Third-Degree Stochastic Dominance and the von-Neumann-Morgenstern Independence Property

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Abstract

This paper is an investigation of the third-degree stochastic dominance order which has been introduced in the context of risk analysis and is now receiving an increased attention in the area of inequality measurement. After observing that this partial order fails to satisfy the von Neumann-Morgenstern property in the space of random variables, we introduce strong and local third-degree stochastic dominance. We motivate these two new binary relations and offer a complete and simple characterizations in the spirit of the Lorenz characterization of the second-degree stochastic order. The paper compares our results with the closest literature.

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1 Introduction

Stochastic orders lie at the crossroads of several fields ranging from portfolio analysis to inequality measurement. Given an underlying space of choices $\mathbb{X}$, they are defined as the intersection of a family $\mathcal{F}$ of complete orders over $\mathbb{X}$. Any particular complete order $\succsim$ in the family $\mathcal{F}$ represents the preference of an individual or social decision-maker over $\mathbb{X}$ and the stochastic order therefore identifies the pairs of choices on which all the decision makers in the class $\mathcal{F}$ unanimously agree. The most common application is the case where $\mathbb{X}$ is the set of probability distributions on the real numbers: each member of is interpreted either as a lottery over monetary gains or losses or as an income distribution. In all the family consists of a subclass of complete orderings satisfying von Neumann-Morgenstern independence property and therefore expected utility maximizers:[1] the first stochastic order $\succsim_1$ asks for the von Neumann-Morgenstern utility to be increasing, the second stochastic order $\succsim_2$ asks in addition to the first property for the von Neumann Morgenstern utility function to be concave, the third stochastic order $\succsim_3$ asks in addition to these two properties for the marginal utility to be convex. The main task is of course to sort out a simple characterization of these various nested stochastic orders but a first thing must be noted: since the von Neumann-Morgenstern independence property is preserved by intersection, all these stochastic orders will satisfy the von Neumann-Morgenstern property too.

The set X of probability distributions over the real numbers is in a one to one relationship with the set X^t of nondecreasing and right-continuous real valued random variables over the unit interval $[0,1]$. To each such random variable let P be its probability law and to each probability distribution over $\Re$, let $X(t) = \underset{F(x)\leq t}{Sup}\ x$ where $F(x) = P(]-\infty, x])$. Given this one to one relationship, we can therefore transpose any complete or partial order $\succsim$ over X into an order $\succsim^t$ over X^t and vice versa. Therefore we can define along these lines $\succsim_1^t$, $\succsim_2^t$ and $\succsim_3^t$. Note also that X^t is a convex

[1] A remarkable characterization of the class of preorders satisfying the von Neumann-Morgenstern independence axiom (together with some regularity axioms) but not necessarily complete has been derived by Baucells and Shapley (1998) and Dubra, Maccheroni and Ok (2004). They show that any such preorder can be represented as the intersection of a finite family of von Neumann-Morgenstern utility functions.

cone in a linear vector space and therefore the von Neumann-Morgenstern independence property is mathematically well defined on X^t as well. We have pointed out that $\succsim_1$, $\succsim_2$ and $\succsim_3$ both satisfy the von Neumann-Morgenstern independence property. There is not reason to think a priori that $\succsim_1^t$, $\succsim_2^t$ and $\succsim_3^t$ will satisfy this property as well. Following well know characterizations of $\succsim_1$ and $\succsim_2$, we have surprisingly that $\succsim_1^t$ and $\succsim_2^t$ satisfy the independence property as well. The point of departure of this paper is the recognition that this is not true anymore for $\succsim_3^t$ and to take this as the mean reason why many things discontinuously change when we reach the third order.

In the first section we gather the main definitions used in this paper without being very explicit about the partial orderings that we are considering. In section 2, we introduce some general notions on partial preorders defined on a convex subset of real vector spaces. In section 3, we introduce the three first stochastic orders over the subset of nondecreasing random variables taking discrete values. We ask the following question in relation to the von Neumann-Morgenstern independence property. Take two such random variables x and y. When is it the case that for any third option z and any λ in $[0,1]$, $\lambda x + (1-\lambda)z \succsim_3^t \lambda y + (1-\lambda)z$? Our first main results states that this will happen if and only if $x \succsim_2^t y$ i.e. there is no subrelation of $\succsim_3^t$ other than $\succsim_2^t$ satisfying the von Neumann-Morgenstern independence property. We then turn to examine a weakened version of the von Neumann-Morgenstern 's test. Precisely we ask: When is it the case that for any λ in $[0,1]$, $x \succsim_3 \lambda x + (1-\lambda)y \succsim_3 y$? We call strong third-degree stochastic dominance this binary relation and we offer a complete characterization of strong third-degree stochastic dominance. The main point in the characterization is its simplicity: it consists of a simple finite list of inequalities very much in the spirit of the classical Lorenz inequalities. Turning to inequality measurement, we then compare our result to some results on inequality measurement with third degree dominance and show exactly why our test is a covariance test strictly more demanding than the classical variance test.

In section 4, we introduce the notion of local stochastic dominance. We say that $x \succsim_i y$ locally, denoted $x \succsim_i^L y$ if any move from x in the direction $y-x$ leads to an improvement in the sense of $\succsim_i$ as long as the intensity of the move is small enough. After noting that local and "global" stochastic

dominance coincide for the first and second stochastic preorders, we show that "global" third-degree stochastic dominance implies local stochastic dominance but that the reverse implication does not hold in general. We offer an almost full simple characterization "à la Lorenz" of local third-degree stochastic dominance and we illustrate how this can be used in the evaluation of policy reforms.

Finally in section 4, we indicate how our results for discrete distributions extend to any distribution and we show why inverse third-degree stochastic dominance implies strong third-degree stochastic dominance when we restrict to some specific smaller cones of random variables. The proofs of all lemmas and propositions are relegated to the appendix.

The results of this paper lead to a better understanding of the first three stochastic preorders. For the first two stochastic orders strong, global, local and inverse are all the same. When we reach the third order these equivalences do not hold anymore: strong implies global which implies local but the reverse implications are false in general; further global and inverse are logically unrelated. Our claim is that the root of this brutal change is the fact that the third-degree stochastic preorder does not satisfy the von Neumann-Morgenstern independence property.

Some Related Literature

Before proceeding with the body of our analysis, let us briefly discuss the relationship of this work to the most closely related literature. Third degree stochastic dominance was introduced in the context of deciding between uncertain prospects and characterized by Whitmore (1970). The property that the third derivative of the von Neumann-Morgenstern utility function has a positive sign has been investigated by Menezes, Geiss and Tressler (1980). Is is strictly less demanding than the classical property asking for the Arrow-Pratt 's measure of risk aversion to decrease with the level of wealth. We could therefore define a new stochastic order for the class of utility functions exhibiting declining risk aversion; Bawa (1975) demonstrates that this new stochastic order coincides with third-degree stochastic dominance as long as the distributions have the same first moment. Bawa produces also a finite algorithm to test for third-degree stochastic dominance. Fishburn (1982, 1985) derive some nice mathematical results on third-degree stochastic dominance.

In the area of inequality measurement, third-degree stochastic dominance has received in the

last decade an increased attention because it offers the obvious advantage of leading sometimes to conclusive judgments[2] about the evolution of inequality in income distribution in situations where the Lorenz curves intersect; it prevents the practitioner from deriving conclusions which would be too sensitive to the choice of a particular inequality index. Early contributions by Atkinson (1973) and Kolm (1976) points out the necessity of supplementing the Lorenz criterion by considerations echoing third-degree stochastic dominance; Kolm's principle of diminishing transfers asks that an inequality index decreases more under the effect of a progressive transfer when the transfer takes place at the bottom part of the income distribution, as opposed to the top part. The more important contributions on this topic are due to Shorrocks and Foster (1987) and Davies and Hoy (1995). The main result in Shorrocks and Foster states that if x and y are two income distributions with the same first moment, then $x \succsim_3 y$ if and only if we can move from y to x by a finite sequence of transfers which are either progressive or variance preserving composite transfers of the following type: some individual i transfers money to some individual j poorer than him and some individual k with an income at least equal to the income of i transfers money to some individual l richer than him. They also show that if the Lorenz curves of x and y intersect only once, then $x \succsim_3 y$ if and only if $x_i > y_i$ where i is the first index for which $x_i \neq y_i$ and the variance of x is less or equal to the variance of y; in words in the case of single crossing of the Lorenz curves, third-degree stochastic dominance is equivalent to the combination of the Rawls's principle and the variance principle. Davies and Hoy (1995) consider the more general case of Lorenz curves with multiple intersections and show that third-degree stochastic dominance amounts to the comparison of variances for truncated income distributions; the truncation points are the intersection points where x intersects y from below.

The main merit of these two contributions is the identification of the key role of the variance in third-degree stochastic dominance. The variance principle is far from being an incontrovertible principle and it is not difficult to come out with examples where the variance principle is not

[2]See Davis and Hoy (1995) and Shorrocks and Foster (1987) for some evidence on real data that third-degree is useful to supplement the Lorenz order. Trannoy and Lugand (1992) make use of third-degree stochastic dominance in their analysis of some French data.

conclusive but where some other principles are conclusive. Take for example the case where $x = (21, 80, 999980, 100000020)$ and $y = (1, 100, 1000000, 100000000)$. The Lorenz curve of x intersects the Lorenz curve of y once from above but the variance of x is greater than the variance of y. In that case however, inverse third-degree stochastic dominance leads to the conclusion that inequality has been reduced when moving from y to x since the Gini index for x is equal to the Gini index for y; the principle of minimal inequality aversion in Le Breton (1994) leads also to the same conclusion for low values of the minimal degree of inequality aversion. This example shows that it is not the case that all excentric inequality conclusions are excluded by third-degree stochastic dominance.

Finally, it should be pointed out that even if the two contributions discussed above improve substantially our understanding of third-degree stochastic dominance, they do not contain a characterization of third-degree stochastic dominance which would have the transparency of Lorenz inequalities; as noted by Shorrocks and Foster "No analogue of Lorenz dominance is known to be equivalent to the third-order stochastic dominance". In our opinion the search of such inequalities is not motivated by the necessity of having a finite algorithm (such algorithms exist) but by the interest of expressing third-degree stochastic dominance $x \succsim_3 y$ by a finite list of inequalities only involving the Lorenz vectors attached to x and y. Our paper provides such characterizations for local third-degree stochastic dominance and strong third degree stochastic dominance but not for third-degree stochastic dominance.

2 Preliminaries

In this section we introduce some general notions on partial preorders defined on real vector spaces. Let K be a convex subset of a real vector space X and $\succsim$ be a preorder over K. We denote respectively by $\succ$ the strict relation induced by $\succsim$ and by $\sim$ the indifference relation induced by $\succsim$. Let $\succsim^*$ and $\succsim^{**}$ be the two subrelations of $\succsim$ defined as follows. Let $x, y \in K$

$$x \succsim^* y \text{ iff } x \succ \lambda x + (1-\lambda)y \succsim y \text{ for all } \lambda \in]0,1[$$

and

$$x \succsim^{**} y \text{ iff } \lambda x + (1-\lambda)z \succsim \lambda y + (1-\lambda)z \text{ for all } \lambda \in]0,1] \text{ and all } z \in K.$$

The latter subrelation $\succsim^{**}$ refines $\succsim$ by asking not only that $x \succsim y$ but also that for any third option z, the mixture $\lambda x + (1-\lambda)z$ is preferred to $\lambda y + (1-\lambda)z$. The former subrelation $\succsim^{*}$ only asks for a sort of intermediateness property: the mixture $\lambda x + (1-\lambda)y$ must always be between x and y. It is trivial to see that $\succsim^{**} \subseteq \succsim^{*}$.

Of some interest are of course the preorders for which either $\succsim^{**} = \succsim$ or $\succsim^{*} = \succsim$. Note first that $\succsim^{**} = \succsim$ is equivalent to:

$$x \succsim y \Rightarrow \lambda x + (1-\lambda)z \succsim \lambda y + (1-\lambda)z \text{ for all } \lambda \in]0,1] \text{ and all } z \in K.$$

This property on $\succsim$ is exactly the independence property of von-Neumann-Morgenstern expected utility theory. Therefore if the preorder is complete and continuous, then it is well known that $\succsim$ satisfies the independence property if and only if there is a continuous linear functional over X which is a utility representation of $\succsim$ over K.

Here we do not assume that $\succsim$ is complete. For each subset $A \subseteq X$, consider the binary relation $\succsim_A$, defined over K as follows:

$$x \succsim_A y \text{ iff } x = y + z \text{ for some } z \in A.$$

It is immediate to show that $\succsim_A$ is a preorder if and only if $0 \in A$ and A is stable under addition. Note that if $A \cap (-A) = \{0\}$, then $x \sim_A y$ if and only if $x = y$. In that case it follows easily that $\succsim_A^{**} = \succsim_A$ if and only if A is a convex cone pointed on 0. Therefore we can generate a large class of partial preorders $\succsim$ for which $\succsim^{**} = \succsim$. Of course if we insist on $\succsim_A$ to be complete, then the cone A must be a half space.

Similarly, $\succsim^{*} = \succsim$ is equivalent to:

$$x \succsim y \Rightarrow x \succ \lambda x + (1-\lambda)y \succsim y \text{ for all } \lambda \in]0,1[\,.$$

It is simple to see that $\succsim_A^* = \succsim_A$ if and only if A is convex. Therefore within the class $\succsim_A$, $\succsim_A^*$ $= \succsim_A$ if and only if $\succsim_A^{**} = \succsim_A$. This does not hold for an arbitrary partial order $\succsim$.

To conclude note that $\succsim^{**}$ is an order but $\succsim^*$ is not transitive in general (however $\succsim^*$ is of course acyclic) as simply illustrated by the two indifference curves of a complete preorder depicted on figure 1 below

Insert Figure1 here

3 Stochastic Orders: A Characterization of $\succ_3^{t*}$ and $\succ_3^{t**}$

From now on, we focus on the family of stochastic dominance orders. These partial orders are defined on subsets of probability distributions over the real numbers. We limit our attention to discrete probability distributions i.e. to probability distribution P of the following type:[3]

$$P = \sum_{j=1}^{n} p_i \delta_{x_j}, \text{ where } x_1 \leq x_2 \leq ..... \leq x_n, \; p_i \geq 0 \; \forall i = 1, ...n \text{ and } \sum_{i=1}^{n} p_i = 1,$$

P can be interpreted as the uncertain prospect or lottery where the worst outcome is x_1 and has probability p_1, the next worst outcome is x_2 and has probability p_2 and so on. P can also be interpreted as an income distribution in a society. The society is divided into n groups from the poorest group denoted by 1 to the richest group denoted by n. In that interpretation, x_i and p_i denotes respectively the mean outcome and the percentage of the population in group i. We denote by $\mathcal{P}$ the set of discrete probability distributions.

To define the first three stochastic orders over $\mathcal{P}$, we need the following family of utility functions. $\mathcal{U}_1$ denotes the set of non decreasing real valued functions over $\Re_+$; $\mathcal{U}_2$ denotes the set of non decreasing and concave real valued functions over $\Re_+$ and $\mathcal{U}_3$ denotes the set of differentiable real valued functions over $\Re_+$ whose first derivative is non negative, non increasing and convex. Then for all $P = \sum_{j=1}^{n} p_i \delta_{x_j}$ and $Q = \sum_{j=1}^{m} q_i \delta_{y_j}$ and all $s = 1, 2, 3$:

[3] for all $t \in \Re$, δ_t denotes the Dirac mass in t.

$$P \succsim_s Q \text{ iff } \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{m} q_i u(y_i) \text{ for all } u \in \mathcal{U}_s.$$

The classical results on stochastic dominance are summarized in the following proposition. Let E_P and F_P denote respectively the first moment of P and the distribution function of probability P, i.e. for all $t \in \Re$, $F_P(t) = P(]-\infty, t]$.

Proposition 1 *Let P, $Q \in \mathcal{P}$. Then:*

$P \succsim_1 Q$ *iff* $F_P(t) \leq F_Q(t)$ *for all* $t \in \Re$,

$P \succsim_2 Q$ *iff* $\int_{-\infty}^{t} F_P(u)du \leq \int_{-\infty}^{t} F_Q(u)du$ *for all* $t \in \Re$,

$P \succsim_3 Q$ *iff* $\int_{-\infty}^{t}\int_{-\infty}^{r} F_P(u)dudr \leq \int_{-\infty}^{t}\int_{-\infty}^{r} F_Q(u)dudr$ *for all* $t \in \Re$ *and* $E_P \geq E_Q$.

Any discrete probability distribution can be approximated by a distribution where the probabilities p_i are all equal. We limit our attention to those with support in $\Re_+$ and we denote by $\mathcal{P}_n$ the subset of such probabilities whose support consist of at most n points. The set $\mathcal{P}_n$ is in a one to one relationship with the cone K_n defined as follows.

$$K_n = \left\{x \in \Re_+^n : x_1 \leq x_2 \leq ..... \leq x_n\right\}.$$

The stochastic orders on $\mathcal{P}_n$ are transported as follows on K_n. For all $x, y \in K_n$ and all $s = 1, 2, 3$ let:

$$x \succsim_s^t y \text{ if and only if } \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j} \succsim_s \frac{1}{n}\sum_{j=1}^{n} \delta_{y_j}$$

i.e.

$$x \succsim_s^t y \text{ iff } \sum_{i=1}^{n} u(x_i) \geq \sum_{i=1}^{n} u(y_i) \text{ for all } u \in \mathcal{U}_s.$$

For all $x \in K_n$ and all $j = 1, ..., n$, let $X_j = \sum_{i=1}^{j} x_i$. In what follows, we will refer to X as being the *Lorenz vector*[4] attached to x. The following proposition can be deduced from Proposition

[4]Strictly speaking it should be called the generalized Lorenz vector (Shorrocks (1983)) since the Lorenz vector refers to the normalized vector where each of the X_i is divided by X_n.

1 or demonstrated directly. The second part is due to Hardy, Littlewood and Polya (1934).[5]

Proposition 2 *Let $x, y \in K_n$. Then:*

$x \succsim_1^t y$ iff $x_j \geq y_j$ for all $j = 1, ...n$ and $x \succsim_2^t y$ iff $X_j \geq Y_j$ for all $j = 1, ...n$.

It follows from Proposition 1 that both $\succsim_1$, $\succsim_2$ and $\succsim_3$ satisfy the von Neumann-Morgenstern independence property and therefore $\succ_1 = \succ_1^* = \succ_1^{**}$, $\succ_2 = \succ_2^* = \succ_2^{**}$ and $\succ_3 = \succ_3^* = \succ_3^{**}$. It follows from Proposition 2 that both $\succsim_1^t$ and $\succsim_2^t$ are cone preorders. Precisely, $\succsim_1^t = \succsim_{A_1}$ and $\succsim_2^t = \succsim_{A_2}$ where $A_1 = \{x \in \Re^n : x_i \geq 0 \ \forall i = 1, ..., n\}$ and $A_2 = \{x \in \Re^n : X_j \geq 0 \ \forall j = 1, ..., n\}$. Therefore, from section 2, they satisfy the von Neumann-Morgenstern independence property and then $\succ_1^t = \succ_1^{t*} = \succ_1^{t**}$and $\succ_2^t = \succ_2^{t*} = \succ_2^{t**}$. Note that $\mathcal{P}_n$ is not convex in $\mathcal{P}$ and that for $x, y \in K_n$ and $\lambda \in]0, 1[$, $\lambda(\frac{1}{n}\sum_{j=1}^n \delta_{x_j}) + (1-\lambda)\frac{1}{n}\sum_{j=1}^n \delta_{y_j}$ is not the same as $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda x_i + (1-\lambda)y_i}$ and the meaning of convex addition differs in the two spaces.[6] The following simple example shows that the third-degree dominance stochastic order $\succsim_3^t$ fails to satisfy even the weak form of the von Neumann-Morgenstern independence property.

Example 1 *Let $n = 4$, $x = (4, 6, 11, 14)$ and $y = (2, 10, 11, 12)$. We can verify that $x \succ_3^t y$.*

Consider now the utility function u defined as follows:

$$u(t) = \begin{cases} 10t - \frac{t^2}{2} \text{ if } t \leq 10 \\ 50 \text{ if } t > 10. \end{cases}$$

It is easy to verify that $u \in U_3$. For all $\lambda \in [0, 1]$, let $W(\lambda) = \sum_{j=1}^n u(\lambda x_j + (1-\lambda)y_j)$. We obtain:

$$W(\lambda) = 168 + 16\lambda - 10\lambda^2.$$

Since $W(0.8) > W(1)$, we don't have $x \succ_3^{t*} y$.

[5] See Marshall and Olkin (1979).

[6] In fact $\mathcal{P}_n$ is not convex in $\mathcal{P}$; $\lambda P + (1-\lambda)Q$ simply denotes the composite lottery where the lotteries P and Q are drawn with probabilities λ and $1-\lambda$. When P and Q stand for the income distributions x and y, I dont see any immediate interpretation of the convex addition $\lambda P + (1-\lambda)Q$ in terms of income distribution while the convex addition $\lambda x + (1-\lambda)y$ is easy to interpret.

We now turn to the characterization of $\succsim_3^{t**}$ and $\succsim_3^{t*}$.

Proposition 3 $\succsim_3^{t**} = \succsim_2^t$.

Proposition 3 is rather intriguing.

Proposition 4 Let $x, y \in K_n$. Then $x \succsim_3^{t*} y$ if and only if $\sum_{j=1}^{k}(x_{j+1} - x_j)(X_j - Y_j) \geq 0$ for all $k = 1, ..., n-1$ and $X_n \geq Y_n$.

Proposition 4 provides a simple full characterization of strong third-degree stochastic dominance. The finite list of inequalities in Proposition 4 is very much in the spirit of the Lorenz inequalities. They consist in comparing weighted partial sums of the Lorenz coordinates of the two distributions under scrutiny, where the Lorenz coordinate of rank j is weighted by the nonnegative coefficient $(x_{j+1} - x_j)$. Since $\succsim_3^{t*}$ is a subrelation of $\succsim_3^t$, we deduce:

Corollary 1 If $\sum_{j=1}^{k}(x_{j+1} - x_j)(X_j - Y_j) \geq 0$ for all $k = 1, ..., n-1$ and $X_n \geq Y_n$ then $x \succsim_3^t y$.

Corollary 1 provides a simple sufficiency test for third-degree stochastic dominance to hold. We now contrast this simple condition with was has been obtained in the literature. To this end, consider two distributions x and y in K_n and define an intersection index as any value of j for which either $X_{j-1} > Y_{j-1}$ and $X_j \leq Y_j$ or $X_{j-1} < Y_{j-1}$ and $X_j \geq Y_j$.[7] Label the intersection indices by order of appearance. It is a trivial exercise to observe that the inequalities in Corollary 1 hold if and only if $\sum_{j=1}^{k}(x_{j+1} - x_j)(X_j - Y_j) \geq 0$ for the first intersection label and all intersection indices k with an even label. For all x and y in K_n, denote by $\|x\|$ and $\|y\|$ their respective Euclidean norms, by $\overline{x}$ and $\overline{y}$ their respective means and by $< x, y >$ their Euclidean scalar product. In the case where there is a single crossing i.e. two intersection indices, we obtain the following corollary.

Corollary 2 Let $x, y \in K_n$ with $\overline{x} = \overline{y}$ and assume that the Lorenz curve of x intersects that of y once from above. Then $x \succ_3^{t*} y$ if and only if $\| x \|^2 \leq < x, y >$.

[7] With the convention $X_0 = Y_0 = 0$.

The proof follows from the fact that under single crossing, the unique intersection index i with an even label satisfies $x_j = y_j$ for all $j = i+1, ...n$ and can therefore, without loss of generality be assumed to be equal to n. Since $\overline{x} = \overline{y}$, it is easy to verify that $\sum_{j=1}^{n}(x_{j+1} - x_j)(X_j - Y_j) \geq 0$ if and only if $\sum_{j=1}^{n} x_i^2 \geq \sum_{j=1}^{n} x_i y_i$. Corollary 2 can be contrasted with Theorem 3 in Shorrocks and Foster (1987). Under the assumptions of Corollary 2, they show that $x \succ_3^t y$ if and only if $\| x \|^2 \leq \| y \|^2$. Since from the Cauchy-Schwarz's inequality, $< x, y > \leq \| x \| \| y \|$ we see immediately why $x \succ_3^{t*} y$ is strictly more demanding than $x \succ_3^t y$. Third-degree stochastic dominance is a variance test whereas strong third-degree stochastic dominance is a covariance test. In fact in the particular case where $\| x \| = \| y \|$, we cannot have $x \succ_3^{t*} y$. Indeed from Corollary 2, if $x \succ_3^* y$, then $\| x \|^2 \leq < x, y >$. From Cauchy-Schwarz's inequality we deduce $\| x \|^2 \leq < x, y > \leq \| x \| \| y \| = \| x \|^2$and therefore $< x, y > = \| x \| \| y \|$. But Cauchy-Schwarz 's inequality is an equality if and only if $x = \lambda y$ for some $\lambda \in \Re$. Since $\overline{x} = \overline{y}$, we obtain $x = y$. Therefore if x results from y by variance preserving transfers, then we cannot obtain $x \succ_3^* y$. This suggests that transfers will lead to a reduction of inequality in the sense of strong third-degree stochastic dominance only if they strictly reduce the variance. In Figure 2

Insert Figure 2 here

we represent all the feasible distributions of a given total income among three agents as points of a Kolm's triangle (such that the height of the triangle is equal to the total income). For an income distribution B, the irregular hexagon with vertex B contains all the vectors that dominate B according to SSD. Under the single crossing condition of Corollary 2, the circle passing through B contains all the points that dominate B according to TSD. Then, $A \succ_3^{t*} B$ is equivalent to $\|A\|^2 < \langle AB \rangle$. This condition leads to $\left\|A - \bar{A}\right\|^2 < (A - \bar{A})(B - \bar{A})$, which gives

$$\left\|A - \bar{A}\right\| < \left\|B - \bar{A}\right\| \cos\beta, \tag{1}$$

where β is the angle generated by the vectors $A - \bar{A}$ and $B - \bar{B}$.

Condition (1) can be geometrically expressed as $OA < OC$, where C is the projection of B on the half-line starting from O and passing through A. The higher part of box 1 in Figure 2 illustrates

that $\succ_3^{t*}$ satisfies the independence property: for any point G of the segment between A and B, we get $OG < OC'$. This evidence can be opposed to the violation of the independence property that occurs to TSD in the lower part of the box 1, where $L \succ_3^t H$ and both vectors are strictly dominated by D in the sense of TSD.

3.1 Discussion

The different policy implications of strong third-degree stochastic dominance and TSD are clarified in the following example.

Example 2 *Let n =3 to model the case where the society is divided into three classes of equal size: the poor class indexed by 1, the middle class indexed by 2 and the rich class indexed by 3. This should be considered as an interpolation of the true "continuous" income distribution obtained by linear interpolation of the Lorenz curve in $\frac{1}{3}$ and $\frac{2}{3}$. Let $y = (y_1, y_2, y_3)$ denotes the current income distribution and consider a public policy leading to the income distribution $x = (y_1 + \delta, y_2 - \delta - \Delta, y_3 + \Delta)$ where δ and Δ are positive numbers and such that $2\delta + \Delta < y_2 - y_1$ i.e. a policy improving the situation of the poor and rich classes at the expense of the middle class. We use Shorrocks and Foster's Theorem 3 and Corollary 2 to obtain:*

$$x \succ_3^t y \text{ if and only if } \delta(y_2 - y_1) - \Delta(y_3 - y_2) \geq \delta^2 + \Delta^2 + \delta\Delta$$

and

$$x \succ_3^{t*} y \text{ if and only if } \delta(y_2 - y_1) - \Delta(y_3 - y_2) \geq 2\delta^2 + 2\delta\Delta + 2\Delta^2.$$

Let $r(y) \equiv 1 - \frac{\frac{y_3+y_1}{2}}{y_2}$ be a relative measure of the gap between the middle class mean income and the average of the extreme classes mean incomes in the original distribution y. In the case where $\delta = \Delta$, the two conditions above simplify to:

$$x \succ_3^t y \text{ if and only if } r(y) \geq \frac{3\Delta}{2y_2}$$

and

$$x \succ_3^{t*} y \text{ if and only if } r(y) \geq \frac{3\Delta}{y_2}.$$

These two conditions illustrate the differences between third-degree stochastic dominance and strong third-degree stochastic dominance. We see that not only $r(y)$ must be strictly positive for both to hold but also that the total relative burden of the transfer program on the middle class should not exceed $\frac{4r(y)}{3}$ for third-degree stochastic dominance and should not exceed $\frac{2r(y)}{3}$ for strong third-degree stochastic dominance.

This example is also useful to illustrate a peculiar fact of strong third-degree stochastic dominance. Indeed let i be the first index for which $X_i < Y_i$. Then if $x_k = x_1$ for all $k = 1, ..., i$, we deduce from Proposition 4 that $x \succ_3^* y$ does not hold. This means that a transfer policy leading to a perfect equalization among the first i groups with a reduction of the total share of the first i groups cannot pass the test of strong third-degree stochastic dominance. Let $y = (1, 81, 100)$, $\delta = 39$ and $\Delta = 2$ in the above specification. Here $r(y) \approx 0,376$ i.e. we are in a situation where the middle class is on the rich side.[8] *By applying the condition above, we can verify that $x \succ_3^t y$ but not $x \succ_3^{t*} y$. A careful reader will check that for the following utility function in U_3*

$$u(t) = \begin{cases} 100t - \frac{t^2}{2} \text{ if } t \leq 100 \\ 5000 \text{ if } t > 100 \end{cases}$$

the income distribution $\lambda x + (1-\lambda)y$ where $\lambda = \frac{3082}{3202}$ is superior to x. It should be noted however that λ is very close to 1 i.e. x is not far from being the best outcome for this utility function.

Let us still use the Figure 2 to stress this important difference among $\succ_3^{t*}$ and TSD. Starting from the income distribution B, according to TSD, the regressive transfer from the median class to the richest one which can be "compensated" by a progressive transfer from the median class to the poor one reaches its maximum when the incomes of the two less endowed classes are equalized ($A = D$). On the contrary, under $\succ_3^{t*}$, the regressive transfer in favour of the rich class CH that can be balanced (in terms of social welfare) by a progressive transfer from the median class to the

[8] Note that $r(y) \leq \frac{1}{2}$ for all income distributions y.

poor one first increases and then decreases with β, becoming 0 when $\beta = \frac{\pi}{2}$. Furthermore, CH reaches its max when $\alpha = \beta$.[9] What is really important is to reduce to a half the gap between the median and the poor class, rather than totally fill such a gap.

4 Local Stochastic Dominance

Stochastic dominance orders have a global character: we can compare any pair of lotteries or income distributions possibly very far apart from each other. In this section, we introduce the concept of local stochastic dominance to address questions of the following type. Suppose that at some point in time the income distribution of a society is described by $y \in K_n$ and we ask ourselves whether the situation would improve if we move locally in the direction $\xi \equiv x - y$ where $x \in K_n$. By locally, we mean that there exists some $\overline{\lambda} > 0$ such that $y + \lambda\xi$ is an improvement for all $\lambda \leq \overline{\lambda}$. Improvement can be defined in several ways and here we will limit ourselves to the stochastic orders $\succsim_s^t$ for $s = 1, 2, 3$. Precisely we will say that the direction $\xi \in \Re^n$ is a $\succsim_s^t$ direction of improvement at $y \in K_n$ if there exists $\overline{\lambda} > 0$ such that:

$$y + \lambda\xi \ \succsim_s^t y \text{ for all } \overline{\lambda} \in [0, \lambda].$$

This leads to the following definition of local stochastic dominance:

$x \succsim_s^{tl} y$ iff there exists $\overline{\lambda} > 0$ such that: $y + \lambda(x - y) \succsim_s^t y$ for all $\lambda \in \left[0, \overline{\lambda}\right]$.

The difference between "global" and local stochastic dominance appears clearly in the above definition since we only ask for improving local changes in the direction $x - y$ instead of asking of moving all the way from y to x. Local and "global" stochastic dominance coincide for the first and second degrees.[10]

Proposition 5 Let $x, y \in K_n$. Then $x \succsim_s^t y$ if and only if $x \succsim_s^{tl} y$ for $s = 1, 2$.

[9] In fact, $CH = OB \sin\beta \sin\alpha$, that is equal to $OB\frac{1}{2}\cos(\beta - \alpha) - \frac{1}{2}\cos(\beta + \alpha)$. Since β and α belong to the interval $[0, \frac{\pi}{2}]$ and $(\beta + \alpha)$ is fixed, CH is maximized for $\beta = \alpha$.

[10] The proof of this proposition follows closely the standard arguments in stochastic dominance and is omitted.

This equivalence does not hold at the third degree and we deduce therefore from proposition 4 that local third-degree stochastic dominance is *strictly less demanding* than third-degree stochastic dominance. The following proposition states a necessary condition[11] for local third-degree stochastic dominance.

Proposition 6 Let $x, y \in K_n$. If $x \succsim_3^{tl} y$ then $X_n \geq Y_n$ and for all $k = 1, ..., n-1$, $\sum_{j=1}^{k}(y_{j+1} - y_j)(X_j - Y_j) \geq 0$.

The condition is simple and also very much in the spirit of the Lorenz inequalities. They consist in comparing weighted partial sums of the Lorenz coordinates of the two distributions under scrutiny where the Lorenz coordinate of rank j is now weighted by the nonnegative coefficient $(y_{j+1} - y_j)$ instead of the coefficient $(x_{j+1} - x_j)$. The cone of improving directions at y is the set of vectors ξ in $\Re^n$ such that:

$$\sum_{i=1}^{n} \xi_i \geq 0 \text{ and } \sum_{j=1}^{k}\sum_{i=1}^{j}(y_{j+1} - y_j)\xi_i > 0 \text{ for all } k = 1, ...n-1.$$

Since $\succsim_3^t$ is a subrelation of $\succsim_3^{tl}$, we deduce the following necessity test for third-degree stochastic dominance:

Corollary 3 If $x \succsim_3^{tl} y$ then, $\sum_{j=1}^{k}(y_{j+1} - y_j)(X_j - Y_j) \geq 0$ for all $k = 1, ..., n-1$ and $X_n \geq Y_n$.

As we did in Section 3, we can always see how this results specializes in the case where the Lorenz curves display a specific pattern.

Corollary 4 Let $x, y \in K_n$ with $\overline{x} = \overline{y}$ and assume that the Lorenz curve of x intersects that of y once from above. Then if $x \succ_3^{t*} y$ then $< x, y > \leq \| y \|^2$.

The proof is identical to the proof of Corollary 2. Corollary 4 can be contrasted with Theorem 3 in Shorrocks and Foster (1987). We already know that under the assumptions of Corollary 4,

[11] These conditions are in fact almost sufficient. We have just to be careful in the case where some of the inequalities $\sum_{j=1}^{k}(y_{j+1} - y_j)(X_j - Y_j) = 0$ as in such a case we need to move to an higher order.

third-degree stochastic dominance is the variance test $\| x \|^2 \leq \| y \|^2$, whereas local third-degree stochastic dominance is again a covariance test. From the Cauchy-Schwarz's inequality, we see immediately that this covariance test is indeed less demanding than the variance test. Therefore local third-degree stochastic dominance may be conclusive in situations where the variance of x is strictly greater than the variance of y. To see by how much the two criteria differ, let us consider Example 2 examined in the previous section.

Example 3 *Let n=3 and as in Example 2 let* $y = (y_1, y_2, y_3)$ *denotes the current income distribution and consider a public policy leading to the income distribution* $x = (y_1 + \delta, y_2 - \delta - \Delta, y_3 + \Delta)$ *where* δ *and* Δ *are positive numbers and such that* $2\delta + \Delta < y_2 - y_1$ *i.e. a policy improving the situation of the poor and rich classes at the expense of the middle class. We already know that:*

$$x \succ_3^t y \text{ if and only if } \delta(y_2 - y_1) - \Delta(y_3 - y_2) \geq \delta^2 + \Delta^2 + \delta\Delta.$$

Instead, Corollary 3 leads to:

$$x \succ_3^{tl} y \text{ if and only if } \delta(y_2 - y_1) - \Delta(y_3 - y_2) \geq 0$$

In the case where $\delta = \Delta$*, the two conditions above simplify to:*

$$x \succ_3^t y \text{ if and only if } r(y) \geq \frac{3\Delta}{2y_2}$$

and

$$x \succ_3^{t*} y \text{ if and only if } r(y) \geq 0.$$

For $y = (1, 81, 100)$*, the cone of improving directions is the set of vectors* $\xi \in \Re^3$ *such that:*

$$\begin{aligned} \xi_1 &\geq 0 \\ 99\xi_1 + 19\xi_2 &\geq 0 \\ \text{and } \xi_1 + \xi_2 + \xi_3 &\geq 0. \end{aligned}$$

Picture 2 can also be used to illustrate $\succ_3^{tl}$. Since $A \succ_3^{tl} L$ implies $\langle AL \rangle < \|L\|^2$, we get

$$\left\|A - \bar{A}\right\| \cos\vartheta < \left\|L - \bar{A}\right\|. \tag{2}$$

Then, it is easy to see that $\succ_3^{tl}$ refines TSD. When the income of the poorest class approaches that of the median one, the regressive transfers from the median to the rich class which are consistent with $\succ_3^{tl}$ (as points near to KD in Figure 2) exceed the regressive transfers allowed by TSD.

We would like to argue that the local stochastic dominance point of view seems very much appropriate to examine public policy reforms along the lines pioneered by Feldstein (1976), Guesnerie (1977), Weymark (1981) and others in their analysis of Pareto improving commodity taxation reforms. A Pareto improving direction of reform is exactly defined as a direction leading to a welfare improvement for everybody in the neighborhood of the original policy. To see, how it applies here, let $y(\theta) = (\ y_1(\theta), ..., \ y_n(\theta))$ be the current income distribution of an economy where θ denotes the vector of policy decisions by the public sector. Then Proposition 6 gives a practical criterion to decide whether moving from θ to $\theta + d\theta$ leads to a third-degree stochastic dominance improvement. Suppose for instance that θ is one dimensional. Then moving from θ to $\theta + d\theta$ leads to an improvement if and only if:

$$\sum_{j=1}^{k}(y_{j+1}(\theta) - y_j(\theta))(\sum_{i=1}^{j}\frac{dx_i(\theta)}{d\theta}) \geq 0 \text{ for all } k = 1, ..., n-1 \text{ and } \sum_{i=1}^{n}\frac{dx_i(\theta)}{d\theta} \geq 0.$$

5 Extensions

In the preceding sections, we limited our attention to probability distributions with a discrete support but the notion of stochastic orders can be extended easily to the broader family of bounded

(i.e.; with compact support) probability distributions over $\Re_+$. Given two such distributions P and Q, define for all i=1,2,3.

$$P \succsim_i Q \text{ iff } \int_{\Re_+} u(x)P(dx) \geq \int_{\Re_+} u(x)Q(dx) \text{ for all } u \in \mathcal{U}_i.$$

A probability distribution P over $\Re_+$ is entirely characterized by its distribution function F defined as $F(x) = P([0,x])$. Characterizations of the stochastic orders $\succsim_i$have been provided in terms of distribution functions but these characterizations are not very meaningful in the area of inequality measurement where alternative characterizations in terms of "inverse distribution functions" are usually privileged. Recall that given a probability distribution P over $\Re_+$, there exists a unique random variable x over $[0,1]$ increasing, right continuous and with P as probability law when $[0,1]$ is endowed with the Lebesgue measure.[12] Like in the preceding sections, we can now investigate characterizations over the cone D of increasing and right continuous random variables in $L^\infty([0,1])$. The following characterizations of $\succsim_1$ and $\succsim_2$ are well known. Given x and y in the cone D:

$$x \succsim_1 y \text{ iff } x(t) \geq y(t) \text{ for all } t \in [0,1]$$

$$x \succsim_2 y \text{ iff } X(t) \geq Y(t) \text{ for all } t \in [0,1],$$

where $X(t) \equiv \int_0^t x(s)ds$ and $Y(t) \equiv \int_0^t y(s)ds$. Muliere and Scarsini (1989) have considered the stochastic order $\succsim_3'$ defined as follows:

$$x \succsim_3' y \text{ iff } \widehat{X}(t) \geq \widehat{Y}(t) \text{ for all } t \in [0,1]$$

where $\widehat{X}(t) \equiv \int_0^t X(s)ds$ and $\widehat{Y}(t) \equiv \int_0^t Y(s)ds$. The stochastic order $\succsim_3$ is different from the stochastic order $\succsim_3$ and we don't have a simple characterization of $\succsim_3$ over D. It is straightforward to extend the preceding propositions concerning $\succsim_3^*$ and $\succsim_3^{**}$ to the cone D.[13]

[12] This random variable is often designated under the name right inverse of P or F and denoted accordingly F^{-1}.

[13] This technical exercise is left to the reader.

We would like to conclude by a simple proposition which helps to understand why the stochastic orders $\succsim_3$ and $\succsim_3'$ differ in general.

Proposition 7 Let x be in D and with a continuous second derivative such that $x''(t) \leq 0$ for all t in $[0,1]$. Then for any y in D such that $x \succsim_3' y$ and $X(1) \geq Y(1)$, $x \succsim_3^* y$ and therefore $x \succsim_3 y$.

Proposition 7 points out the root of the gap between $\succsim_3'$and $\succsim_3$. Indeed, within the cone D, we control the sign of the first derivative but not the sign of the second. This suggests that it could be valuable to explore the stochastic order $\succsim_3$over the cone of "concave" random variables.

6 Appendix

Proof of Proposition 3: Since $\succsim_2 \subset \succsim_3$ and $\succsim_2^{**} = \succsim_2$, we obtain $\succsim_2 \subset \succsim_3^{**}$. We now prove that $\succsim_3^{**} \subset \succsim_2$. Let $x, y \in K$ with $x \succsim_3^{**} y$, i.e.

$$F_{u,z}(\lambda) = \sum_{i=1}^{n} u(\lambda x_i + (1-\lambda) z_i) - \sum_{i=1}^{n} u(\lambda y_i + (1-\lambda) z_i) \geq 0 \text{ for all } z \in K, u \in \mathcal{U}_3 \text{ and } \lambda \in [0,1]\,. \quad (3)$$

Since $F_{u,z}(0) = 0$, (1) implies:

$$F'_{u,z}(0) = \sum_{i=1}^{n} u'(z_i)(x_i - y_i) \geq 0 \text{ for all } z \in K \text{ and } u \in \mathcal{U}_3. \quad (4)$$

Let

$$u_t(w) = \begin{cases} tw - \frac{w^2}{2} \text{ if } w \leq t \\ \frac{t^2}{2} \text{ if } w > t \end{cases}$$

$z_i = t - \epsilon$ for all $i = 1, ..., k$ and $z_i = t + \epsilon$ for all $i = k+1, ..., n$ for some $t > 0$ and $0 < \epsilon < t$. Since $u_t \in \mathcal{U}_3$, we deduce from (2)

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$$

as desired.□

Proof of Proposition 4: Let $x, y \in K$; $x \succsim_3^* y$ is equivalently formulated as:

$$F_u(1) \geq F_u(\lambda) \geq F_u(0) \text{ for all } u \in \mathcal{U}_3 \text{ and } \lambda \in [0,1] \tag{5}$$

where:

$$F_u(\lambda) = \sum_{i=1}^{n} u(\lambda x_i + (1-\lambda) y_i).$$

To proceed, we need the following claims:[14]

Claim 1: (5) is equivalent to $F_u'(1) \geq 0$ for all $u \in \mathcal{U}_3$

Proof of Claim 1: Necessity follows immediately from (5). To prove sufficiency let $u \in \mathcal{U}_3$. Since

$$F_u''(\lambda) = \sum_{i=1}^{n} u''(\lambda x_i + (1-\lambda) y_i)(x_i - y_i)^2$$

it follows that F_u is concave on $[0,1]$. Therefore, if $F_u'(1) \geq 0$ then $F_u'(\lambda) \geq 0$ and (5) follows.□

Claim 2: Every $u \in \mathcal{U}_3$ is the uniform limit of positive linear combinations of functions u_t where

$$u_t(w) = \begin{cases} tw - \frac{w^2}{2} \text{ if } w \leq t \\ \frac{t^2}{2} \text{ if } w > t. \end{cases}$$

By combining the two claims and noting that $u_t'(w) = Max(t-w, 0)$ for all $t > 0$, we deduce that $x \succsim_3^* y$ if and only if:

$$\Phi(t) \equiv F_{u_t}'(1) = \sum_{i=1}^{n} Max(t - x_i, 0)(x_i - y_i) \geq 0 \text{ for all } t > 0$$

[14] The proof of claim 2 appears in Le Breton (1987).

Φ is linear on the intervals $[0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ and $[x_n, +\infty[$. Therefore $\Phi(t)$ for all $t > 0$ if and only if:

$$\Phi(x_k) \geq 0 \text{ for all } k = 2, ..., n \text{ and } \Phi'(t) \geq 0 \text{ on } [x_n, +\infty[.$$

Note that:

$$\Phi(x_k) = \sum_{i=1}^{k-1} (x_k - x_i)(x_i - y_i).$$

By using the Abel's trick, we deduce:

$$\Phi(x_k) = \sum_{i=1}^{k-1} (x_{i+1} - x_i)(X_i - Y_i).$$

The last inequality in Proposition 4 follows from the fact that $\Phi(t) = t(X_n - Y_n) - \sum_{i=1}^{n} x_i(x_i - y_i)$ and therefore $\Phi'(t) = X_n - Y_n$.□

Proof of Proposition 6: Let $x, y \in K$; if $x \succsim_3^{tl} y$ then we deduce:

$$F'_u(0) \geq 0 \text{ for all } u \in \mathcal{U}_3. \tag{6}$$

Proceeding as in the proof of Proposition 4, we deduce that (4) holds if and only if:

$$\Psi(t) \equiv \sum_{i=1}^{n} Max(t - y_i, 0)(x_i - y_i) \geq 0 \text{ for all } t > 0$$

the rest of the proof follows the last step in the proof of Proposition 4 and is omitted.□

Proof of Proposition 7: Let $x, y \in K$. As in the proof of Proposition 4, $x \succsim_3^{*} y$ is equivalent to:

$$F'_u(1) \geq 0 \text{ for all } u \in \mathcal{U}_3$$

where:

$$F_u(\lambda) = \int_0^1 u(\lambda x(t) + (1-\lambda)y(t))dt.$$

Let u be three times continuously differentiable.[15] Since

$$F'_u(1) = \int_0^1 u'(x(t))(x(t) - y(t))dt$$

integrating by parts, we obtain:

$$\begin{aligned} F'_u(1) &= u'(x(1))(X(1) - Y(1)) - u''(x(1))(\widehat{X}(1) - \widehat{Y}(1) \\ &\quad + \int_0^1 \left(u'''(x(t))x'(t)^2 + u''(t)x''(t)(\widehat{X}(t) - \widehat{Y}(t)\right) dt \end{aligned}$$

Since $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$ and $x'' \leq 0$, the conclusion follows. □

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[15] This is without loss of generality since any u in $\mathcal{U}_3$ can be uniformly approximated with such functions within the class $\mathcal{U}_3$. A proof of this fact and similar approximation statements can be found in Le Breton (1986).

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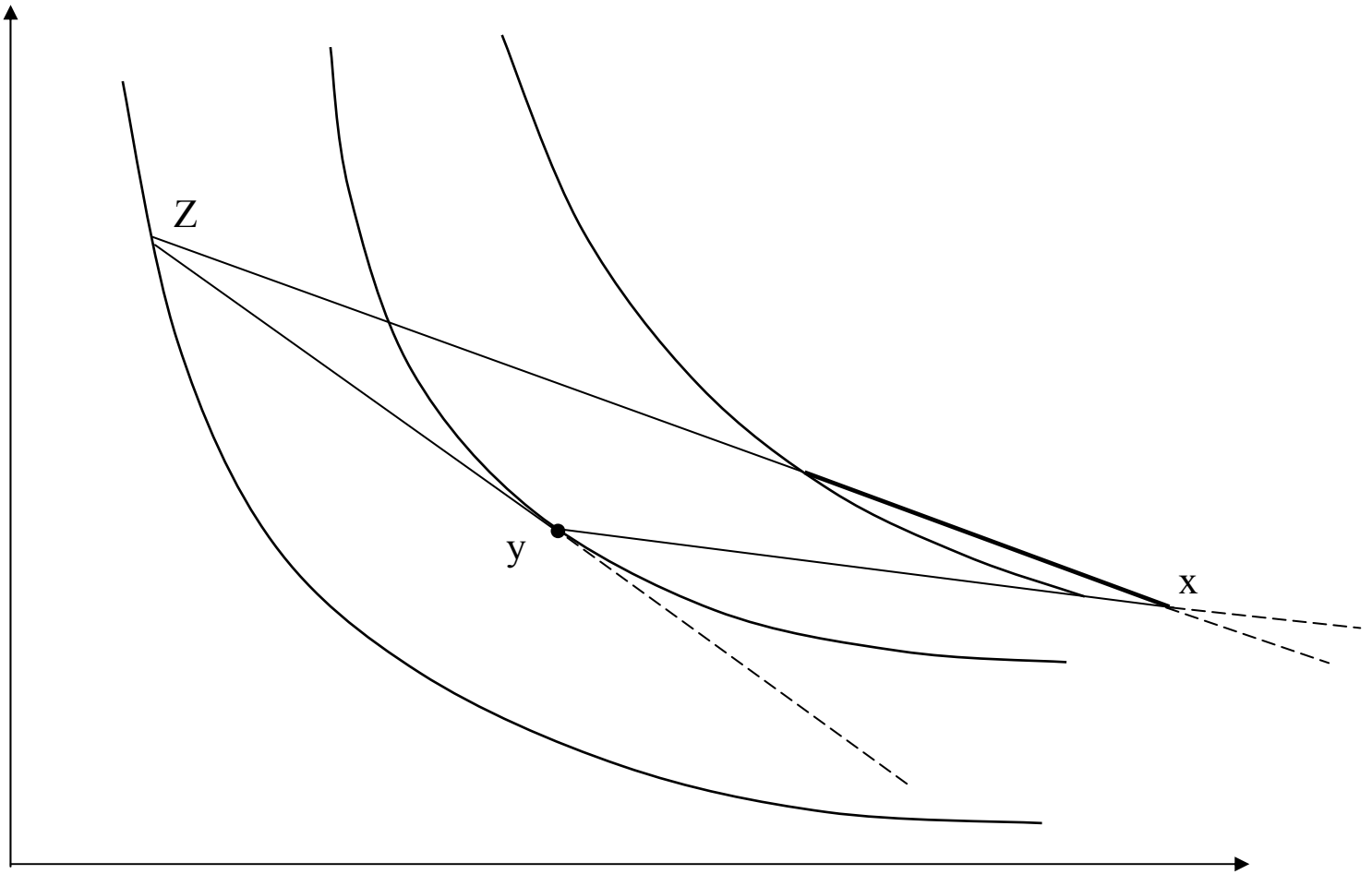
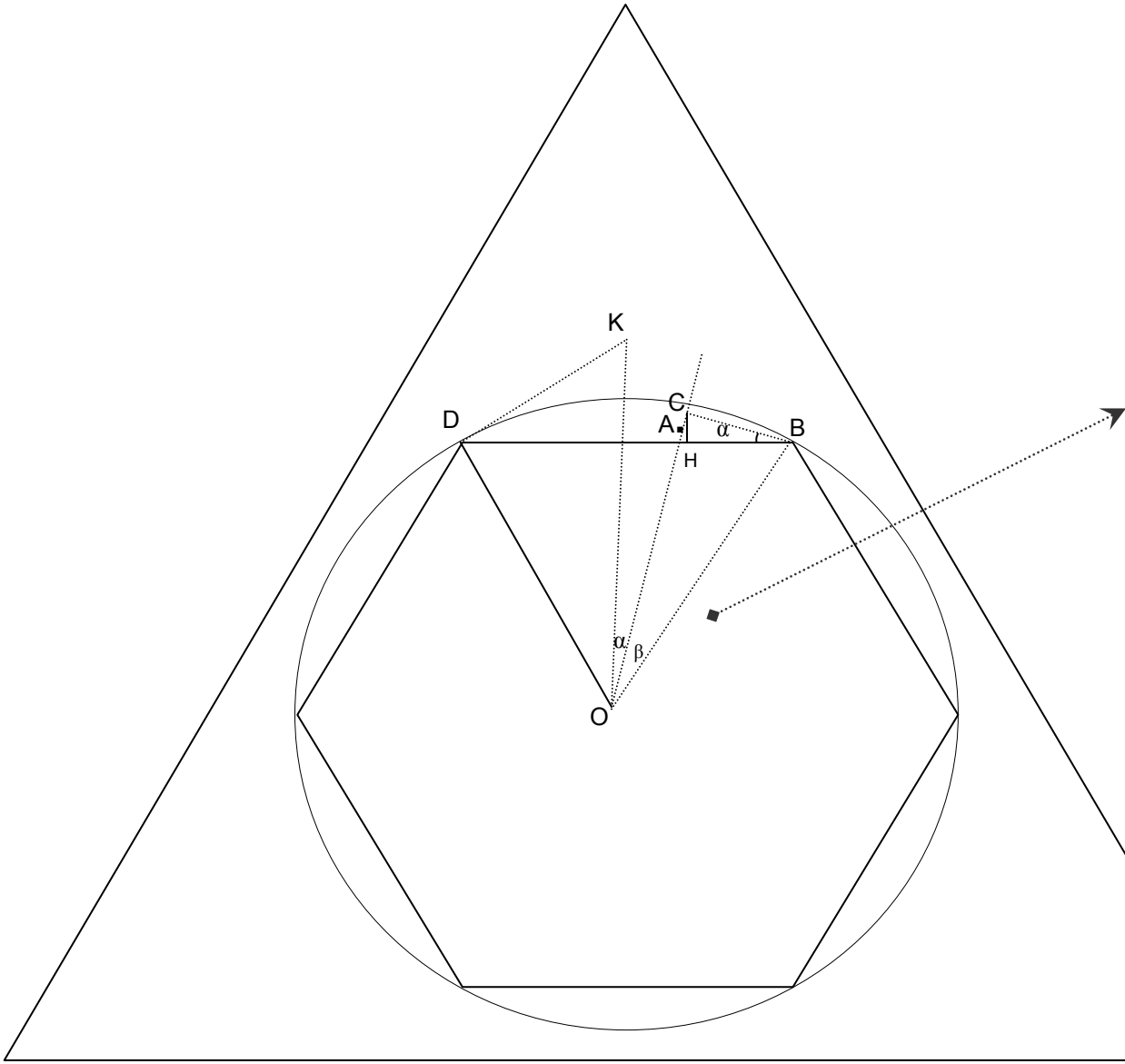
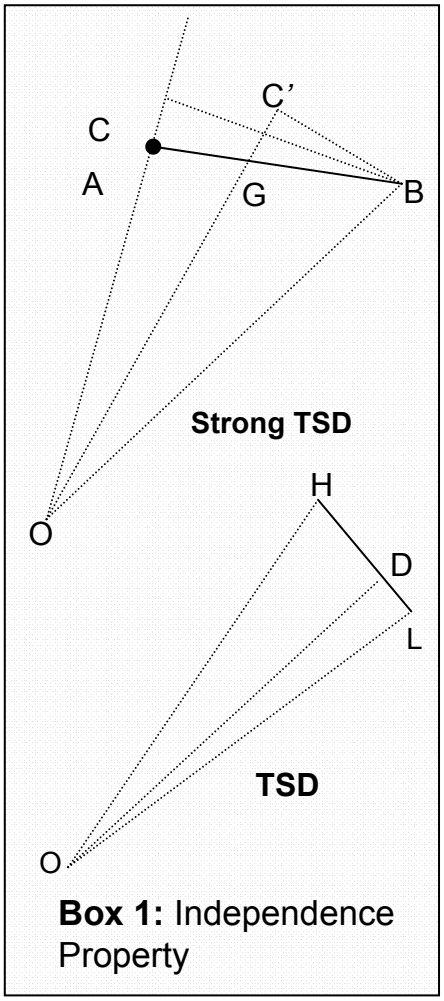


Figure 1





Box 1: Independence Property

Figure 2