8)[11]. There are also chaotic oscillations in the stock of new fields available for exploration at the beginning of each regime I phase (Figure 6). Furthermore, as (11) predicts, there is a tight connection between that initial stock and the length of the time spent in period 1 (Fig. 7); the regime I cycle lasts for 2 periods if the initial stock of knowledge is $> \approx 0.6$, and for 1 period otherwise.

Figure 9 reports the average rate of innovation during the time spent in regime II. We see that it exhibits irregular fluctuations. We also see (Figure

$$s' = T + 2INT\left(\frac{\ln\frac{\gamma\nu}{\mu_{T(\mu_c)+1}(1+\gamma nI^*)-\gamma\nu}}{2\ln(\gamma\bar{n}I^*)} + \frac{1}{2}\right).$$

If $\mu_T < \frac{\gamma\nu}{1+\gamma\bar{n}I^*}$, then

$$s' = 2INT\left(\frac{\ln\frac{\gamma\nu}{\mu_{T(\mu_c)+1}(1+\gamma nI^*)-\gamma\nu}}{2\ln(\gamma\bar{n}I^*)}\right) + 1.$$

As the initial measure μ_T changes, the duration of phase II will jump, and so will $\mu_{s'}$. Therefore, we also have discontinuities in the function which related $\mu_{s'}$ to μ_T.

[11] These figures report the 70 first cycles after the initial one.

10), that cycles where a longer time is spent in regime II, have a lower rate of innovation. Intuitively, if a large number of researchers produce new fields, it is more likely that the economy reverts to regime I in the following period in order to exploit the potential of these new fields[12].

Relative to that benchmark simulation, we can perform some exercices. Figures 11 and 12 report the structure of cycles when we reduce the decreasing returns parameter from $\beta = 0.3$ to $\beta = 0.2$.[13] We see that overall, the economy spends more time in regime I and less time in regime II. In a cycle, regime I last between 1 and 5 periods, although that is quite often just 1 period, and regime II typically does not exceed 2 periods, although there are very rare occurences of cycles where the economy spends 3 periods in regime II.

In fact, while there is a maximum duration for the regime I phase, if the dynamics are truly chaotic one will have (very rare) regime II phases of arbitrary length. The reason is that the initial values of μ will span all the $[0, \gamma v]$ interval, becoming sometimes arbitrarily close to the unstable steady state value $\bar{\mu}$.

4.4 With citation premium

Two main difficulties arise when trying to extend the previous analysis to the case $\theta > 0$. First, current exploitation of an existing field now depends on the future evolution of the field. Thus, $n_t(\omega)$ depends on λ_t and $n_{t+1}(\omega)$ in eq. (2). And second, the value of writing a paper in a new field V_t will

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[12]Another interesting property of that simulation, is that cycles where regime I lasts for two periods, are such that the economy only spends 1 period in regime II. The explanation could be as follows: at the end of such cycles, fields are quite exhausted, and the value of working in new fields in regime II is quite high. Thus a large mass of innovation will take place during a short period of time, after which people revert to exploiting the new fields.

However, this explanation is incomplete, since longer cycles are also those with a higher total initial potential. And that regularity is not robust to parameter changes.

[13]Given the richer results, simulation are reported over 140 cycles rather than 70.

generally vary with t.

In the Appendix, we show that the first issue can be addressed by backward induction from the last period of exploitation. Consider fields being exploited in a regime I episode, lasting from $s + 1$ to $T - 1$. Their exploitation ends at T, the first new period in regime II. At T, there are no more future citations for these fields, hence $n_T(\omega)$ only depends on V_T. This determines the citation premium at $T - 1$ and $n_{T-1}(\omega)$. Reasoning backward, we show that the analysis for $\theta = 0$ can be replicated through an appropriate change of variable on the shadow value of time λ_t. In the end, research dynamics in regime I episodes only depend on two factors: the mass of fields originally available, μ_{s+1}, and the value of invention at the end, V_T. Especially, eqs. (11) and (13) directly extend. Duration of regime I phase is now given by

$$ T - s - 1 = INT(I^*(V_T)\frac{\bar{n}\mu_s}{\nu}) $$

while the mass of new fields invented at T is

$$ \mu_T = \gamma\upsilon(1 - DEC(I^*(V_T)\frac{\bar{n}\mu_s}{\upsilon})) $$

Next, we describe how the value of writing a paper in a new field, V_t, evolves. This evolution differs within regimes and at regime transitions. Within regime I, changes in V_t do not matter, as new fields are never invented. Within regime II, it is easy to see that $V_t = \Phi(V_{t+1})$. The final, key equation is provided by the transition from regime II to regime I. Let s be a last period in regime II. The citation premium for new fields invented at s depend on the extent of their exploitation at $s + 1$, the first period of the new regime I phase. In turn, this mainly depends on the mass of new fields invented at s, μ_s. More precisely, we show in Appendix that:

$$ V_s = \Phi(I^{*-1}(\frac{\nu}{\mu_s\bar{n}})) $$

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Finally, we show that an equilibrium indeed exists. The proof is complicated by the fact that the function describing the evolution of μ is not continuous, nor contracting. We carefully construct sequences avoiding the discontinuities.

We are now in a position to analyze how the parameters of interest affect the equilibrium. We do it in two steps: first, we look at the case where (9) does not hold, and perform local comparative statics around the steady state. Second, we consider how the structure of cycles is affected by the model parameters when (9) holds.

5 Comparative statics

In this section, we perform local comparative statics around a regime II steady state. In particular, we are interested in how the equilibrium level of invention, $\bar{\mu}$, is affected by

- The citation premium θ,

- The distribution of field quality $f(\omega)$; in particular: how does the riskiness of invention, measured by the variance of $f(.)$, affect the equilibrium allocation of effort between innovation and exploitation?

- The strength of decreasing returns β.

5.1 The effect of the citation premium

Equation (7) clearly implies that $\bar{V}$ is an increasing function of θ. Furthermore, one can straightforwardly check that $dI^*/d\bar{V} < 0$. Hence, $d\bar{\mu}/d\bar{V} > 0$. Consequently,

PROPOSITION 2 – More research input is devoted to new fields, the higher the citation premium θ.

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This result is not totally obvious. In principle, the citation premium increases incentives to work both in new fields and in existing fields. However, in this equilibrium, existing fields are only exploited during one period; thus one earns no citation premium on them. An increase in θ thus clearly increases the value of working on new fields.

5.2 The role of research uncertainty

Next, we look at the role of uncertainty in research; we want to know how the variance of ω – or any mean-preserving spread parameter denoted by σ – affects the arbitrage between working in new fields vs. existing fields. As we shall see, option values intervene in two conflicting ways.

We first note that $I^*(V)$ can be written as $E(z(\omega))$, where $z(\omega) = \max(e^{\frac{\omega(i)-V}{\beta}} - 1, 0)$ is a convex function of $\omega(i)$. By Jensen's inequality, a mean-preserving spread in the distribution of ω raises $I^*(V)$ for any given V. If V were to remain unchanged, or move by only a little, μ would actually fall: more research uncertainty reduces the incentives to work in new fields.

If $\theta = 0$, it is actually true that $\bar{V}$ does not change in response to a mean-preserving change in the distribution of ω, for it is equal to $\gamma\bar{\omega}$. Similarly, eq. (7) implies that for θ arbitrarily small the change in $\bar{V}$ can be made arbitrarily small. Therefore:

PROPOSITION 3 – For θ small enough a mean-preserving spread in the distribution of ω reduces μ_∞.

$$\frac{\partial \mu_\infty}{\partial \sigma} < 0.$$

Uncertainty increases the value of existing fields because one can select those of them with the highest potential. A greater variance of ω means that it is more valuable to work in the top field, while the value of working in the

bottom fields is unchanged, because these fields are abandoned anyway. In contrast, the value of writing the first paper in an unknown field is increased is the field turns out to be good, but reduced if it turns out to be bad – if θ is small, then that value will roughly equal $\omega(i)$, regardless of the fate of the field after its invention. Hence greater uncertainty increases the value to work in known fields relative to unknown, new fields.

Against that logic, runs the fact that uncertainty increases the value of new fields, because of the citation premium. That is apparent from (7): a mean-preserving spread in ω increases its RHS. The option value of working in an existing field only if it is good enough also affects the value of working in new fields through the citation premium. When uncertainty goes up, researchers gain from their good ideas being cited more, but do not lose from their bad, uncited ideas, being cited less. In other words, the higher the citation premium, the less risk-averse the researchers.

PROPOSITION 4 – A mean preserving spread in ω increases $\bar{V}$:

$$\frac{\partial \bar{V}}{\partial \sigma} > 0.$$

The effect is larger, the larger the citation premium:

$$\frac{\partial^2 \bar{V}}{\partial \sigma \partial \theta} > 0.$$

Consequently, the larger the citation premium, the lower the negative effect of uncertainty on the research input into new fields:

$$\frac{\partial^2 \mu_\infty}{\partial \theta \partial \sigma} > 0.$$

An interesting question is: can the reduction in risk aversion induced by the citation premium be so strong as to overturn the direct effect of uncertainty, so that one would have $\frac{\partial \mu_\infty}{\partial \sigma} > 0$?[14]

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[14]We ran simulations, not shown here, with uniform distributions for ω. In these simulations, the overall effect of uncertainty on μ was always negative.

5.3 The role of decreasing returns

A similar trade-off appears regarding the effect of β. It is easy to see that $I^*(V)$ is a decreasing function of β. Thus, when $\theta = 0$ or θ is small enough, μ increases when β increases. Since existing fields lose their value more quickly, while the rewards from invention are (almost) unchanged, researchers devote more time to invention. As with uncertainty, an opposite effect comes into play through the citation premium. Observe that $\bar{V}$ is decreasing in β when $\theta > 0$. The citation premium is lower since with a higher β fields are being less exploited. Consequently, the value of writing a paper in a new field is lower (they will be cited less), which tends to counteract the first effect.

6 Comparative dynamics

Given the highly nonlinear nature of our cycles, it is difficult to establish comparative dynamics results. However, Proposition 5 establishes some results about the impact of the citation premium on the likelihood and structure of cycles. Typically, the casual idea that a greater citation premium makes "fads" more important and therefore cycles more likely, is not supported by the model. The reason is that the value of new fields goes up with the citation premium, which reduces the attractivity of existing fields, thus making it less likely that instability arises in (10). As the next section shows, however, a larger citation premium makes fads more likely in an indeterminacy sense.

PROPOSITION 5 – (i) The equilibrium is less likely to be cyclical, the greater θ.

(ii) Conditional on the initial mass of invented fields, the economy spends less time in the regime I phase for $\theta > 0$ than for $\theta = 0$. Furthermore, if the amount of time spent in regime I is the same, then more invention takes place at the beginning of the subsequent regime II phase, if $\theta > 0$.

PROOF – See Appendix.

7   Indeterminacy and "sunspots"

The greater θ, the more expectations about future citations have a strong effect on the decision to work on a given field. By analogy with the literature on indeterminacy, we can speculate that there are multiple equilibria for θ large enough. That is actually the case. The following result shows that there is local indeterminacy around the regime II steady state for large enough values of θ.

PROPOSITION 6 – Assume

$$\frac{\gamma\theta}{\beta}(1 - F(\bar{V})) > 1$$

and

$$I^*(\bar{V}) < \frac{1}{\gamma\bar{n}}.$$

Then there exists a continuum of equilibria indexed by any initial value $V_0 = \bar{V} + v_t$, for v_t sufficiently small.

PROOF – See Appendix.

This indeterminacy affects the value of invention. If scientists think that opening new fields brings a higher payoff, they devote more effort to invention. The mass of papers in new fields is higher. This increases subsequent research in the better of these new fields. The citation premium originally associated to the new fields is thus effectively larger, which confirms the original expectation. In short, expecting invention to bring a high payoff can be a self-fulfilling prophecy.

8 Some welfare results

Due to the complexity of our model, it is not easy to make a thorough comparison between the equilibrium and the social optimum. However, it is possible to compare the steady state in regime II to its equivalent for the social planner. That is what we do in this section.

In order to perform a welfare analysis, a criterion is needed. There are many options since our model only specifies the value of innovation to researchers. An ample literature discusses the appropriability problems associated with research. Here we want to use our model to focus on only one market failure, which is that the stock of knowledge created by researchers is durable and will benefit future generations beyond their lifetime. We then show that absent a citation premium the value of a new field in the equilibrium steady state is lower than at the optimum steady state, and that an optimal "pigovian" citation premium can be introduced so as to induce the socially optimal level of fundamental. We provide a formula for computing this citation premium.

The social welfare function we use is as follows. At each date t there is a stock of knowledge K_t, which grows because of the introduction of new fields and because of improvements in existing fields. We assume that the increase in the stock of knowledge is equal to the intrinsic value of all papers written at date t. Thus, the intrinsic value perceived by each researcher captures well their contribution to the knowledge stock. Researchers only fail to internalize the fact that their contribution increases the stock of knowledge forever. They get rewards from the flow of ideas they produce while society gets rewards from the stock of ideas.

We capture that with an intertemporal social welfare function given by

$$SW = \sum_{t=0}^{+\infty} \frac{K_t}{(1+\phi)^t},$$

where K_t is the stock of knowledge at t and ϕ the social discount rate, which can conveniently be interpreted as an inverse measure of the weight put on future generations. The evolution of the knowledge stock is then given by, in regime II,

$$K_t = K_{t-1} + \mu_t \int \omega f(\omega) d\omega + \mu_{t-1} \int_\omega \left(\int_{\bar{n}}^{n_t(\omega)} (\omega - \beta(\ln z - \ln \bar{n})) dz \right) f(\omega) d\omega.$$

The first integral is the initial value of the fields invented at date t. The second integral is the contribution of the improvements made during t to the fields invented at $t-1$. Note that we integrate the marginal contribution of all papers ranked between $\bar{n}$ and n_t. This guarantees that researchers internalize the congestion externality they exert upon others by moving, through their contribution, the state of the field down the marginal value curve. In other words, the intrinsic value of writing a paper in a field with potential ω is equal to its marginal effect on K, $(\omega - \beta(\ln n_t - \ln \bar{n}))$.

This equation may be rewritten

$$K_t = K_{t-1} + \mu_t \bar{\omega} + \mu_{t-1} \int_\omega \left((\omega + \beta) \left(n_t(\omega) - \bar{n} \right) - \beta n_t \left(\ln n_t - \ln \bar{n} \right) \right) f(\omega) d\omega. \tag{15}$$

It can easily be shown that, as in the equilibrium, given the fraction of researchers who work in new fields, it is optimal to allocate the others so as to equate their intrinsic marginal value across active fields. Otherwise, one could reallocate the research effort across existing fields to get a higher value of the last term in (15). Consequently, at each date there exists a critical field ω_t^* such that $n_t(\omega) = \bar{n}$ for $\omega < \omega_t^*$ and $n_t(\omega) = \bar{n} e^{\frac{\omega - \omega_t^*}{\beta}}$ for $\omega \geq \omega_t^*$. In steady state, ω_t^* will be constant through time. Using this property, the evolution equation for knowledge can be rewritten as

$$K_t = K_{t-1} + \mu_t \bar{\omega} + \mu_{t-1} \Gamma(\omega_t^*),$$

with

$$\Gamma(\omega^*) = \bar{n} \int_{\omega^*}^{+\infty} \left[(\beta + \omega^*) e^{\frac{\omega - \omega^*}{\beta}} - (\beta + \omega) \right] f(\omega) d\omega.$$

The social planner's problem can be rewritten recursively by introducing the value function

$$V(\mu_{t-1}, K_{t-1}) = \max(K_t + \frac{1}{1 + \phi} V(\mu_t, K_t)).$$

Maximization takes place with respect to x_t, the fraction of research allocated to new fields. We thus have

$$\mu_t = \gamma \upsilon x_t, \tag{16}$$

while the resource constraint allows to compute ω_t^* as a function of x. Aggregating the number of papers written in existing fields, we get

$$\upsilon(1 - x) = \mu_{t-1} \bar{n} I^*(\omega_t^*). \tag{17}$$

PROPOSITION 7 – The steady-state, welfare maximizing value of ω_t^* is determined by the following equation:

$$\omega^* = \Psi(\omega^*),$$

where $\Psi(.)$ is a decreasing function defined by

$$\Psi(\omega^*) = \gamma \bar{\omega} + \frac{\bar{n}\gamma}{1 + \phi} \int_{\omega^*}^{+\infty} \Delta(\omega - \omega^*) f(\omega) d\omega, \tag{18}$$

and where $\Delta(.)$ is a positive, increasing, convex function defined by $\Delta(x) = \beta(e^{x/\beta} - 1) - x$.

PROOF – See Appendix.

The critical level ω^* is the social opportunity cost of working in an existing field rather than a new field. Its equivalent in the analysis of the equilibrium is

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V_t, which is equal to $\bar{V}$, the fixed point of Φ in the equilibrium. Furthermore, (5) and (6) show that a market economy will allocate employment across existing fields in exactly the same way as the social optimum if $\bar{V} = \omega^*$. Since the resource constraints (16) and (17) are the same in the equilibrium case and the optimum case, all that is needed to compare the equilibrium with the optimum is to compare the fixed point of Φ with that of Ψ. If they coincide, then the equilibrium steady state is identical to the social optimum steady state. Confronting (7) with (18) we then get that the two fixed points coincide provided the citation premium is equal to

$$\theta^* = \frac{\bar{n}\beta}{1+\phi} \frac{\int_{\omega^*}^{+\infty} \Delta(\omega - \omega^*)f(\omega)d\omega}{\int_{\omega^*}^{+\infty}(\omega - \omega^*)f(\omega)d\omega}.$$

This citation premium goes down with ϕ, which means that it must be higher when the social planner cares more about future generations.[15] That is because the social planner puts more weight on subsequent improvements of a new field, the lower ϕ. The value of these subsequent improvements–which raises the value of a new field beyond its contemporaneous effect $\bar{\omega}$– is internalized by the inventor only through the citation premium. Thus it must go up when ϕ goes down.

9   Conclusion

This paper has developed a simple model of the allocation of effort between fundamental research, which invents new fields, and applied research, which improves existing fields. Despite the model's simplicity, our results are quite rich.

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[15]To see this, simply rewrite (8) as $\bar{V} = \Phi(\bar{V}; \theta)$, $\Phi'_1 < 0$, $\Phi'_2 > 0$, and (18) as $\omega^* = \Psi(\omega^*, \phi)$, $\Psi'_1 < 0, \Psi'_2 < 0$. The welfare maximizing value of θ, θ^*, is the unique solution to $\omega^* = \Phi(\omega^*; \theta)$. Hence, $\partial\theta^*/\partial\omega^* > 0$. Since ω^* falls with ϕ, so does θ^*.

We were able to characterize the cyclical dynamics of the economy and derive a necessary and sufficient condition for cycles to arise. We have shown that indeterminacy may also appear, and that the citation premium makes the equilibrium less cyclical, but at the same time makes indeterminacy more likely.

We have also established some comparative statics for a steady-state in regime II, and to compare this steady state to the welfare optimum. We were able to highlight the role of the option value in determining the optimal and equilibrium allocation of research between the two activities.

10   Appendix:

10.1   Proof of proposition 1

λ_t is the shadow cost of a paper at t. Clearly one must have $\lambda_t > \gamma\bar{\omega}$, which is a lower bound for V_t, the value of inventing a new field.

At date t, a field i is exploited if and only if

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}(i)) < \lambda_t,$$

in which case n_t is determined by

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

A. One cannot forever remain in regime I

As if the $\theta = 0$ case, we first prove that one cannot stay forever in regime I. We do so by contradiction.

Assume the economy is always in regime I from t on. Call μ the total mass of existing fields, indexed by i, and $dF(i)$ their density. Assume a field is exploited at t. Then

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t > \gamma\bar{\omega}; \text{ therefore,}$$

$$\ln n_{t+1}(i) \geq \frac{\beta + \theta}{\theta} \ln n_t(i) + \frac{\gamma\bar{\omega}}{\theta} - \frac{\omega(i)}{\theta} - \frac{\beta \ln \bar{n}}{\theta}.$$

A sufficient condition for $n_{t+1}(i) > n_t(i)$ is therefore

$$n_t(i) > \bar{n}e^{\frac{\omega(i) - \gamma\bar{\omega}}{\beta}} = k(i) \tag{19}$$

. Hence, if a field is exploited at t and satisfies (19), then it will be exploited at $t + 1$. Since $n_{t+1}(i)$ in turn also satisfies (19), one has $n_{t+2}(i) > n_{t+1}(i)$, and so on. Therefore:

If a field satisfies (19) and is exploited at t, i.e. $n_t(i) > n_{t-1}(i)$, then it is exploited forever.

Assume that at date t there is a set Λ with strictly positive measure of active fields such that (19) holds. Then, for each of these fields, $n_{t+s+1}(i) = n_{t+s}(i) \left(\frac{n_{t+s}(i)}{k(i)}\right)^{\frac{\beta}{\theta}}$. The quantity $n_{t+s+1}(i) - n_{t+s}(i)$ is growing without bounds, which contradicts the requirement that $\mu \int_\Lambda (n_{t+s+1}(i) - n_{t+s}(i))dF(i) \leq \nu$. Consequently, such a set cannot exist. Let then Λ_t be the set of all active fields at t. It must be that (19) is violated almost everywhere over Λ_t. Let $\kappa_t(i) = \max(k(i) - n_t(i), 0) \geq 0$. We have that

$$\int_{\Lambda_t} n_t(i)dF(i) = \int_{\Lambda_t} n_{t-1}(i)dF(i) + \frac{\nu}{\mu}$$

Furthermore, as $n_{t-1}(i) < n_t(i) < k(i)$ almost everywhere:

$$
\begin{aligned}
\int_{\Lambda_t} \kappa_t(i)dF(i) &= \int_{\Lambda_t} (k(i) - n_t(i))dF(i) \\
&= \int_{\Lambda_t} (k(i) - n_{t-1}(i))dF(i) - \frac{\nu}{\mu} \\
&= \int_{\Lambda_t} \kappa_{t-1}(i)dF(i) - \frac{\nu}{\mu}.
\end{aligned}
$$

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On the other hand, $n_t(i) = n_{t-1}(i)$ for $i \notin \Lambda_t$, implying $\kappa_t(i) = \kappa_{t-1}(i)$. Therefore:

$$\int_\Omega \kappa_t(i)dF(i) = \int_\Omega \kappa_{t-1}(i)dF(i) - \frac{\nu}{\mu}.$$

We have constructed a sequence of positive functions whose integral eventually becomes negative, which is a contradiction.

B. Characterizing dynamics in regime I

Let then T be the date when regime I ends: at date T one is in regime II. Let V_T be the value of working in a new field at T. By assumption, all fields invented prior to T are obsolete after $T+1$. Hence T must be the last period when fields active during the regime I phase are exploited. An existing field is active at T iff

$$\omega(i) - \beta(\ln n_{T-1} - \ln \bar{n}) > V_T,$$

in which case n_T is determined by

$$\omega(i) - \beta(\ln n_T - \ln \bar{n}) = V_T.$$

Consider now a date $t < T$ in the regime I phase. Denoting by λ_t the shadow cost of a paper, a field is active iff

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}) > \lambda_t$$

In which case

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

We now construct a sequence $\hat{\lambda}_t$ such that the following property holds:

PROPERTY P1 – A field is active iff $\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) > \hat{\lambda}_t$, in which case $\omega(i) - \beta(\ln n_t - \ln \bar{n})) = \hat{\lambda}_t$.

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The sequence is constructed by backward induction, starting from $t = T$. We clearly can pick $\hat{\lambda}_T = V_T$. Now, assume P1 holds for $t' > t$.

Assume $\lambda_t > \hat{\lambda}_{t+1}$. Then all active fields at t must also be active at $t+1$. To prove so, suppose there is a field i active at t and inactive at $t+1$. Then it must be that $n_{t+1}(i) = n_t(i)$. So that $\omega(i) - \beta(\ln n_t - \ln \bar{n})) \leq \hat{\lambda}_{t+1}$ and $\omega(i) - \beta(\ln n_t - \ln \bar{n})) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t = \omega(i) - \beta(\ln n_t - \ln \bar{n}))$, which violates the assumption that $\lambda_t > \hat{\lambda}_{t+1}$.

Then, a field is active at t if and only if

$$\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) + \theta(\ln \bar{n} + \frac{\omega(i) - \hat{\lambda}_{t+1}}{\beta} - \ln n_{t-1}) > \lambda_t$$

If we define $\hat{\lambda}_t = \frac{\theta \hat{\lambda}_{t+1} + \beta \lambda_t}{\theta + \lambda}$, we get that this equation is equivalent to $\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) > \hat{\lambda}_t$, and we can check that we then have $\omega(i) - \beta(\ln n_t - \ln \bar{n})) = \hat{\lambda}_t$.

Assume $\lambda_t \leq \hat{\lambda}_{t+1}$. Consider a field active at both t and $t+1$. Then we must have $\omega(i) - \beta(\ln n_t(i) - \ln \bar{n})) < \omega(i) - \beta(\ln n_t - \ln \bar{n})) + \theta(\ln n_{t+1}(i) - \ln n_t(i))$ $= \lambda_t \leq \hat{\lambda}_{t+1} = \omega(i) - \beta(\ln n_{t+1}(i) - \ln \bar{n}))$, implying that $n_{t+1}(i) < n_t(i)$, which cannot be. Therefore, all fields active at t must be inactive at $t+1$, in which case we just pick up $\hat{\lambda}_t = \lambda_t$.

To summarize, the $\hat{\lambda}_t$ sequence can be constructed as

$$\begin{aligned} \hat{\lambda}_T &= V_T; \\ \hat{\lambda}_t &= \min(\frac{\theta \hat{\lambda}_{t+1} + \beta \lambda_t}{\theta + \beta}, \lambda_t). \end{aligned} \tag{20}$$

Let T_0 be the initial period of that phase in regime I. Denoting by μ_{T_0-1} the measure of exploitable fields, we can now get the evolution of the $\hat{\lambda}_t$s.

We have that

$$\mu_{T_0-1} \int_{\omega(i) > \hat{\lambda}_{T_0}} (\bar{n} e^{\frac{\omega(i) - \hat{\lambda}_{T_0}}{\beta}} - \bar{n}) = \nu,$$

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or equivalently

$$\mu_{T_0-1}\bar{n}I^*(\hat{\lambda}_{T_0}) = \nu. \tag{21}$$

This equation allows to compute $\hat{\lambda}_{T_0}$ as a function of μ_{T_0-1}. At date T_0+1, there are two kinds of fields: those which were exploited at T_0, whose value of $n_{T_0}(i)$ satisfies (P1), and those which were not, such that $n_{T_0}(i) = \bar{n}$. If $\hat{\lambda}_{T_0+1} \geq \hat{\lambda}_{T_0}$, no field can be exploited at $t+1$, which is not possible. Therefore it must be that $\hat{\lambda}_{T_0+1} < \hat{\lambda}_{T_0}$. One can then compute $\hat{\lambda}_{T_0+1}$ as

$$\mu_{T_0-1}\int_{\omega(i)>\hat{\lambda}_{T_0}} (\bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0+1}}{\beta}} - \bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0}}{\beta}})dF(i) + \mu_{T_0-1}\int_{\hat{\lambda}_{T_0}>\omega(i)>\hat{\lambda}_{T_0+1}} (\bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0+1}}{\beta}} - \bar{n})dF(i) = \nu,$$

or equivalently

$$\mu_{T_0-1}\bar{n}(I^*(\hat{\lambda}_{T_0+1}) - I^*(\hat{\lambda}_{T_0})) = \nu. \tag{22}$$

Similarly, assuming $\hat{\lambda}$ is falling between T_0 and t, at the beginning of $t+1$, fields can be split between those which were never exploited, so that $n_t(i) = \bar{n}$ and those which were exploited at t, such that $n_t(i) = \bar{n}e^{\frac{\omega(i)-\hat{\lambda}_t}{\beta}}$. Again, it must be that $\hat{\lambda}_{t+1} < \hat{\lambda}_t$, so that the $\hat{\lambda}$s must fall by induction. And they must again satisfy

$$\mu_{T_0-1}\bar{n}(I^*(\hat{\lambda}_{t+1}) - I^*(\hat{\lambda}_t)) = \nu. \tag{23}$$

Given that $\hat{\lambda}_{t+1} < \hat{\lambda}_t$, it must be that $\lambda_t > \hat{\lambda}_{t+1}$, so that active fields at t remain so until the end of regime I.

In regime I, the value of working on a new field must not exceed the value of working in existing fields. Consider a field invented at t. As this field would be infinitesimal, it would not make other fields obsolete at $t+1$. Its value at t is

$$\begin{aligned} W(i) &= \omega(i) + \theta(\ln n_{t+1}(i) - \ln\bar{n}) \\ &= \omega(i) + \frac{\theta}{\beta}(\omega(i) - \hat{\lambda}_{t+1}), \text{ if } \omega(i) > \hat{\lambda}_{t+1}, \end{aligned}$$

and

$$W(i) = \omega(i)$$

if not. Therefore, the value of working on a new field at t is equal to

$$
\begin{aligned}
V_{Nt} &= \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_{\omega(i) > \hat{\lambda}_{t+1}} (\omega(i) - \hat{\lambda}_{t+1}) \right] \\
&= \Phi(\hat{\lambda}_{t+1}).
\end{aligned}
$$

Hence for the economy to be in regime I, the following must hold:

$$\Phi(\hat{\lambda}_{t+1}) < \lambda_t. \tag{24}$$

C. Characterizing dynamics in regime II.

We now characterize the dynamics in regime II. Because a positive measure of new fields is invented at every period, fields invented at t are at most only exploited at $t + 1$. Such a field i will be exploited at $t + 1$ if and only if

$$\omega(i) > V_{t+1},$$

in which case

$$\ln n_{t+1} = \ln \bar{n} + \frac{\omega(i) - V_{t+1}}{\beta}.$$

Consequently, the value of a new field at t is

$$W(i) = \omega(i) + \frac{\theta}{\beta}(\omega(i) - V_{t+1}), \ \text{if } \omega(i) > V_{t+1},$$

and

$$W(i) = \omega(i)$$

if not. Integrating, we get that

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$$V_{Nt} = \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_{\omega(i) > V_{Nt+1}} (\omega(i) - V_{t+1}) \right] \qquad (25)$$
$$= \Phi(V_{t+1}).$$

This defines the dynamics of V_{Nt} in regime II. Denoting now by μ_t the measure of fields invented at t, the input into working in existing fields at $t + 1$ is

$$\nu_{At+1} = \mu_t \int_{\omega(i) > V_{Nt+1}} (\bar{n} e^{\frac{\omega(i) - V_{t+1}}{\beta}} - \bar{n})$$
$$= \bar{n} I^*(V_{t+1}) \mu_t.$$

Therefore the dynamics of μ are given by

$$\mu_{t+1} = \gamma(\nu - \bar{n} I^*(V_{t+1}) \mu_t). \qquad (26)$$

To remain in regime II this formula must yield a positive value of μ_{t+1} throughout.

D. The transition from regime I to regime II

Consider now the first period in regime II, T. We focus on the case where $V_T < \hat{\lambda}_{T-1}$. The other possibility will be ruled out further below. That inequality is equivalent to

$$I^*(\hat{\lambda}_{T-1}) < I^*(V_T) \qquad (27)$$

Then, fields active during regime I are still exploited at T. The total input into active fields at T is given by

$$\mu_{T_0 - 1} \bar{n} (I^*(V_T) - I^*(\hat{\lambda}_{T-1})) = \nu_{AT}.$$

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For this to be consistent with equilibrium, we need that $\nu_{At} < \nu$, that is:

$$I^*(\hat{\lambda}_{T-1}) > I^*(V_T) - \frac{\nu}{\mu_{T_0-1}\bar{n}}. \tag{28}$$

Because of (22), for a given V_T and a given μ_{T_0-1}, there exists at most a unique value of T such that (28) and (27) simultaneously hold. Therefore, the duration (and characteristics) of the regime I phase are entirely pinned down by the initial measure of exploitable fields μ_{T_0-1} and the terminal value V_T. The phase lasts at least one period if and only if $I^*(\hat{\lambda}_{T_0}) < I^*(V_T)$, or equivalently

$$I^*(V_T) > \frac{\nu}{\mu_{T_0-1}\bar{n}}. \tag{29}$$

Otherwise, there cannot be a phase in regime I (we are in the special case where $T = T_0$.)

If (29) holds, then the initial measure of invented fields at the beginning of regime II is

$$\mu_T = \gamma(\nu - \mu_{T_0-1}\bar{n}(I^*(V_T) - I^*(\hat{\lambda}_{T-1}))). \tag{30}$$

The economy then evolves as described above.

E. The transition from regime II to regime I

Next, consider the value of inventing a new field at date $T_0 - 1$. It is given by

$$V_{T_0-1} = \Phi(\hat{\lambda}_{T_0}) \tag{31}$$

This defines a negative relationship between $\hat{\lambda}_{T_0}$ and V_{T_0-1}. At the same time, (21) defines $\hat{\lambda}_{T_0}$ uniquely as an increasing function of μ_{T_0-1}. Thus, there must be a one-to-one, decreasing relationship between μ_{T_0-1} and V_{T_0-1} :

$$V_{T_0-1} = \Phi(I^{*-1}\left(\frac{\nu}{\mu_{T_0-1}\bar{n}}\right)). \tag{32}$$

F. Ruling out the case $V_T > \hat{\lambda}_{T-1}$.

The above argument about the decreasing sequence $\hat{\lambda}_t$ does not apply to its terminal value V_T, since it rests on the argument that all labor goes into existing fields. Let us now examine the case $V_T > \hat{\lambda}_{T-1}$. If this holds, no existing field is exploited at T. Thus, it must be that $\mu_T = \gamma\nu$. Because of (20), it must also be that $\lambda_{T-1} = \hat{\lambda}_{T-1} > V_{T-1} = \Phi(V_T)$. Therefore, $V_T > \Phi(V_T)$, implying that $V_T > \bar{V}$, where $\bar{V}$ is the steady state value V in regime II; $\bar{V} = \Phi(\bar{V})$.

Assume the economy is still in regime II at $T+1$. Then, (26) implies that

$$\mu_{T+1} = \gamma(\nu - \bar{n}I^*(V_{T+1})\gamma\nu).$$

Since $V_{T+1} = \Phi^{-1}(V_T) < \bar{V}$, $I^*(V_{T+1}) > I^*(\bar{V})$. A sufficient condition for the RHS of this equation to be negative is thus

$$\bar{n}I^*(\bar{V})\gamma > 1 \tag{33}$$

When constructing a cyclical equilibrium, we will assume that (33) holds. In this case, the economy cannot be be in regime II at $T+1$.

Assume the economy is in regime I at $T+1$, and that (33). Then T is the last period in regime II before regime I. V_T must therefore satisfy (32) for $\mu_T = \gamma\nu$

$$V_T = \Phi(I^{*-1}\left(\frac{1}{\gamma\bar{n}}\right)).$$

Using (33) again, we see that $\bar{V} < I^{*-1}(\frac{1}{\gamma\bar{n}})$, implying $\bar{V} = \Phi(\bar{V}) > \Phi(I^{*-1}(\frac{1}{\gamma\bar{n}})) = V_T$, which contradicts the observation that $V_T > \bar{V}$. Under assumption (33), it can therefore never be that $V_T > \hat{\lambda}_{T-1}$.

G. Constructing an equilibrium

Let t be a period in regime II and s the last period in regime II before t. Given V_t and the number of fields invented in the previous regime II episode,

μ_s, we can compute exactly whether or not there will be a period in regime I between s and t.

We know that if (29) holds, i.e. if $V_t < I^{*-1}(\frac{\nu}{\bar{n}\mu_s})$, there is a regime-I episode. Its duration must satisfy (28) and (27), and the new mass of fields invented is given by (30). Finally, the $\hat{\lambda}$ sequence must satisfy (23). Putting these things together, we see that the duration of the cycle must be equal to $INT(\frac{\mu_s\bar{n}}{\nu}I^*(V_t))$ and that the new value of μ_t must be equal to

$$\mu_t = \gamma\nu(1 - DEC\left(\frac{\mu_s\bar{n}}{\nu}I^*(V_t)\right)) = m(\mu_s, V_t). \tag{34}$$

If (29) is violated, i.e. if $V_t > I^{*-1}(\frac{\nu}{\bar{n}\mu_s})$, there is no regime I period between s and t. One must then have $t = s + 1$. μ_t is computed using regime II dynamics, i.e. (26), which, given that $\frac{\mu_s\bar{n}}{\nu}I^*(V_t) < 1$, is equivalent to (34).

Therefore, it must be that

$$\mu_t = m(\mu_s, V_t).$$

The discontinuity points of $m(\mu_s, .)$ are given by $I^*(V) = \frac{k\upsilon}{\mu_s\bar{n}}$, i.e. they are precisely equal to the successive values of $\hat{\lambda}_t$ during the regime I phase. The value of k corresponding to the discontinuity point immediately above V_t must be equal to the duration of the regime I phase. As $V_t \geq \gamma\bar{\omega}$, this duration cannot exceed $\mu_s\bar{n}I^*(\gamma\bar{\omega})/\upsilon \leq \gamma\bar{n}I^*(\gamma\bar{\omega})$.

To continue the construction of the equilibrium, we must pick the value of V_t such that in the last period in regime II, the equilibrium condition for the transition from regime II to regime I, (32), holds. We prove that such a V_t exists as follows.

For $u = t, ..., T$, an *admissible* sequence of pairs $\{(x_u, y_u), u = t, ..., T\}$ is a sequence of real numbers such that

$$
\begin{aligned}
x_u &= \gamma(\nu - \bar{n}x_{u-1}I^*(y_u)), u > t \qquad (35)\\
y_u &= \Phi(y_{u+1}), t \leq u < T;\\
y_T &\in (\gamma\bar{\omega}, (\gamma + \frac{\theta}{\beta})\bar{\omega})
\end{aligned}
$$

Note that, a priori, we do not rule out negative values for x_u.

An admissible sequence is *feasible* iff

$$x_u \geq 0, \forall u; \tag{36}$$

$$x_t = m(\mu_s, y_t). \tag{37}$$

If we find a feasible sequence such that

$$y_T = \Phi(I^{*-1}\left(\frac{\nu}{x_T \bar{n}}\right)), \tag{38}$$

then we can construct a phase in regime II during $T - t$ periods starting from T, such that the transitional condition (32) holds, by choosing $\mu_t = x_t$ and $V_t = y_t$.

A feasible sequence is *maximal* iff

$$\bar{n} x_T I^*(\Phi^{-1}(y_T)) > \nu. \tag{39}$$

That condition implies that there cannot be another feasible sequence $\{(x'_u, y'_u), u = t, ..., T'\}$ such that $T' > T$ and $(x'_u, y'_u) = (x_u, y_u)$ for all $u \leq T$, because the implied value of x_{T+1} would be negative.

PROPERTY P2 – Any feasible sequence is such that

$$\bar{n} x_u I^*(\Phi^{-1}(y_u)) \leq \nu, \text{ for all } u < T.$$

PROOF – For $u < T$ one must have $0 \leq x_{u+1} = \gamma(\nu - \bar{n} x_u I^*(y_{u+1})) = \gamma(\nu - \bar{n} x_u I^*(\Phi^{-1}(y_u)))$. QED.

Let us now denote by $K = \{I^{*-1}(\frac{k\nu}{\mu_s \bar{n}}), k = 1, ...\}$ the set of discontinuity points of $m()$. Then for any admissible sequence such that $y_t \notin K$, x_u and y_u are locally C^1 functions of y_t. Furthermore, as y_t varies, the following properties hold:

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PROPERTY P3 – Assume $y_t \notin K$. Then:

(i) $\frac{dy_u}{dy_t} > 0$ if $u - t$ is even, < 0 if $u - t$ is odd.

(ii) $\frac{dy_u}{dy_t}\frac{dx_u}{dy_t} > 0$.

PROOF – Property (i) derives trivially from the fact that $y_u = \Phi(y_{u+1})$. Property (ii) can be proved by induction. It is clearly true for $u = t$, as $\partial m/\partial V_N > 0$. Assume it holds for $u - 1$. Differentiating (35), we get

$$dx_u = -dx_{u-1}\frac{\bar{n}\gamma}{\nu}I^*(y_u) - \frac{\bar{n}\gamma}{\nu}x_{u-1}I^{*'}(y_u)dy_u.$$

By assumption, $sign(dx_{u-1}) = sign(dy_{u-1}) = -sign(dy_u)$. Consequently, the first term has the same sign as dy_u, and so does the second, as $I^{*'}() < 0$. QED.

Thus, as the initial value y_t varies, subsequent contemporaneous values of x and y move in the same direction.

PROPERTY P4 – Assume that there exist two feasible sequences $\{(x_{0u}, y_{0u}), u = t, ..., T\}$, and $\{(x_{1u}, y_{1u}), u = t, ..., T\}$ such that for some integer k :

$$I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < y_{0t} < y_{1t} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}}).$$

Then there exists a family of mappings $X_u(y)$ (resp. $Y_u(y)$) from $[y_{0t}, y_{1t}]$ to $[x_{0u}, x_{1u}]$ (resp. $[y_{0u}, y_{1u}]$) such that

(i) $X_u(y)$ and $Y_u(y)$ are continuously differentiable in y

(ii) $\{(X_u(y), Y_u(y)), u = t, ..., T\}$ is feasible;

(iii) $X_t(y) = m(\mu_{s-1}, y); Y_t(y) = y$

(iv) $X_u(y_{0t}) = x_{0u}; Y_u(y_{0t}) = y_{0u}; X_u(y_{1t}) = x_{1u}; Y_u(y_{1t}) = y_{1u}$

(v) $X'_u > 0, Y'_u > 0$ for $u - t$ even, and $X'_u < 0, Y'_u < 0$ for $u - t$ odd.

PROOF – The condition implies that y_{0t} and y_{1t} are between two consecutive discontinuity points. $m(\mu_s, .)$ is therefore C^1 over $[y_{0t}, y_{1t}]$. We can

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then construct $Y_u()$ recursively as $Y_t(y) = y$ and $Y_u(y) = \Phi^{-1}(Y_{u-1}(y))$; similarly, $X_u()$ is constructed recursively as $X_t(y) = m(\mu_{s-1}, y)$, $X_u(y) = \gamma(\nu - \bar{n}X_{u-1}(y)I^*(Y_u(y)))$. Thus, (i),(iii) and (iv) trivially hold. The monotonicity properties (v) in turn are a consequence of property P3. Finally, the feasibility property (ii) is a consequence of monotonicity: while (37) holds by construction, (36) derives from the fact that $X_u(y)$ is between x_{0u} and x_{1u}, which are both nonnegative. QED.

With these properties in hand, we are now able to construct a feasible sequence such that (38) holds. Denoting again by $\bar{V}$ the fixed point of $\Phi()$, and assuming (33) holds, there necessarily exists a *maximal* sequence such that $y_u = \bar{V}$, for all u. Again, that is because the dynamics of x are then unstable if (33) holds. Call $\tilde{x}_u$ the values of x_u in that sequence. Because of (39), we must have

$$\bar{V} > \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T}))$$

Consider now the *admissible* sequence of equal length T, $\{(\hat{x}_u, \hat{y}_u), u = t, ..., T\}$, such that $\hat{y}_T = \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T}))$. It can be easily constructed by iterating $\Phi()$ backwards on $\hat{y}_T$, yielding some $\hat{y}_t$, and then computing the corresponding values of x_u by applying (35). Three possibilities arise:

G1. The admissible sequence is feasible and $\hat{y}_t$ is such that $I^{*-1}(\frac{kv}{\mu_s\bar{n}}) < \hat{y}_t < \bar{V} < I^{*-1}(\frac{(k-1)v}{\mu_s\bar{n}})$ (for $T - t$ even), or $I^{*-1}(\frac{kv}{\mu_s\bar{n}}) < \bar{V} < \hat{y}_t < I^{*-1}(\frac{(k-1)v}{\mu_s\bar{n}})$ (for $T - t$ odd).

In this case, we note that property P4 can be applied, using the feasible sequences $\{(\tilde{x}_u, \bar{V})\}$ and $\{(\hat{x}_u, \hat{y}_u)\}$ as our boundaries. Because of the monotonicity property (v), it must be that $\hat{x}_T < \tilde{x}_T$, since $\hat{y}_T < \bar{V}$. Therefore $\hat{y}_T = \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T})) < \Phi(I^{*-1}(\frac{\nu}{\bar{n}\hat{x}_T}))$. Hence:

$$\hat{y}_T < \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T})).$$

Therefore, the function $Y_T(y) - \Phi(I^{*-1}(\frac{\upsilon}{\bar{n}X_T(y)}))$, which is continuous, becomes positive and negative as y varies between $\hat{y}_t$ and $\bar{V}$. There exists $y^* \in [\hat{y}_t, \bar{V}]$, such that it is equal to zero. The sequence $\{(X_u(y^*), Y_u(y^*), u = t, ..., T\}$ is feasible because of property (ii), and satisfies (38).

G2. The admissible sequence is not feasible, but $\hat{y}_t$ satisfies $I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \hat{y}_t < \bar{V} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}})$ (for $T - t$ even), or $I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \bar{V} < \hat{y}_t < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}})$ (for $T - t$ odd).

In this case, lack of feasibility must be due to the fact that (36) is violated. We can then construct a maximal sequence $\{(\hat{x}_u, \hat{y}_u), u = t, ..., T_2\}$ for some $T_2 < T$.[16]

Because of Property (P2), we have that

$$\bar{V} \leq \Phi(I^{*-1}(\frac{\upsilon}{\tilde{x}_{T_2}\bar{n}})).$$

Because of the maximality condition (39), we have that

$$\hat{y}_{T_2} > \Phi(I^{*-1}(\frac{\upsilon}{\hat{x}_{T_2}\bar{n}})).$$

As $\{(\hat{x}_u, \hat{y}_u), u = t, ..., T_2\}$ is now feasible, we can apply the same reasoning as in case G1, but between t and T_2 instead of t and T.

G3. The admissible sequence does not satisfy $I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \hat{y}_t < \bar{V} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}})$ (for $T - t$ even), or $I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \bar{V} < \hat{y}_t < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}})$ (for $T - t$ odd).

Assume $T - t$ is even. Then it must be that $\hat{y}_t < I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \bar{V}$. Consider now $y_m = I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) + \eta$, for $\eta > 0$ arbitrarily small. Note that $\Phi^{-1}(y_m)$ exists, as $y_m \in [\hat{y}_t, \bar{V}]$. The two-period sequence $\{(x_{mt}, y_{mt}), (x_{mt+1}, y_{mt+1})\}$ defined

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[16]Furthermore $T - T_2$ has to be odd. That is because, by construction, $\hat{x}_u < \tilde{x}_u$ and $\hat{\omega}_t < \bar{V}$ if $T - T_2$ is even (due to Property P3), implying that $\bar{n}\hat{x}_{T_2}I^*(\Phi^{-1}(\hat{\omega}_u)) \leq \bar{n}\tilde{x}_{T_2}I^*(\Phi^{-1}(\bar{V})) \leq \nu$, where the last inequality is due to (P2). That violates the maximality condition (39).

by $x_{mt} = (m(\mu_s, y_m), y_{mt} = y_m, x_{mt+1} = (\gamma(\nu - \bar{n}m(\mu_s, y_m)I^*(\Phi^{-1}(y_m)),$ $y_{mt+1} = \Phi^{-1}(y_m))$, is clearly feasible, since $y_{mt+1} \in [\bar{V}, \hat{y}_{t+1}]$, $m(\mu_s, y_m)$ is positive and arbitrarily close to zero, and x_{mt+1} is thus arbitrarily close to $\gamma\nu$.

Next, note that the maximality condition holds for $t + 1$, since

$$\bar{n}(\gamma\nu)I^*(\Phi^{-1}(y_{mt+1})) > \bar{n}(\gamma\nu)I^*(\Phi^{-1}(\bar{V})) > \nu,$$

because of (33).

Thus, this 2 period sequence is maximal, and satisfies

$$y_{mt+1} > \Phi(I^{*-1}(\frac{\upsilon}{x_{mt+1}\bar{n}})),$$

while, again because of (P2):

$$\bar{V} < \Phi(I^{*-1}(\frac{\upsilon}{\tilde{x}_{t+1}\bar{n}})).$$

We are again in a position to apply the same continuity argument as in cases G1 and G2.

Assume $T - t$ is odd. Then it must be that $\bar{V} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}}) < \hat{y}_t$. Consider now $y_m = I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}}) - \eta$, for $\eta > 0$ arbitrarily small. The one-period sequence $\{(x_{mt}, y_{mt})\}$ defined by $x_{mt} = m(\mu_s, y_m), y_{mt} = y_m$, is clearly feasible. Again, note that because of (33), the maximality condition holds for t, since

$$\bar{n}(\gamma\nu)I^*(\Phi^{-1}(y_{mt})) > \bar{n}(\gamma\nu)I^*(\Phi^{-1}(\bar{V})) > \nu.$$

We have constructed a maximal sequence which satisfies $y_{mt} > \Phi(I^{*-1}(\frac{\upsilon}{x_{mt}\bar{n}}))$. Property (P2) again implies that $\bar{V} < \Phi(I^{*-1}(\frac{\upsilon}{\tilde{x}_t\bar{n}}))$. Since $I^{*-1}(\frac{k\upsilon}{\mu_s\bar{n}}) < \bar{V} < y_{mt} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s\bar{n}})$, we can again apply the continuity argument as in G1 to construct a one-period sequence matching the equilibrium condition (38).

Thus we are always able to construct a feasible sequence such that the equilibrium condition (38) holds.

The only equilibrium condition that remains to be checked is condition (24) in the regime I phase. Noting that, during that phase, $\lambda_u = \hat{\lambda}_u + \frac{\theta}{\beta}(\hat{\lambda}_u - \hat{\lambda}_{u+1})$, that condition is equivalent to

$$\Phi(\hat{\lambda}_{u+1}) < \hat{\lambda}_u + \frac{\theta}{\beta}(\hat{\lambda}_u - \hat{\lambda}_{u+1}). \qquad (40)$$

Observe now that we have constructed the value of V_t at the end of phase I in such a way that V_t falls in the same $[I^{*-1}(\frac{kv}{\mu_s \bar{n}}), I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}})]$ as $\bar{V}$. Since $I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}}) = \hat{\lambda}_{t-1}$, it follows that $\hat{\lambda}_{t-1} > \bar{V} > \Phi(\hat{\lambda}_{t-1})$; since the $\hat{\lambda}$ are a decreasing sequence, we also have that $\hat{\lambda}_u > \bar{V} > \Phi(\hat{\lambda}_u)$ for all u. Consider now the function defined by $\Lambda(\lambda, y) = \lambda + \frac{\beta}{\lambda}(\lambda - y) - \Phi(y)$. Clearly, $\Lambda(\hat{\lambda}_u, \hat{\lambda}_u) = \hat{\lambda}_u - \Phi(\hat{\lambda}_u) > 0$. Furthermore, as $\gamma < 1$, $\frac{\partial \Lambda}{\partial y} = -\frac{\theta}{\beta} + \gamma\frac{\theta}{\beta}(1 - F(y)) < 0$. Thus, $\Lambda(\hat{\lambda}_u, y) > 0$ for all $y < \hat{\lambda}_u$. In particular, $\Lambda(\hat{\lambda}_u, \hat{\lambda}_{u+1}) > 0$, which is equivalent to (A22).

That concludes the proof that a cyclical equilibrium exists.

H. Constructing an equilibrium when (33) does not hold.

We have constructed a cyclical equilibrium when (33) holds. If it does not, we can easily construct an equilibrium in regime II which converges to the steady state. To do so, start from an inherited mass of invented fields μ_s at the beginning of the (potentially empty) phase in regime I, and choose $V_t = \bar{V}$, $\mu_t = m(\mu_s, \bar{V})$. This defines an initial phase in regime I whose length equals the number of discontinuity points of $m(\mu_s, .)$ above $\bar{V}$. Thereafter, the measure of new fields evolves according to

$$\mu_{u+1} = \gamma(\nu - \bar{n}I^*(\bar{V})\mu_u),$$

and, given that (33) is violated and that $m(\mu_s, \bar{V}) \in [0, \gamma v]$, it is straightforward to check that μ_u will converge to its steady-state value by damped oscillations, remaining in the feasible $[0, \gamma v]$ interval. Furthermore, the largest

discontinuity point of $m()$ is at $I^{*-1}(\frac{\upsilon}{\mu_s \bar{n}}) < I^{*-1}(\frac{1}{\gamma \bar{n}}) < \bar{V}$. Thus the economy in fact does not spend any time in regime I.

I. *Proof of uniqueness when $\theta = 0$.*

If the feasible sequence constructed in G. is unique, then the equilibrium is unique. That is easy to prove in the $\theta = 0$ case. The initial value of V_t, at the beginning of regime II, must be equal to $\gamma \bar{\omega}$. Furthermore, the date at which the economy leaves regime II to revert to regime I is uniquely determined: If it were such that $\bar{n} x_u I^*(\gamma \bar{\omega}) \le \upsilon$, then (29) would be violated for the next regime I phase, since one must have $V = \gamma \bar{\omega}$ at the end of that regime. Therefore, it must be equal to the first date such that $\bar{n} x_u I^*(\gamma \bar{\omega}) > \upsilon$, since the economy cannot feasibly remain in regime II after that date.

10.2 Proof of Proposition 5

(i) Recall, $\bar{V}$ is increasing in θ and I^* is decreasing. Thus, if the condition $I^*(\bar{V}) < \frac{1}{\gamma \bar{n}}$ is satisfied for θ, it is also satisfied for $\theta' > \theta$.

(ii) The duration of the first regime I phase is $INT(I^*(\gamma \bar{\omega})\frac{\bar{n}\mu_{-1}}{\nu})$ when $\theta = 0$ and $INT(I^*(V_T)\frac{\bar{n}\mu_{-1}}{\nu})$ when $\theta > 0$. Since $V_T \ge \gamma \bar{\omega}$, the first regime I phase is necessarily longer when $\theta > 0$. Examining the equation for μ gives the second result.

10.3 Proof of Proposition 6

To prove Proposition 6, just differentiate the dynamics of V_t in regime II, $V_t = \Phi(V_{t+1})$, around the fixed point $\bar{V}$. Denoting by $v_t = V_t - \bar{V}$, we get

$$v_t = \frac{\gamma \theta}{\beta}(1 - F(\bar{V}))v_{t+1}.$$

If $\frac{\gamma \theta}{\beta}(1 - F(\bar{V})) > 1$, then we can construct an equilibrium for any initial value of v_t. QED.

10.4 Proof of Proposition 7

The first order condition for maximization of the value function with respect to x is

$$
\begin{aligned}
0 &= \left(\bar{\omega} \frac{\partial \mu_t}{\partial x} + \mu_{t-1} \Gamma'(\omega^*) \frac{\partial \omega^*}{\partial x} \right) \left(1 + \frac{1}{1+\phi} \frac{\partial V(\mu_t, K_t)}{\partial K} \right) \\
&\quad + \frac{1}{1+\phi} \frac{\partial \mu_t}{\partial x} \frac{\partial V(\mu_t, K_t)}{\partial \mu_t}.
\end{aligned} \tag{41}
$$

The resource constraints allow us to compute the following derivatives:

- $\frac{\partial \mu_t}{\partial x} = \gamma \upsilon$

- $\frac{\partial \omega^*}{\partial x} = -\frac{\upsilon}{I^{*\prime}(\omega^*)\mu_{t-1}\bar{n}}$

  $\frac{\partial \omega^*}{\partial \mu_{t-1}} = -\frac{I^*(\omega^*)}{I^{*\prime}(\omega^*)\mu_{t-1}}.$

Differentiating the value function while ignoring the changes in x because of the envelope theorem allows to compute the following:

- $\frac{\partial V}{\partial K} = \frac{1+\phi}{\phi}.$

- $\frac{\partial V(\mu_{t-1}, K_{t-1})}{\partial \mu_{t-1}} = \left(\Gamma(\omega^*) + \mu_{t-1} \Gamma'(\omega^*) \frac{\partial \omega^*}{\partial \mu_{t-1}} \right) \frac{1+\phi}{\phi}.$

Substituting these formulas into (41) while making use of the steady state assumption, we get

$$
\begin{aligned}
0 &= \left(\bar{\omega} \gamma \upsilon - \Gamma'(\omega^*) \frac{\upsilon}{I^{*\prime}(\omega^*)\bar{n}} \right) \\
&\quad + \frac{1}{1+\phi} \gamma \upsilon \left(\Gamma(\omega^*) - \Gamma'(\omega^*) \frac{I^*(\omega^*)}{I^{*\prime}(\omega^*)} \right).
\end{aligned} \tag{42}
$$

To get to (18), compute the derivatives of Γ and I^* :

- $\Gamma'(\omega^*) = -\bar{n} \frac{\omega^*}{\beta} \int_{\omega^*}^{+\infty} e^{\frac{\omega - \omega^*}{\beta}} f(\omega) d\omega;$

- $I^{*\prime}(\omega^*) = -\frac{1}{\beta} \int_{\omega^*}^{+\infty} e^{\frac{\omega - \omega^*}{\beta}} f(\omega) d\omega$.

Replace all the terms in $\Gamma'(\omega^*)/I^{*\prime}(\omega^*)$ in (42) by $\bar{n}\omega^*$, and replace the term in $\Gamma(\omega^*)$ by the following expression (it can be checked that it is indeed equal to $\Gamma(\omega^*)$):

$$(\beta + \omega^*)\bar{n}I^*(\omega^*) - \bar{n}\int_{\omega^*}^{+\infty} (\omega - \omega^*)\, f(\omega) d\omega.$$

These operations yield equation (18). QED

11   References

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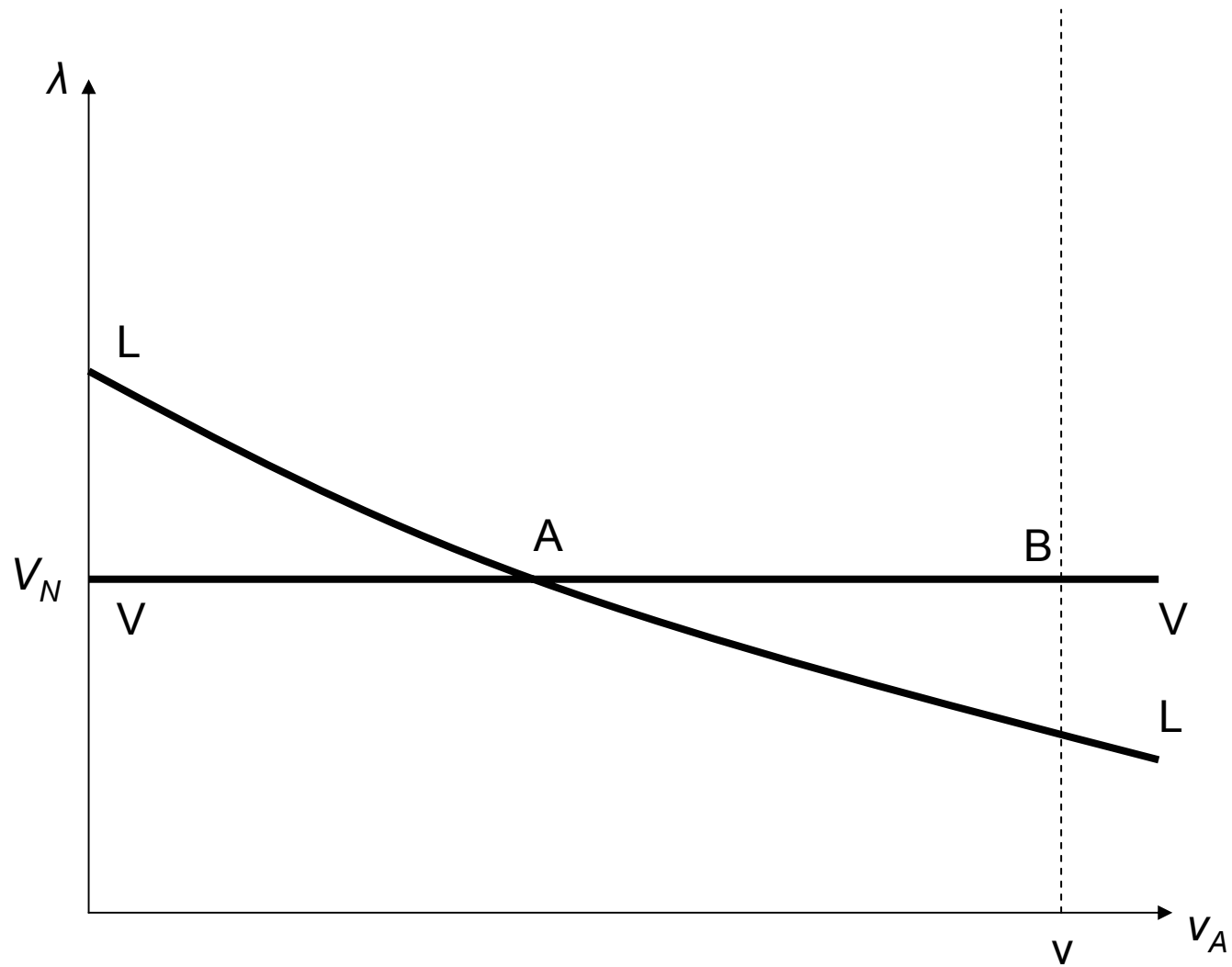
Figure 1 – Equilibrium determination in regime II
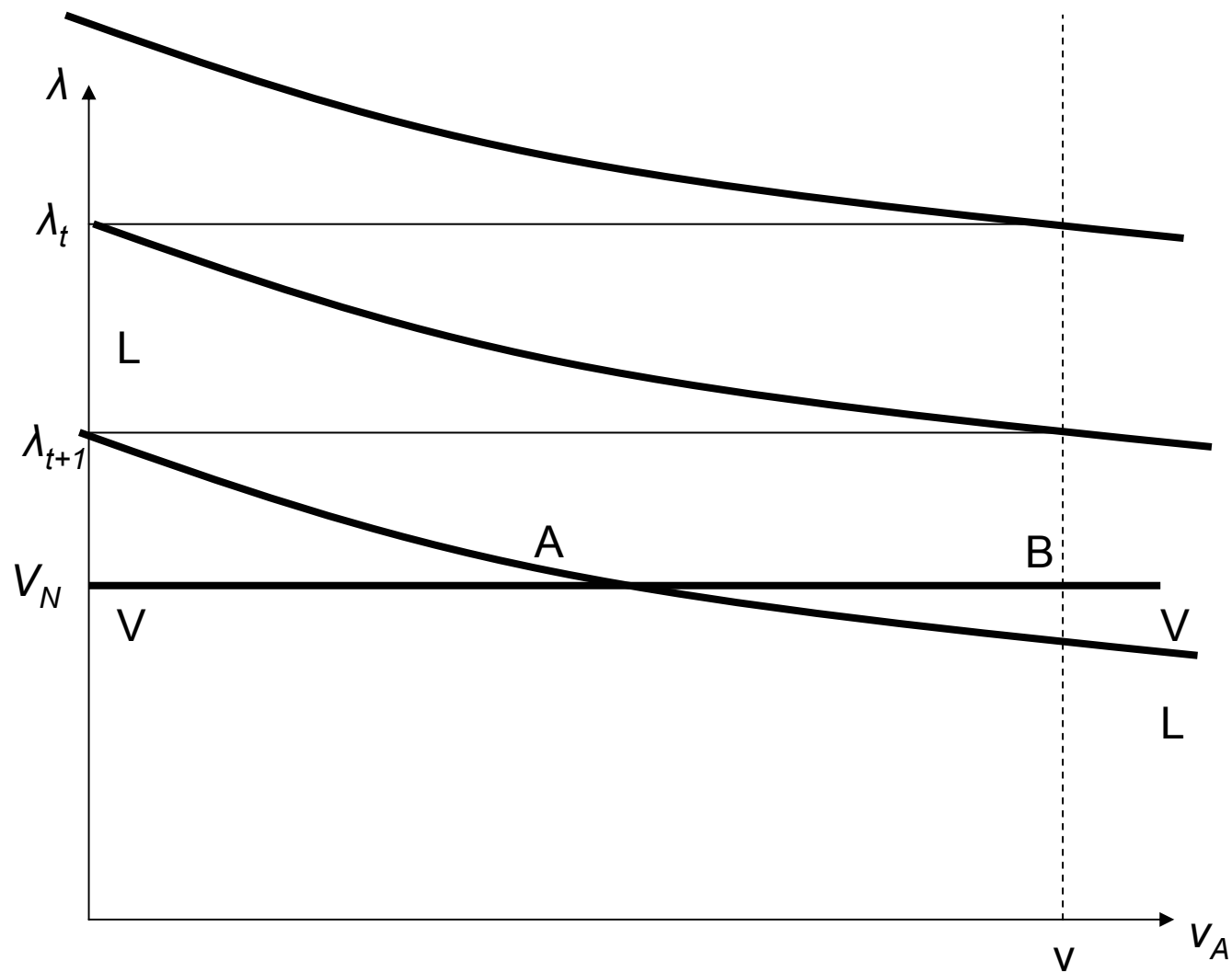
Figure 2 – Equilibrium determination in regime I
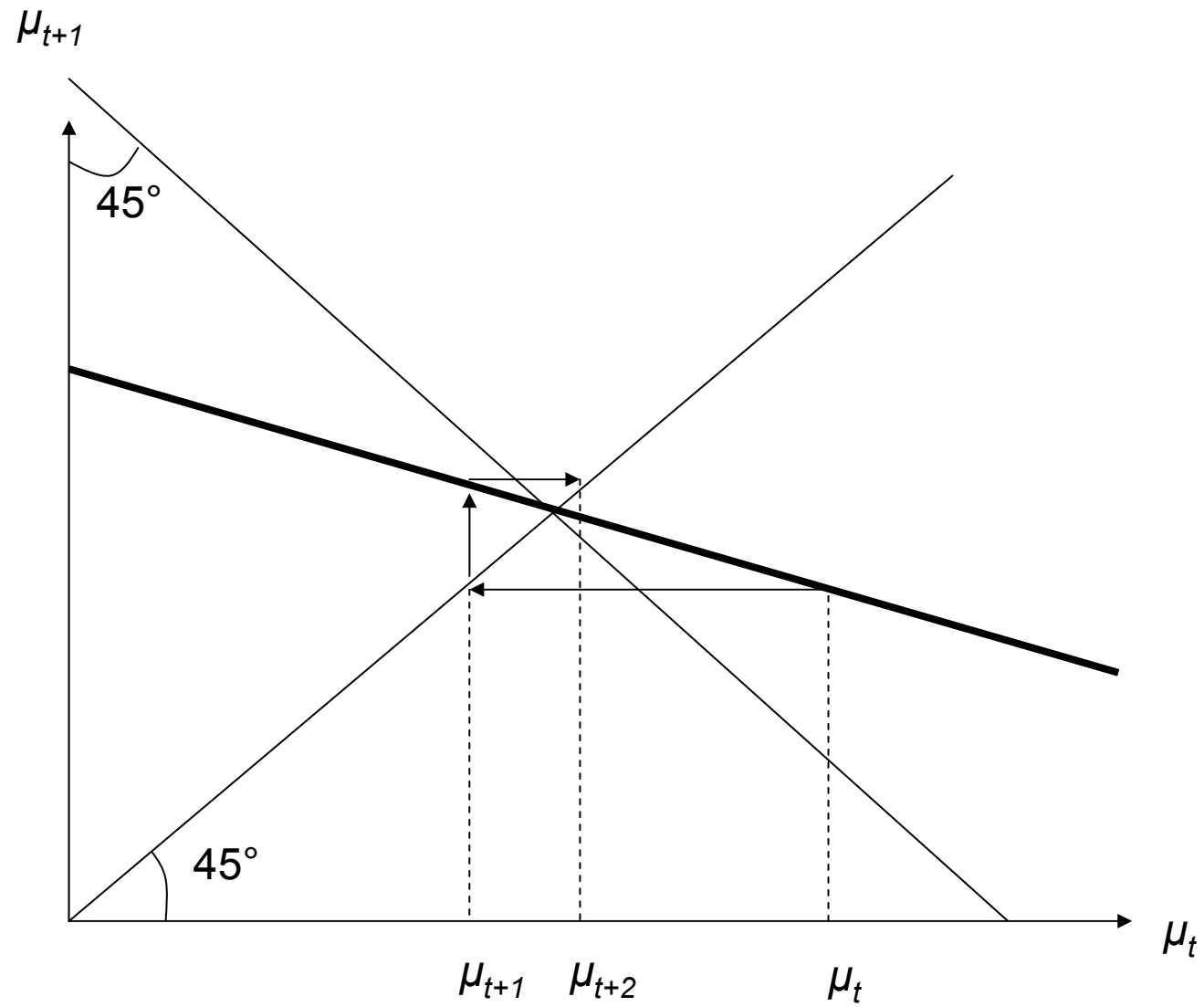
Figure 3 – Convergence to the regime II steady state
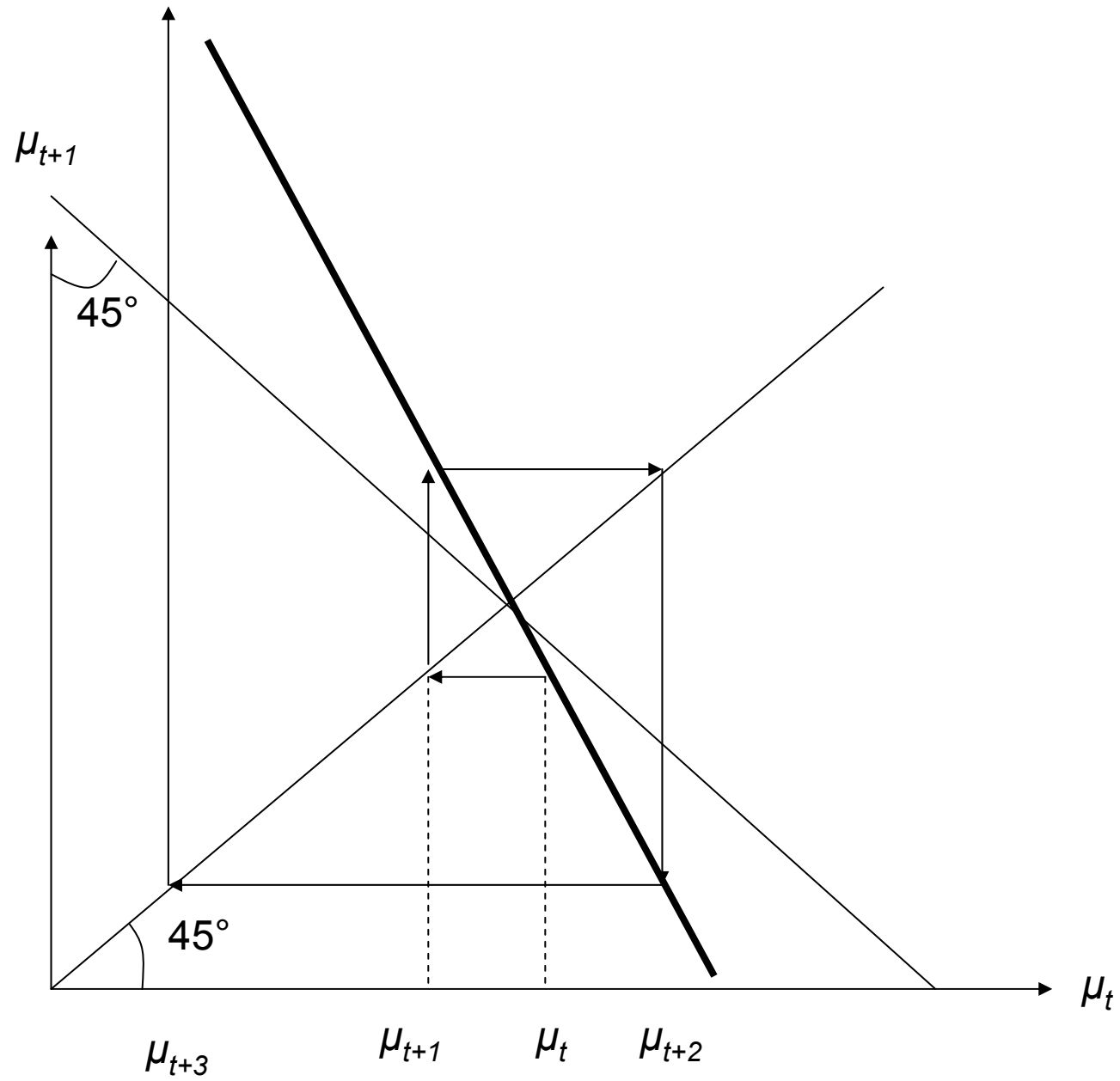
Figure 4 – The economy eventually leaves regime II

Fig. 5: time in I per cycle
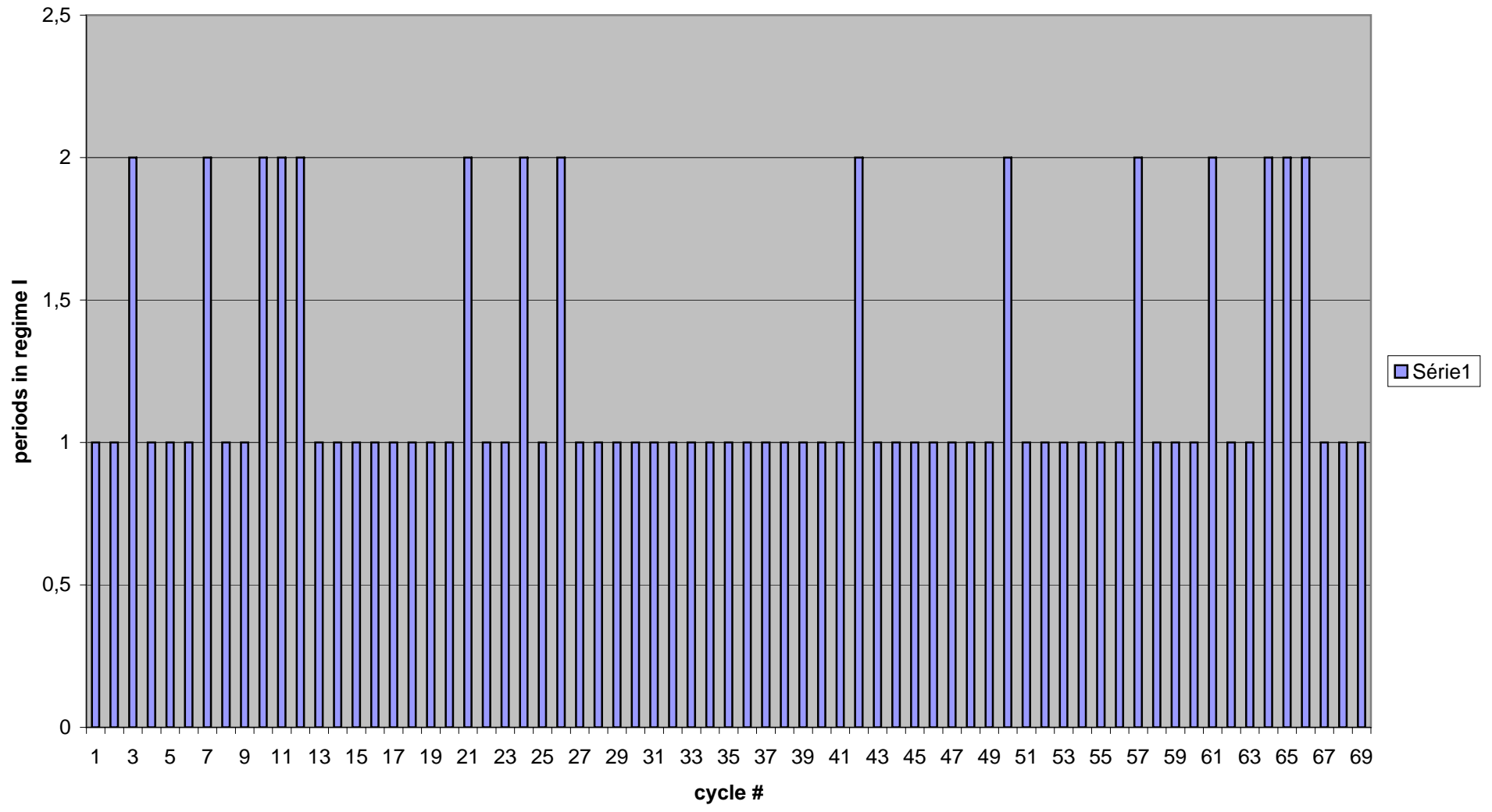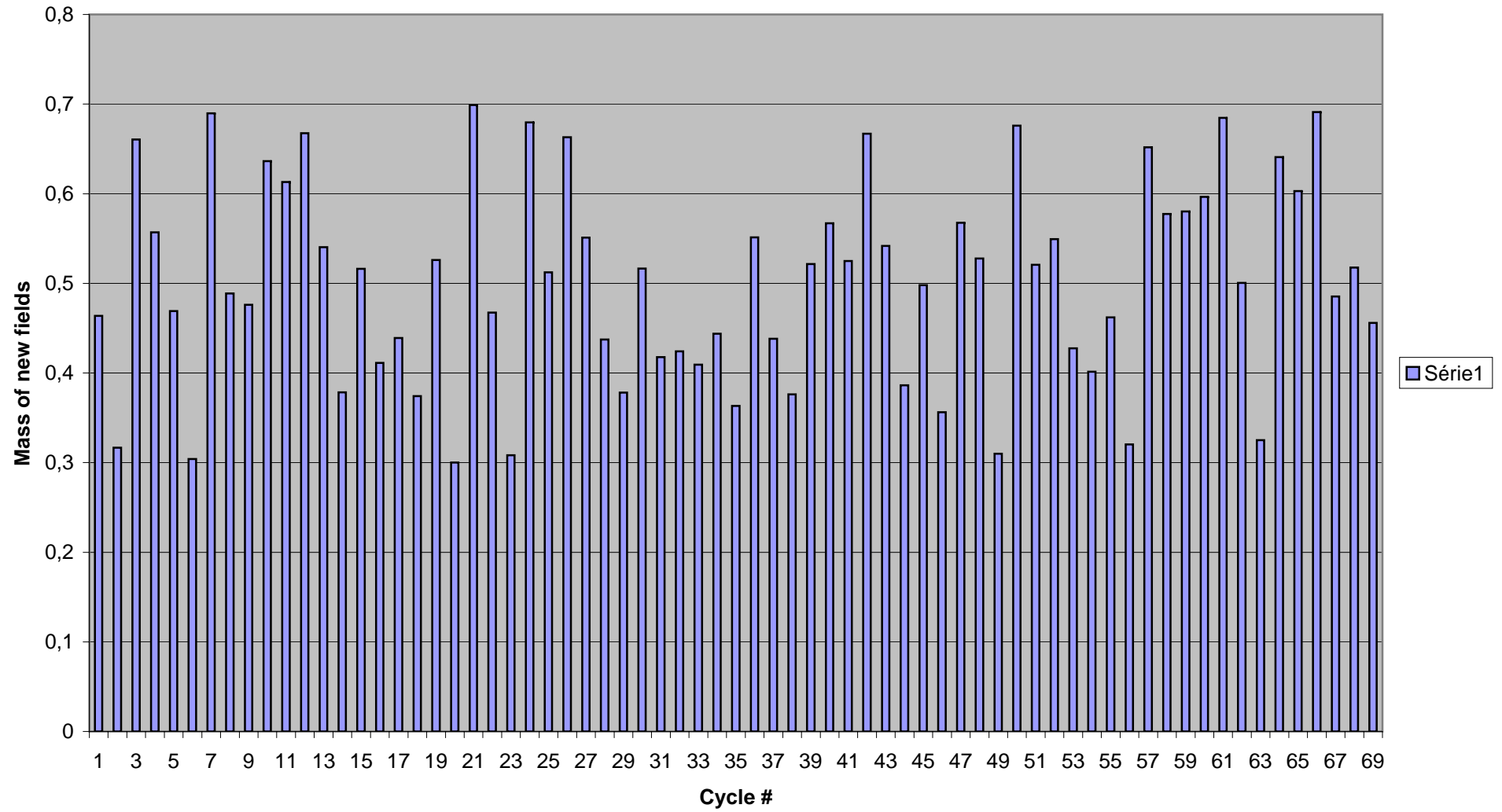
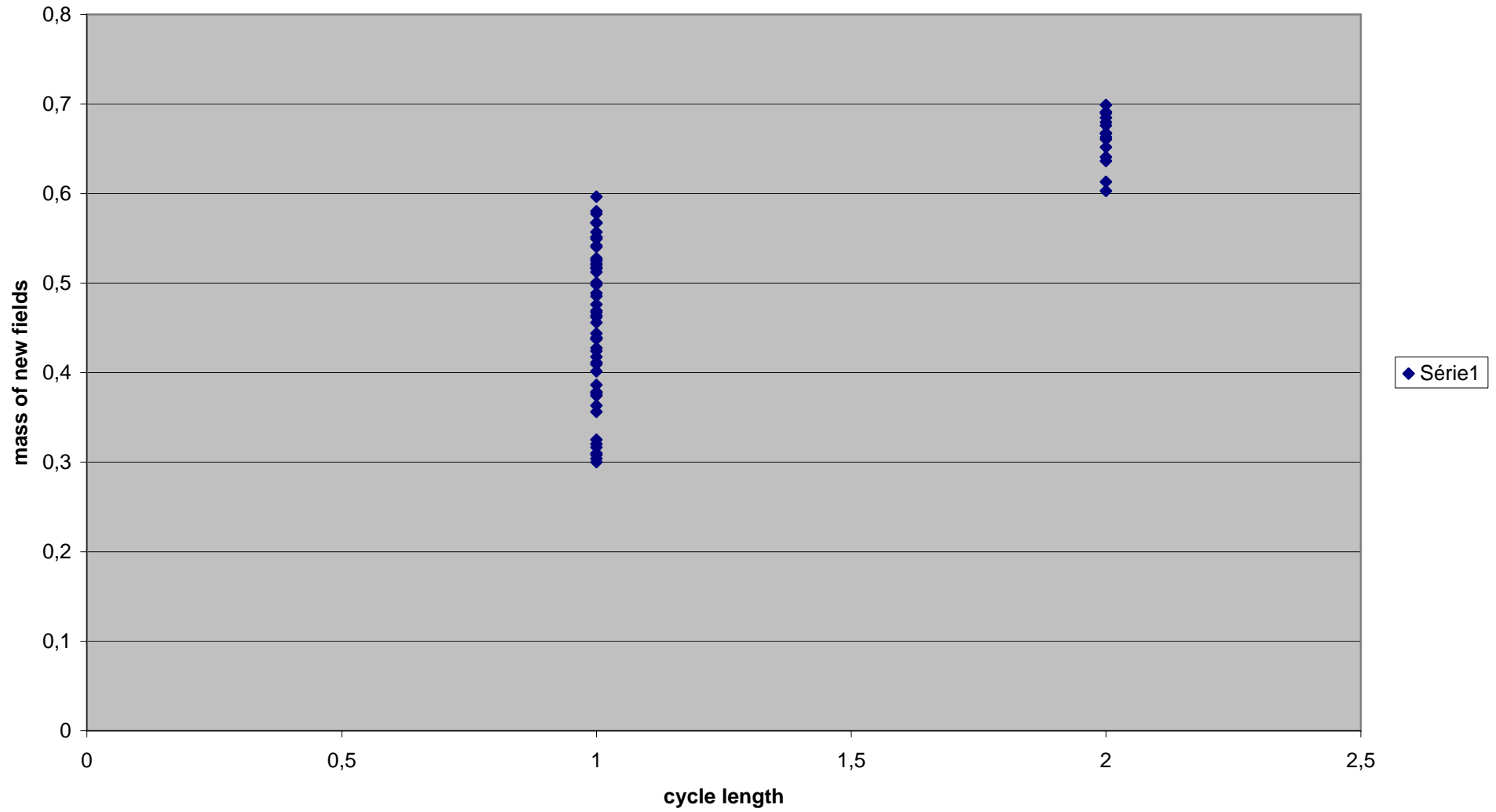**Fig 6: mass of new fields per cycle**
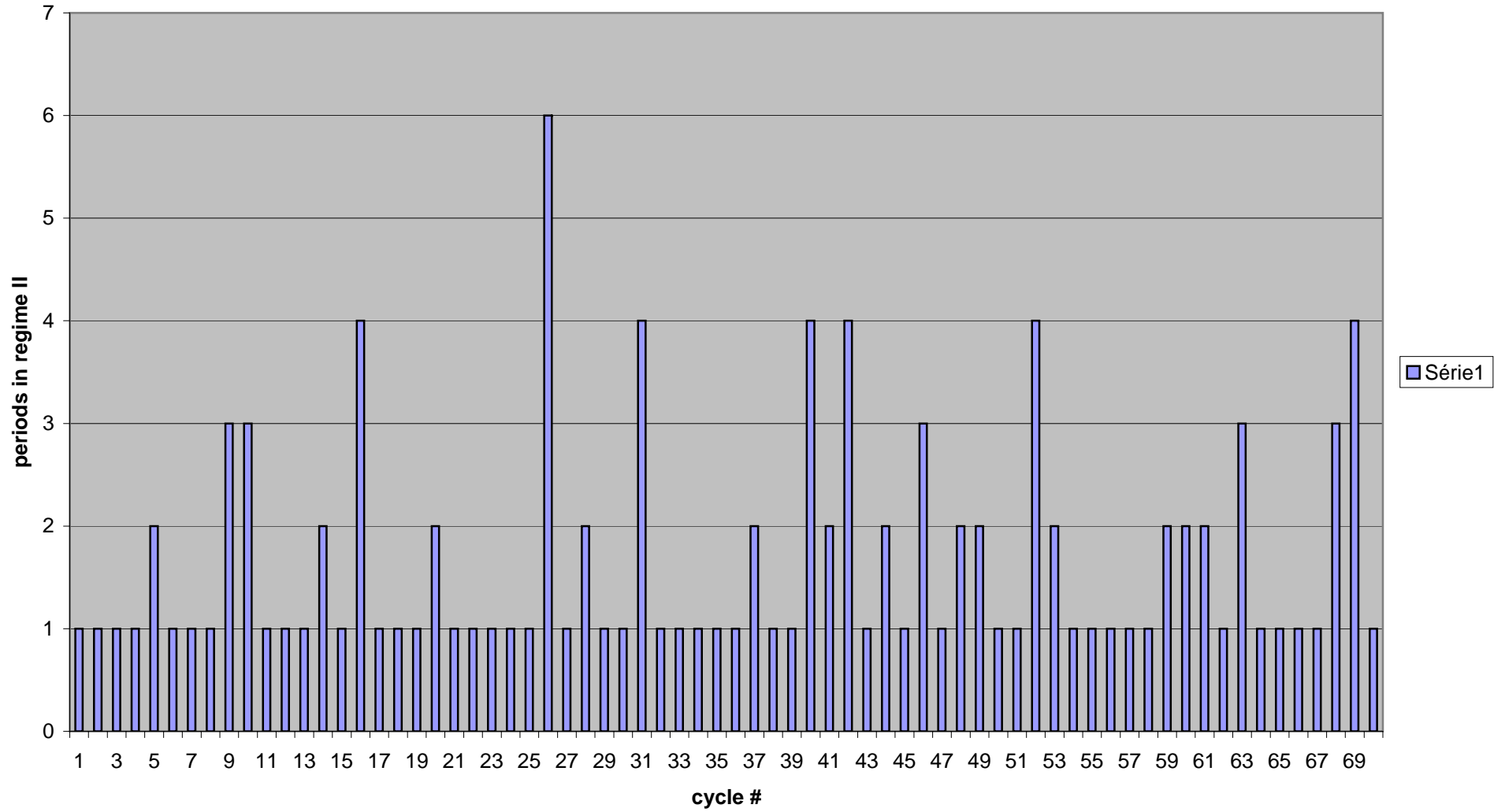
Fig. 7 cycle length and mass of new fields

Fig. 8 time spent in regime II

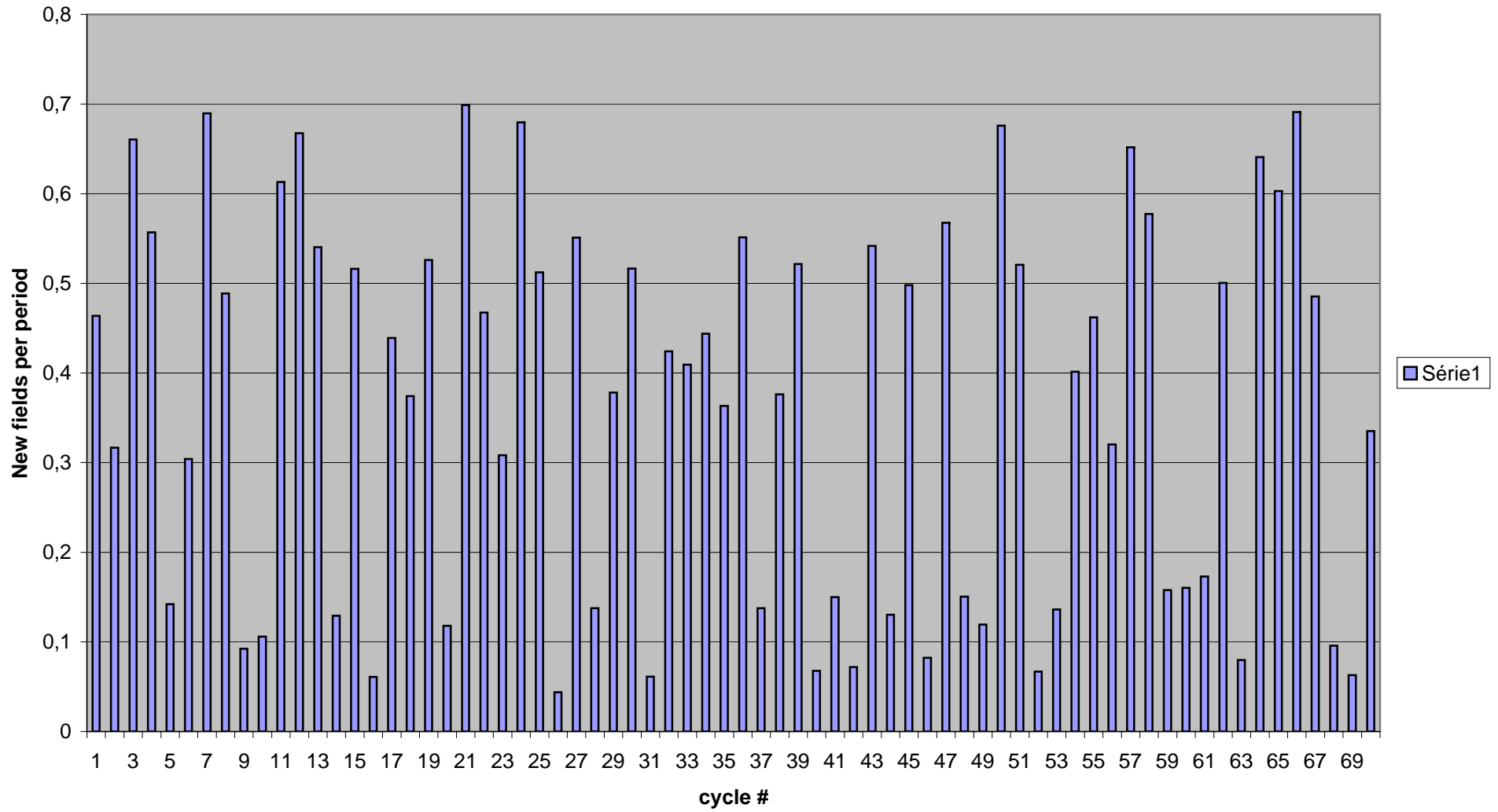**Fig. 9: Average production of new fields in regime II per cycle**

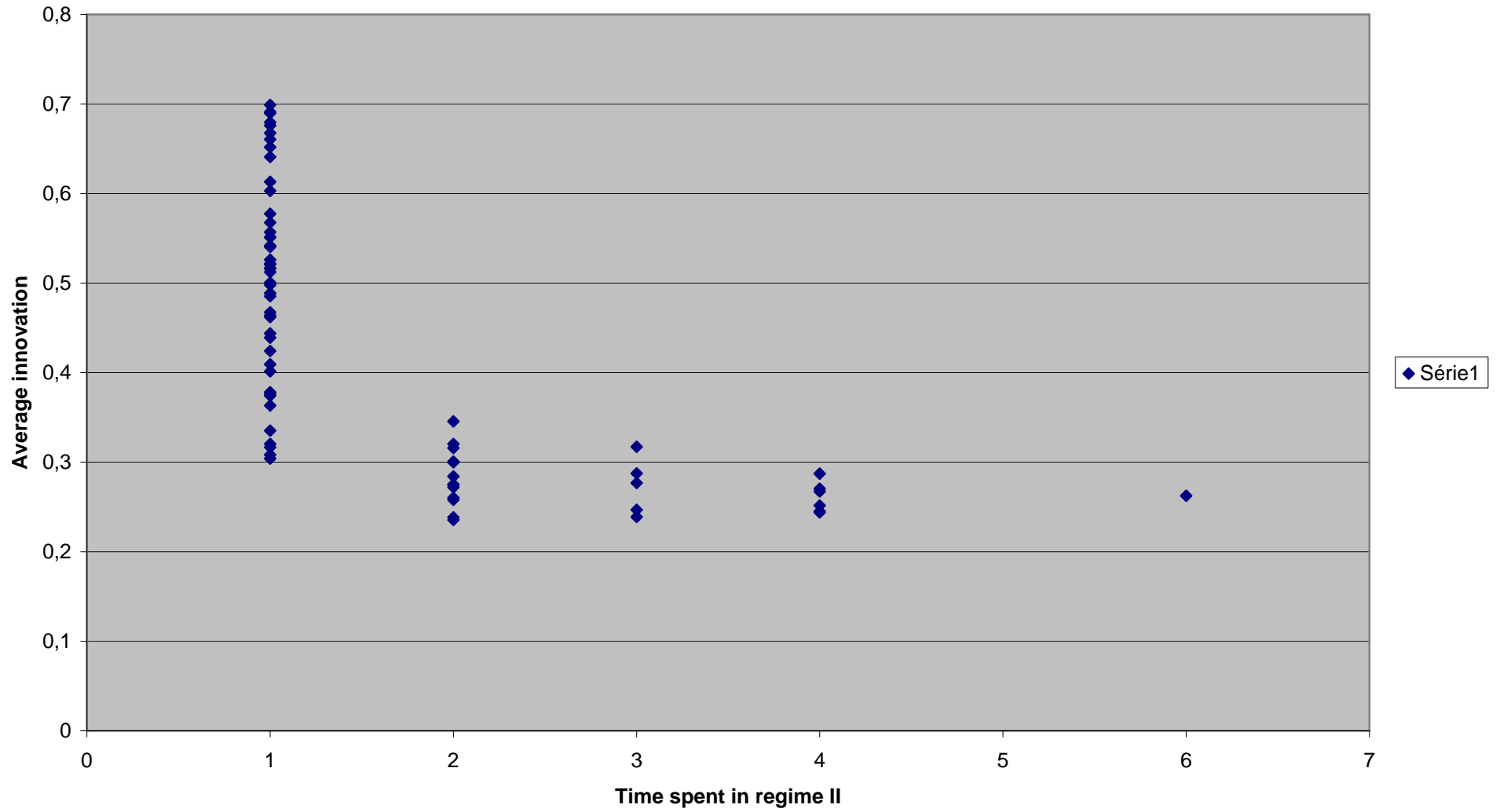Fig. 10: Time in regime II and average innovation
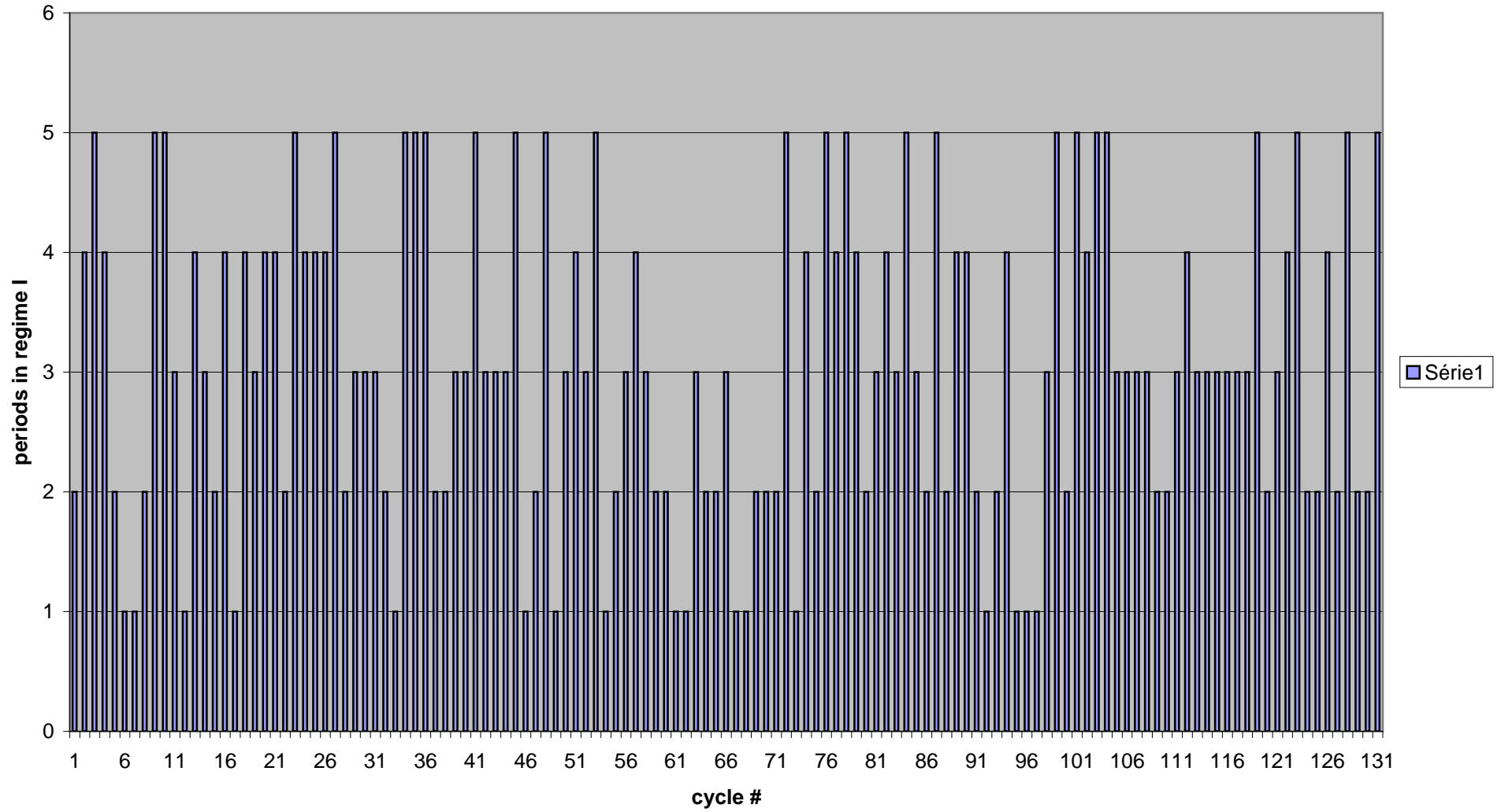
Fig. 11: cycle duration in reg. I, beta = 0.2

Fig. 12, time spent in regime II per cycle, beta = 0.2