Liquidity Risk and Corporate Demand for Hedging and Insurance*

Jean-Charles Rochet†
and
Stéphane Villeneuve‡

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†Toulouse University, (IDEI, GREMAQ) and Toulouse Business School.
‡Toulouse University (GREMAQ).

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Abstract: We analyze the demand for hedging and insurance by a corporation that faces liquidity risk. Namely, we consider a firm that is solvent (i.e. exploits a technology with positive expected net present value) but potentially illiquid (i.e. that may face a borrowing constraint). As a result, the firm's optimal liquidity management policy consists in accumulating reserves up to some threshold and distribute dividends to its shareholders whenever its reserves exceed this threshold.

We study how this liquidity management policy interacts with two types of risk: a Brownian risk that can be hedged through a financial derivative, and a Poisson risk that can be insured by an insurance contract. We derive individual demand functions for hedging and insurance by corporations. We show that there is a finite price above which both demand functions are zero. Surprisingly we find that the patterns of insurance and hedging decisions as a function of liquidity are pole apart: **cash poor firms should hedge but not insure, whereas the opposite is true for cash rich firms**. We also find non monotonic effects of profitability and leverage. This may explain the mixed findings of empirical studies on corporate demand for hedging and insurance: linear specifications are bound to miss the impact of profitability and leverage on risk management decisions.

1 Introduction

Corporate risk management has been the subject of a large academic literature in the last twenty years. This literature aims at filling the gap between the irrelevance results derived from the benchmark of perfect capital markets (Modigliani and Miller, 1958) and the practical importance of risk management decisions in modern corporations.

Several directions have been explored for explaining how and why firms should hedge their risks:[1]

- managerial risk aversion (Stulz, 1984),
- tax optimization (Smith and Stulz, 1985),
- cost of financial distress (Smith and Stulz, 1985),
- cost of external financing (Stulz, 1990; Froot, Scharfstein and Stein, 1993).[2]

A few papers have applied these ideas to model corporate demand for insurance.[3]

The testable implications derived from these models are different, but there is now a consensus among financial economists that profitability and leverage should be important determinants of firms' hedging and insurance policies. All of the above theories predict indeed that more profitable firms should hedge less and that more leveraged firms should hedge more. However this is not confirmed by the data. Indeed, although the empirical literature (see for example Tufano 1996 and Geczy et al. 1997) typically finds that liquidity is an important determinant of hedging (more liquid firms hedge less), leverage does not seem to have a clear and robust impact on hedging decisions.

The main objective of this paper is to show that when liquidity management and risk management decisions are endogenized simultaneously, the theoretical impact of profitability and leverage is non monotonic: the firms that gain the most from actively managing their risks are not the less profitable nor the most indebted. Moreover when insurance decisions are explicitly modeled, we find that the optimal patterns of hedging and insurance decisions by firms are exactly opposite: cash poor firms should hedge but not insure, whereas the opposite is true for cash rich firms. Thus the relation between liquidity, leverage and optimal risk management decisions of firms may be more complex than

[1]For a critical assessment of these ideas, see Smith and Stulz (1985).

[2]There are also theories based on performance evaluation (Breeden and Viswanathan, 1996; De Marzo and Duffie, 1995), and the use of proprietary information by managers (De Marzo and Duffie, 1991). However, these theories do not have simple implications on the impact of liquidity and leverage on corporate risk management, the focus of this paper.

[3]See for example Mayers and Smith (1982), (1987) and (1990), or Caillaud et al. (2000).

initially thought. This may explain the mixed findings of empirical studies on corporate demand for hedging and insurance, who typically use linear specifications.

Our model uses a continuous time stationary framework la Merton (1974) or Leland (1998), with the important difference that we focus on liquidity risk rather than solvency risk. Namely, we consider a model similar to Radner and Shepp (1996) and Jeanblanc and Shirayev (1995) where a firm operates a profitable technology but is confronted with unpredictable[4] liquidity shocks. In the benchmark model, the firm does not have access to external financing. Thus, if its shareholders are cash constrained, the firm runs the risk of being forced to liquidate if it makes operating losses. To mitigate this risk, the optimal financial policy of the firm is to accumulate cash balances up to some target level x^* and distribute as dividends all further gains. As we explain below, x^* can be viewed as a measure of the cost of financial frictions.

After presenting a brief survey of empirical evidence on the relation between liquidity, leverage and corporate risk management (Section 2), we recall the properties of the benchmark model in Section 3. Then we study how the optimal financial policy of the firm and the cost of financial frictions interact with hedging and insurance decisions. In Section 4 we study hedging by introducing a Brownian risk that can be hedged by a financial derivative. We show that the optimal value function can be characterized by a system of variational inequalities (Theorem 1) then we exhibit the solution to this system (Propositions 3 and 4). This allows us to measure the gain from hedging, as the reduction in the cost of financial frictions that is obtained through hedging. In Section 4 we study insurance by introducing a Poisson risk than can be covered through an insurance contract. Here also, we characterize the solution by a system of variational inequalities (Theorem 3) that we solve explicitly (Propositions 6, 7 and 8). In Section 6 we introduce the possibility of external financing and show that it decreases dramatically the gains from hedging and insurance. Finally Section 7 derives testable implications on the impact of profitability, risk and leverage on corporate hedging.

2 Liquidity, Leverage, and Corporate Risk Management: Some Empirical Evidence

We will not review the enormous (and fast growing) empirical literature on corporate risk management, but focus instead on the two factors we are interested in, namely liquidity and leverage.[5]

[4]By contrast, in Merton (1978) and Leland (1998) the profitability of the firm is uncertain but cash flows are predictable.

[5]We could not find any study on the impact of profitability on corporate risk management.

Geczy et al. (1997) study a sample of 372 US firms, composed of the Fortune 500 largest firms that have at least one source of foreign exchange exposure. They use a logit model to explore the determinants of the use of currency derivatives. They find no statistically significant relationship between the decision to use currency derivatives and capital structure, even when endogeneity of the latter is taken into account by using a two-stage instrumental variable estimation technique. However they find evidence that higher quick ratios, indicating more internally available funds, are associated with a lower probability of using currency derivative instruments.

Tufano (1996) studies risk management behavior in the US gold mining industry. He finds that managerial compensation (in the form of share ownership or stock options holdings) is a major determinant of the use of derivatives: when managers own shares, firms hedge more, but when managers own options, firms hedge less. They also find that more liquid firms hedge less. However Tufano does not find a strong correlation between leverage and hedging.

Using survey data, Hoyt and Khang (2000) study how the volume of insurance premiums paid by corporations depend on their financial stucture. They find that leverage (measured by the debt-equity ratio) is strongly significant (with a positive coefficient) but surprisingly that the bankruptcy probability (measured by Altman's Z value) is not. Core (1997) studies the use of Directors and Officers liability insurance in Canada on a sample of 222 firms for 1994. He finds, contrarily to Hoyt and Khang (2000), that risk of financial distress is an important determinant of Directors and Officers insurance purchase, but that leverage is not. This is confirmed by Boyer (2003) on a larger sample.

Finally, Haushalter (2000) examines the risk management activities of 100 oil and gas producers for 1992 to 1994. When hedging is measured by a continuous variable, he finds that leverage is strongly positively correlated with hedging. But when he studies separately the decision to hedge and the extend of hedging, he finds that leverage is not statistically significant in any of the probit regressions that look for the determinants of the decision to hedge. However, among the firms who hedge, the extent of hedging is positively correlated with leverage.

3 The Benchmark Model

We consider a firm that exploits a technology characterized by a cash generating process following a drifted Brownian motion:

$$dX_t = \mu dt + \sigma dW_t, \tag{1}$$

where μ and σ are positive parameters, W is a standard Wiener process, and X_t represents the amount of cash owned by the firm at date t. Shareholders are risk neutral and discount the future at rate r.

We assume that the manager of the firm acts in the best interest of its shareholders but that the firm does not have access to external finance. Shareholders cannot inject new funds and the firm cannot issue new securities or borrow from a bank. Thus, whenever X_t becomes negative, the firm is liquidated and the technology is lost forever.[6] In the absence of liquidity constraints (i.e. if the shareholders had unlimited cash holdings) the firm would continue forever,[7] the shareholders injecting money whenever needed and, symmetrically, collecting any cash surplus in the form of dividends. The NPV of a firm starting with a level of cash x would be:

$$V_{FB}(x) = x + E\left[\int_0^{+\infty} e^{-rt} dX_t\right] = x + \frac{\mu}{r}, \tag{2}$$

where the notation V_{FB} stands for the "first best" value of the firm.

In fact this formula can be extended to negative values of x:

$$V_{FB}(x) = \max\left(0, x + \frac{\mu}{r}\right), \tag{3}$$

with the interpretation that unconstrained shareholders would be ready to pay out the initial debt of the company up to the amount $\frac{\mu}{r}$, above which they exert their limited liability option.

For the rest of the paper, we will consider the second best situation where the shareholders of the firm have no cash,[8] in which case the firm is liquidated whenever X_t becomes negative. In that case it becomes optimal for shareholders to accumulate reserves up to some level x^* and to distribute dividends whenever X_t exceeds this level.[9] The value of the firm becomes:

$$V_0(x) = E\left[\int_0^{\tau_0} e^{-rt} d\ell_t | x_0 = x\right], \tag{4}$$

where τ_0 represents the first time that X_t hits zero and $d\ell_t$ represents the local time[10] associated to X_t attaining the threshold x^*. Hence, V_0 can be found explicitly by solving

[6] These extreme assumptions are relaxed in Section 6.

[7] Contrarily to models such as Merton (1978) or Leland (1998) we focus on liquidity risk, and don't consider solvency risk.

[8] In the benchmark model, issuing debt or new equity is also impossible.

[9] See for example Radner and Shepp (1996), Jeanblanc and Shirayev (1995) or Milne and Robertson (1996).

[10] The interpretation is that the optimal reserve is a diffusion reflected at x^*. The local time ℓ_t is the process which ensures the reflection. We refer to Karatzas and Shreve (1991) section 3.6 for a rigourous presentation of the notion of local time (see also the Appendix).

the following boundary value problem:

$$\begin{cases} LV(x) = 0, & 0 \leq x \leq x^* \\ V(0) = 0, & V'(x^*) = 1, \end{cases} \tag{5}$$

where LV represents the differential operator associated to (1), namely:

$$LV(x) = \mu V'(x) + \frac{1}{2}\sigma^2 V''(x) - rV(x). \tag{6}$$

The explicit value of V_0 is given by

$$V_0(x) = \frac{e^{\rho_2 x} - e^{\rho_1 x}}{\rho_2 e^{\rho_2 x^*} - \rho_1 e^{\rho_1 x^*}}, \tag{7}$$

where $\rho_1 < 0 < \rho_2$ are the roots of the characteristic equation:

$$r - \mu\rho - \frac{1}{2}\sigma^2\rho^2 = 0. \tag{8}$$

The optimal value of x^* is obtained by minimizing the denominator of V_0 or equivalently by solving $V_0''(x^*) = 0$:

$$x^* = \frac{1}{\rho_2 - \rho_1} \ln \frac{\rho_1^2}{\rho_2^2}. \tag{9}$$

A more compact way of characterizing the optimal value function V_0 is):

$$\forall x \geq 0, \quad \max\left(LV_0(x), 1 - V_0'(x)\right) = 0. \tag{10}$$

Like before, V_0 can be extended to $x < 0$ by setting $V_0(x) = 0$ (limited liability). All these preliminary results are summarized in the following proposition:

Proposition 1 *(Jeanblanc and Shirayev, 1995) : In the benchmark model with cash constrained shareholders but no external risks, the value of the firm when it holds cash volume x is:*

$$V_0(x) = \max\left(0, \frac{e^{\rho_2 x} - e^{\rho_1 x}}{\rho_2 e^{\rho_2 x^*} - \rho_1 e^{\rho_1 x^*}}\right), \tag{11}$$

where x^* is given by formula (9).

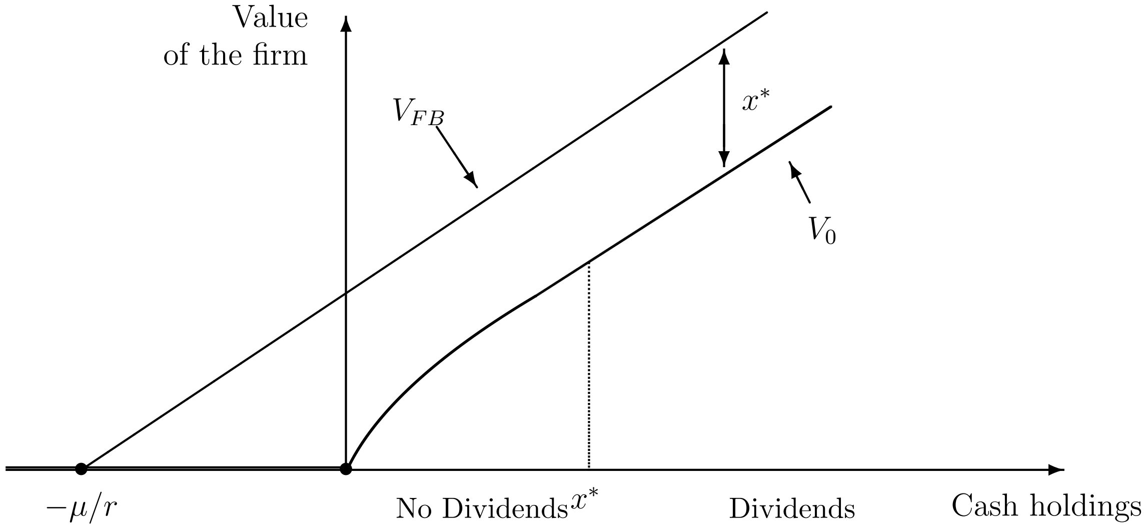

Figure 1: No external risk: The value $V_0(x)$ of the cash constrained firm compared with the first best value $V_{FB}(x)$.

$V_0(x)$ can be extended above x^* by setting, for $x \geq x^*$:

$$V_0(x) = x - x^* + V(x^*), \tag{12}$$

with the interpretation that, if the firm starts with cash reserves x above x^*, the amount $(x - x^*)$ is immediately distributed as dividends. Therefore, for x large enough, the difference $V_{FB}(x) - V(x)$, which measures the cost of financial frictions, is constant. Since $V_0'(x^*) = 1$ and $V_0''(x^*) = 0$, equation (5) implies that

$$V_0(x^*) = \frac{\mu}{r}, \tag{13}$$

which means that the cost of financial frictions $(V_{FB}(x) - V_0(x))$ is equal[11] to x^*, when $x \geq x^*$.

What is noticeable about Figure 1 is that V_0 is **convex** (and not differentiable) at 0 but **concave** for $0 < x < x^*$, and **linear** above x^*. This comes from the interaction between the limited liability option (which generates the convex kink at zero) and liquidation costs (which generate the strict concavity of V in $[0, x^*)$). This pattern will reveal crucial in the sequel.

It is also interesting to look at the comparative statics properties of the cost x^* of financial frictions. Proposition 2 summarizes these properties:

Proposition 2 *: The cost of financial frictions x^* is a single peaked function of μ, and an increasing function of σ^2.*

[11] Since $(V_{FB} - V_0)$ decreases in x, x^* underestimates the cost of financial frictions for $x < x^*$.

Proof of Proposition 2: See the appendix.

The following figures represent the cost of financial frictions x^* as a function of μ and σ^2. Notice that x^* is bounded above by $\frac{\mu}{r}$.

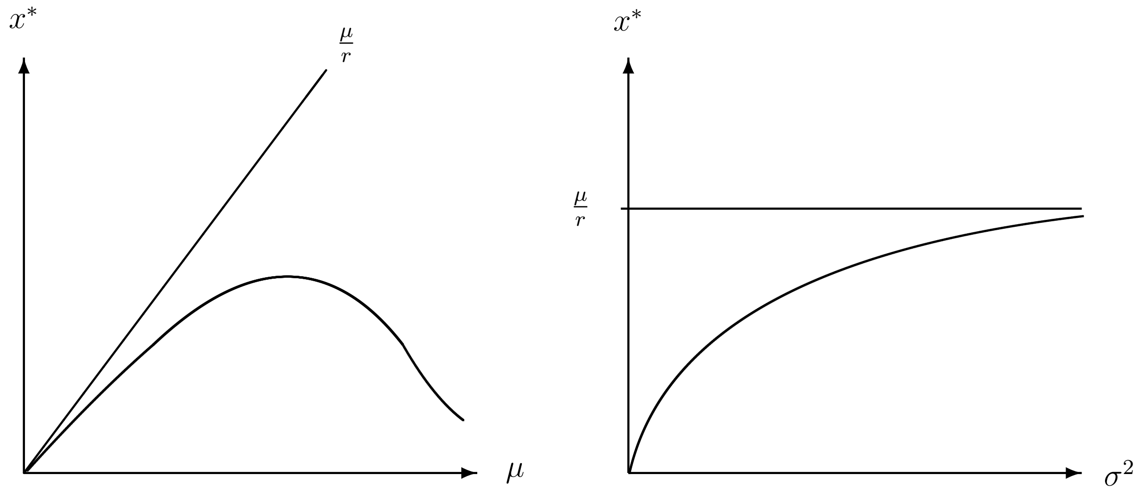

Figure 2: The cost of financial frictions
as a function of μ and σ^2.

Maybe the most striking of these properties is the non-monotonicity of x^* with respect to μ, which measures the profitability of the firm. Highly profitable firms are not really affected by financial frictions because their probability of financial distress is small. Conversely, barely profitable firms have little to lose from failure. It is the intermediate firms that are hurt the most by the risk of failure. The same is true for leverage. Suppose that the firm has to pay a constant coupon flow[12] c, our formulas are immediately adapted by replacing μ by $\mu - c$. Figure 2 then implies a non monotonic influence of c on the cost x^* of financial frictions. The case of the leveraged firm is analyzed in more detail in Section 5.

4 Hedging

We now introduce a first form of external risk, by assuming that the "operating" cash flow process given by (1) is perturbed by a second Brownian motion of volatility σ_R (and zero drift) that can be hedged through a financial derivative. This "external" risk can be interpreted as foreign exchange risk or commodity price risk, the corresponding hedging instruments being currency futures or commodity futures.

[12] As in Leland (1998), we assume that coupon payments are constant over time to maintain stationarity of the financial structure.

Equation (1) becomes

$$dX_t = \mu dt + \sigma dW_t + \sigma_R dW_t^R, \tag{14}$$

when the external risk is not hedged ($h = 0$) and

$$dX_t = \left(\mu - \frac{1}{2}\pi\sigma_R^2\right) dt + \sigma dW_t, \tag{15}$$

when the external risk is hedged[13] ($h = 1$). $\pi \geq 0$ is interpreted as the risk premium or loading factor (cost of hedging per unit of variance) that remunerates the sellers of the futures contract for the risk they take. Without loss of generality, we can assume that the two Brownian motions W and W_R are independent.[14] Let us introduce the differential operators associated to $h = 0, 1$:

$$L(0)V(x) = \mu V'(x) + \frac{1}{2}(\sigma^2 + \sigma_R^2)V''(x) - rV(x), \tag{16}$$

and

$$L(1)V(x) = \left(\mu - \frac{1}{2}\pi\sigma_R^2\right) V'(x) + \frac{1}{2}\sigma^2 V''(x) - rV(x). \tag{17}$$

The optimal value function is obtained by finding the adapted process (h_t, Z_t) (where $h_t \in \{0, 1\}$ represents the hedging decision at date t and Z_t is the cumulative amount of dividends distributed up to date t, a non-negative, non-decreasing[15] right continuous process) that maximizes the expected discounted value of dividends up to liquidation (which occurs at τ, the first time where X_t hits 0). Formally

$$V(x) = \max_{h,Z} E\left[\int_0^\tau e^{-rt} dZ_t | x_0 = x\right]. \tag{18}$$

This is a mixed singular/regular control problem of the type studied by Fleming and Soner (1993), who prove the following result:

Theorem 1 *(Fleming and Soner, 1993): If the value function V defined by (18) is $\mathcal{C}^2$, it satisfies the following variational inequalities:*

$$\forall x > 0 \qquad \max\left(L(0)V(x), L(1)V(x), 1 - V'(x)\right) = 0, \tag{19}$$

together with the initial condition:

$$V(0) = 0. \tag{20}$$

[13]In Section 5 we consider the possibility of partial hedging ($h \in [0, 1]$).

[14]An alternative, but equivalent, formulation is to consider that the hedging instrument is imperfectly correlated with W. Perfect hedging corresponds to the limit case of perfect correlation ($\sigma = 0$ in our model).

[15]The condition that Z_t be non-decreasing corresponds to the assumption that shareholders are cash constrained (non negative dividends). Without this restriction the first best could be attained.

Nevertheless, solving the conditions (19), (20) does not always guarantee the optimality of the solution, since we do not know a priori if this solution is unique. We succeed in our case by constructing a concave solution of (19), (20) and by proving the following verification theorem.

Theorem 2 *Assume there exist a twice continuously differentiable concave function W and a constant x_1 such that*

$$\forall x \leq x_1 \qquad \max\left(L(0)W(x), L(1)W(x)\right) = 0 \text{ and } W^{'}(x) \geq 1, \tag{21}$$

$$\forall x \geq x_1 \qquad W^{'}(x) = 1 \text{ and } \max\left(L(0)W(x), W(1)V(x)\right) \leq 0 \tag{22}$$

together with the initial conditions:

$$W(0) = 0 \text{ and } W^{'}(0) < +\infty \tag{23}$$

then $W = V$. Furthermore, let h^* be the measurable function defined by

$$h^*(x) = 1\!\!1_{\{L(1)W(x) \geq L(0)W(x)\}}$$

and $L_t(x_1)$ the local time at the level x_1 of the diffusion process

$$dX_t = (\mu - \frac{\sigma_R^2}{2}\pi h^*(X_t))dt + \sigma dW_t + \sigma_R(1 - h^*(X_t))dW_t^R,$$

then $W(x) = E\int_0^{\tau_0} e^{-rs} dL_s(x_1)$, where

$$\tau_0 = \inf\{t \ \geq 0,\ X_t \leq 0\}.$$

Proof: See the appendix.

We are now going to characterize V by constructing a solution to (19), (20) that has the same shape as V_0 (see Section 2), i.e. has a convex kink at 0, is concave for x positive but smaller than some threshold x_1 (the target level of cash reserves) and linear for x above x_1. In particular this shape implies that, when $\pi > 0$, hedging is never optimal for x sufficiently close to x_1. Indeed:

$$L(1)V(x) - L(0)V(x) = -\frac{1}{2}\sigma_R^2\left[\pi V'(x) + V''(x)\right],$$

which is negative at x_1 (since $V'(x_1) = 1$ and $V''(x_1) = 0$) and thus, by continuity, for x close to x_1. In fact we establish below the existence of a second threshold $x_0 \in [0, x_1]$ such that the optimal hedging decision is given by:

$$\begin{aligned} h^*(x) &= 1 \quad \text{if } 0 \leq x < x_0 &(24)\\ &= 0 \quad \text{if } x_0 \leq x \leq x_1. &(25)\end{aligned}$$

The limit case $x_0 = 0$ corresponds to no hedging at all.

Here also, the value function V can be obtained explicitly by finding a $\mathcal{C}^2$ solution to the following free boundary problem:

$$\begin{cases} L(1)V(x) = 0 & 0 < x < x_0, \\ L(0)V(x) = 0 & x_0 < x < x_1, \\ V(0) = 0,\ V'(x_1) = 1,\ V''(x_1) = 0. \end{cases}$$

We need to introduce some notation. Let us denote by $\theta_1 < 0 < \theta_2$ the roots of the characteristic equation corresponding to $h = 1$ (hedging):

$$\left(\mu - \frac{1}{2}\pi\sigma_R^2\right)\theta + \frac{1}{2}\sigma^2\theta^2 = r, \tag{26}$$

and by $\gamma_1 < 0 < \gamma_2$ the roots of the characteristic equation corresponding to $h = 0$ (no hedging):

$$\mu\gamma + \frac{1}{2}(\sigma^2 + \sigma_R^2)\gamma^2 = r. \tag{27}$$

Proposition 3 *: The value of the firm corresponding to the optimal hedging problem is characterized by two regimes:*

- $0 \leq x \leq x_0$ *(hedging regime):*

$$V(x) = A\left[e^{\theta_2 x} - e^{\theta_1 x}\right], \tag{28}$$

- $x_0 \leq x \leq x_1$ *(no-hedging, or self insurance regime):*

$$V(x) = Be^{\gamma_1 x} + Ce^{\gamma_2 x}, \tag{29}$$

  where $A,\ B,\ C$ are positive constants.

proof: See the appendix.

Interestingly, the pattern derived in Proposition 3 is confirmed by empirical evidence: both Tufano (1996) and Geczy et al. (1997) find that lower liquidity (measured by a low quick ratio) is a significant determinant of hedging.

The five parameters characterizing V (namely x_0, x_1, A, B, C) are obtained by using the five boundary conditions:

- V, V' and V'' are continuous at x_0,
- $V'(x_1) = 1,\ V''(x_1) = 0.$

In particular we can easily determine the values of x_0 and x_1:

Proposition 4 *: When* $\pi \leq \frac{2\mu}{\sigma^2+\sigma_R^2}$*, the hedging and dividend thresholds are given by*

$$x_0 = \frac{1}{\theta_2 - \theta_1} \ln \frac{\theta_1(\theta_1+\pi)}{\theta_2(\theta_2+\pi)}, \tag{30}$$

$$x_1 = x_0 + \frac{1}{\gamma_2 - \gamma_1} \ln \frac{\gamma_1(\gamma_2+\pi)}{\gamma_2(\gamma_1+\pi)}. \tag{31}$$

Proof: See the appendix.

Two limit cases are interesting:

- When $\pi = 0$ (costless heldging), $x_1 = x_0 = \frac{1}{\theta_2-\theta_1} \ln \left(\frac{\theta_1}{\theta_2}\right)^2$, which coincides in this case with the dividend threshold x^* of Section 3 (where $\sigma_R = 0$). This just means that when hedging is costless, the firm always hedges and we are back to the benchmark model with no external risk.
- When $\pi \geq \frac{2\mu}{\sigma^2+\sigma_R^2}$, $x_0 = 0$: Thus there is a maximum price of hedging above which the firm stops buying any hedging. This maximum price increases with the expected profitability of the firm (μ) and decreases with the volatility of the unhedged cash flow ($\sqrt{\sigma^2+\sigma_R^2}$).

The properties of the optimal hedging pattern are represented in the following figure.

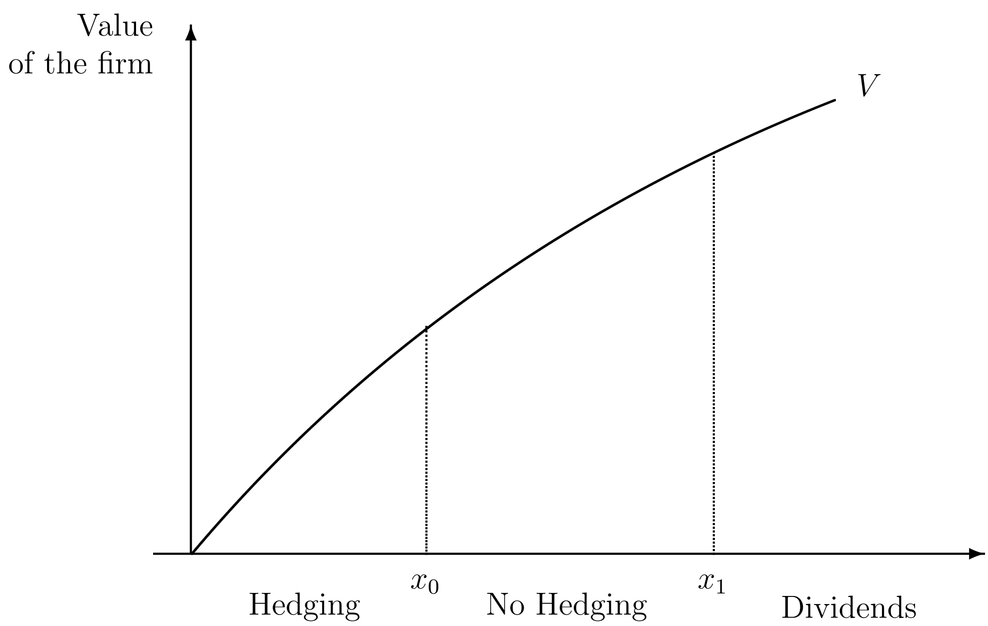

Figure 3: The pattern of optimal hedging decisions.

The next figure represents the gains from hedging.

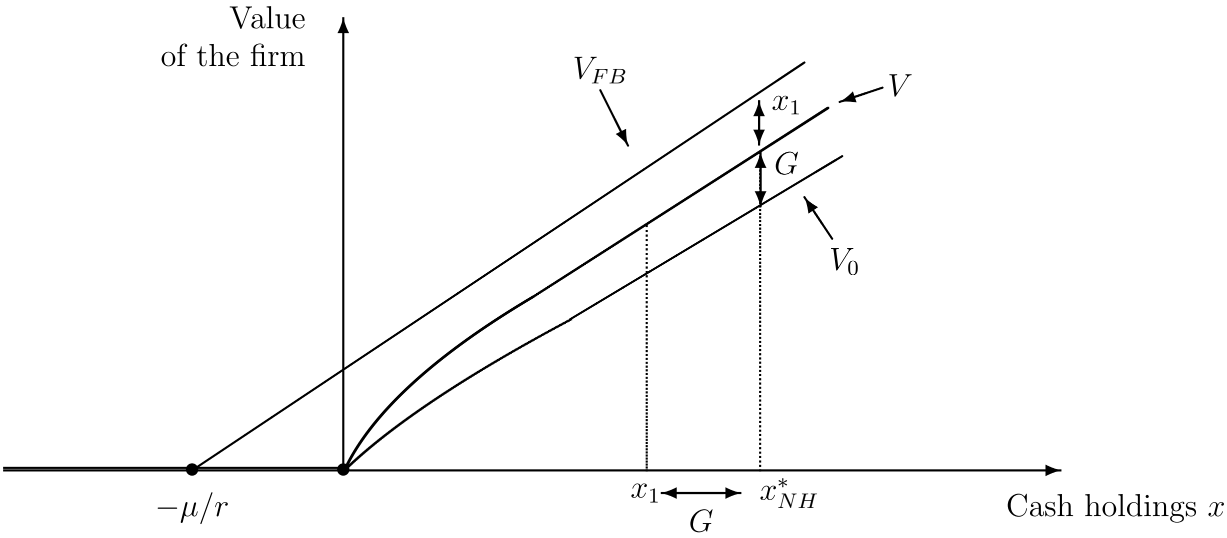

Figure 4: The gains from hedging: the value $V(x)$ of the firm that hedges optimally compared with the value $V_0(x)$ of the firm that does not hedge, and the first best value $V_{FB}(x)$.

Figure 4 shows the value $V(x)$ of the firm that hedges optimally (i.e. whenever $x \leq x_0$). This value $V(x)$ is in between the first best value $V_{FB}(x)$ and the value $V_0(x)$ of the firm who never hedges. Notice that the hedged firm distributes more dividends. The gains from hedging are represented by the difference $G = x^*_{NH} - x_1$ between the cost x^*_{NH} of financial frictions without hedging $\left(x^*_{NH} = \frac{1}{\gamma_2-\gamma_1} \ln \frac{\gamma_1^2}{\gamma_2^2}\right)$, and the cost x_1 of financial frictions with optimal hedging. As we have already noticed, this difference vanishes when $\pi \geq \frac{2\mu}{\sigma^2+\sigma_R^2}$.

Using formulas (30) and (31), we can derive the expression of these gains from hedging, which we denote by G:

$$
\begin{aligned}
G &= \frac{1}{\gamma_2 - \gamma_1} \ln \frac{\gamma_1^2}{\gamma_2^2} - x_1 \\
&= \frac{1}{\gamma_2 - \gamma_1} \ln \frac{\gamma_1(\gamma_1 + \pi)}{\gamma_2(\gamma_2 + \pi)} - \frac{1}{\theta_2 - \theta_1} \ln \frac{\theta_1(\theta_1 + \pi)}{\theta_2(\theta_2 + \pi)},
\end{aligned} \tag{32}
$$

where θ_1, θ_2 and γ_1, γ_2 are defined respectively by (26) and (27). The comparative statics properties of G are complicated, since in particular θ_1 and θ_2 are implicit functions of the loading factor π. However, since the markets for financial derivatives are in general highly competitive, we can reasonably assume that π is close to zero, in which case G converges to:

$$
G_0 = \frac{1}{\gamma_2 - \gamma_1} \ln \frac{\gamma_1^2}{\gamma_2^2} - \frac{1}{\rho_2 - \rho_1} \ln \frac{\rho_1^2}{\rho_2^2}. \tag{33}
$$

Notice that θ_1 and θ_2 converge to ρ_1 and ρ_2 (see Section 2) when π goes to zero. Let us denote by $x^*(\mu, \sigma^2)$ the cost of financial frictions derived in Section 2:

$$x^*(\mu, \sigma^2) = \frac{1}{\rho_2 - \rho_1} \ln \frac{\rho_1^2}{\rho_2^2} = \frac{\sigma^2}{2\sqrt{\mu^2 + 2r\sigma^2}} \ln \left[\frac{\sqrt{\mu^2 + 2r\sigma^2} + \mu}{\sqrt{\mu^2 + 2r\sigma^2} - \mu} \right]. \tag{34}$$

We obtain a simple expression of the gains from (costless) hedging:

$$G_0 = x^*(\mu, \sigma^2 + \sigma_R^2) - x^*(\mu, \sigma^2). \tag{35}$$

The gains from costless hedging are thus equal to the reduction in the cost of financial frictions[16] obtained by decreasing the squared volatility of the cash generating process from $\sigma^2 + \sigma_R^2$ to σ^2. In Section 7, we use this formula to derive testable implications of our model on the determinants of corporate hedging.

5 Insurance

We consider in this section a different type of external risk (say a fire or an accident) which is modelled as a Poisson process: with a probability λdt, the firm incurs a loss m. This risk can be covered by an insurance contract characterized by a flow premium $\lambda[m + \pi m^2]$ where π represents, here again, a loading factor associated to the remuneration of the risk taken by the insurer. Denoting by P a Poisson process of intensity λ, the cash flow dynamics becomes:

$$dX_t = \mu dt + \sigma dW_t - m dP_t, \tag{36}$$

in the absence of insurance ($i = 0$), and

$$dX_t = \{\mu - \lambda(m + \pi m^2)\}dt + \sigma dW_t, \tag{37}$$

if the firm buys insurance ($i = 1$). Assuming that W and P are independent, the associated operators are:

$$D(0)V(x) = \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \lambda[V(x) - V(x - m)] - rV(x), \tag{38}$$

and:

$$D(1)V(x) = \frac{\sigma^2}{2} V''(x) + [\mu - \lambda(m + \pi m^2)]V'(x) - rV(x). \tag{39}$$

Again, the optimal value function is obtained by finding the adapted process (i_t, Z_t) (where $i_t \in \{0, 1\}$ represents the insurance decision at date t and Z_t is the cumulative

[16] Recall that this cost is measured by the amount of cash reserves needed before dividends can be distributed. This is why hedging firms can distribute more dividends.

dividend process) that maximizes the expected discounted value of dividends up to liquidation

$$\bar{V}(x) = \max_{i,Z} E\left[\int_0^\tau e^{-rt} dZ_t | x_0 = x\right]. \tag{40}$$

As in the case of hedging, we obtain $\bar{V}$ through a verification theorem whose proof, similar to that of Theorem 2, is omitted.

Theorem 3 *Assume there exist a twice continuously differentiable concave function W and a constant x_1 such that*

$$\forall x \leq x_1 \qquad \max\left(D(0)W(x), D(1)W(x)\right) = 0, \tag{41}$$

$$\forall x \geq x_1 \qquad W^{'}(x) = 1 \tag{42}$$

together with the initial conditions:

$$W(0) = 0 \text{ and } W^{'}(0) < +\infty \tag{43}$$

then $W = \bar{V}$. Furthermore, let i^* the measurable function defined by

$$i^*(x) = 1\!\!1_{\{D(1)W(x) \geq D(0)W(x)\}},$$

and $L_t(x_1)$ the local time at the level x_1 of the diffusion process:

$$dX_t = \mu - \lambda(m + \pi m^2) i^*(X_t))dt + \sigma dW_t - m(1 - i^*(X_t))dP_t.$$

Then $W(x) = E \int_0^{\tau_0} e^{-rs} dL_s(x_1)$, where

$$\tau_0 = \inf\{t \ \geq 0, \ X_t \leq 0\}.$$

Using the same method as before, we shall construct a solution $\bar{V}$ to these variational inequalities that has the same pattern as V_0 and V: convex kink at 0, then concave. Before going further, we point out an interesting result: insurance is never optimal for x small. Indeed:

$$D(1)\bar{V}(x) - D(0)\bar{V}(x) = \lambda\left[\bar{V}(x) - \bar{V}(x-m) - (m + \pi m^2)\bar{V}'(x)\right].$$

Since $\bar{V}(x-m) = 0$ for $x \leq m$ and $\bar{V}'(0) > 0$, this expression is negative for x sufficiently small.
Next section will be devoted to the case of fair insurance, that is $\pi = 0$.

5.1 Fair Insurance

In the case of fair insurance, we will establish below that the optimal insurance decision is given by:

$$
\begin{aligned}
i^*(x) &= 0 \quad \text{if} \quad 0 \leq x < \bar{x}_0 &(44)\\
&= 1 \quad \text{if} \quad \bar{x}_0 \leq x \leq \bar{x}_1, &(45)
\end{aligned}
$$

where $\bar{x}_0$ and $\bar{x}_1$ are respectively the insurance and the dividend thresholds. Notice that the relation between cash holdings and insurance decisions is the opposite of the relation between cash holdings and hedging decisions: the firms that are poor in cash do not buy insurance but they do hedge. The opposite is true for cash-rich firms.

Like before, the value function $\bar{V}$ can be obtained by finding a $\mathcal{C}^2$ solution to the following free boundary problem:

$$
\begin{cases}
D(0)V(x) = 0 & 0 < x < \bar{x}_0 \\
D(1)V(x) = 0 & \bar{x}_0 < x < \bar{x}_1 \\
V(0) = 0,\ V'(x_1) = 1,\ V''(\bar{x}_1) = 0.
\end{cases}
$$

We need to introduce some notation. By analogy with Section 3, let us denote by $\bar{\gamma}_1 < 0 < \bar{\gamma}_2$ the roots of the characteristic equation corresponding to $i = 1$ (insurance):

$$
(\mu - \lambda m)\gamma + \frac{1}{2}\sigma^2\gamma^2 = r, \tag{46}
$$

and by $\bar{\theta}_1 < 0 < \bar{\theta}_2$ the roots of the characteristic equation corresponding to $i = 0$ (no insurance):[17]

$$
\mu\theta + \frac{1}{2}\sigma^2\theta^2 = r + \lambda. \tag{47}
$$

We are in a position to give the value of the firm corresponding to the optimal fair insurance problem.

Theorem 4 *Assume that $m \leq \frac{\mu}{r+\lambda}$. The optimal return function V is given by*

$$
\bar{V}(x) = \begin{cases}
\bar{A}(e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}) \text{ for } x \leq \bar{x}_0 \\
\bar{B}e^{\bar{\gamma}_1 x} + \bar{C}e^{\bar{\gamma}_2 x} \text{ for } \bar{x}_0 \leq x \leq \bar{x}_1 \\
x + \frac{\mu - \lambda m}{r} - \bar{x}_1 \text{ for } x \geq \bar{x}_1.
\end{cases}
$$

where $\bar{x}_0$ and $\bar{x}_1$ are given by

$$
\bar{x}_0 = \frac{1}{\bar{\theta}_2 - \bar{\theta}_1} \ln\left(\frac{1 - m\bar{\theta}_1}{1 - m\bar{\theta}_2}\right). \tag{48}
$$

$$
\bar{x}_1 = \bar{x}_0 + \frac{1}{\bar{\gamma}_2 - \bar{\gamma}_1} \ln\left(\frac{\bar{\gamma}_1^2(1 - m\bar{\gamma}_2)}{\bar{\gamma}_2^2(1 - m\bar{\gamma}_1)}\right). \tag{49}
$$

[17] This assumes implicitly that $\bar{x}_0 \leq m$, so that $V(x - m) = 0$ in the no-insurance region. This will be checked ex post.

Proof: See the appendix.
Let us point out that the optimal fair insurance problem is characterized by three regimes:

- $0 \leq x \leq \bar{x}_0$ (no insurance regime):

$$\bar{V}(x) = \bar{A}\left[e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}\right], \tag{50}$$

- $\bar{x}_0 \leq x \leq \bar{x}_1$ (insurance regime):

$$\bar{V}(x) = \bar{B}e^{\bar{\gamma}_1 x} + \bar{C}e^{\bar{\gamma}_2 x}, \tag{51}$$

- $x \geq \bar{x}_1$ (dividend payment):

$$\bar{V}(x) = x + \frac{\mu - \lambda m}{r} - \bar{x}_1$$

Theorem 4 shows that fair insurance is bought when the firm is rich ($x > \bar{x}_0$) and the risks are not too high ($m \leq \frac{\mu}{r+\lambda}$). For completeness, we shall examine the case $m \geq \frac{\mu}{r+\lambda}$. Next proposition shows that in that case, the shareholders optimally assume the Poisson risk. The optimal return function has the same form as in the Benchmark case. Hence, and by contrast with the hedging case (see Proposition 4) large risks are not insured even if insurance is fair.

Proposition 5 *Assume that $m \geq \frac{\mu}{r+\lambda}$, the optimal return function $\bar{V}$ is given by:*

$$\bar{V}(x) = \begin{cases} \tilde{A}(e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}) \text{ for } x \leq \tilde{x}_1 \\ x - \tilde{x}_1 + \frac{\mu}{r+\lambda}, \end{cases}$$

where

$$\tilde{x}_1 = \frac{2}{\bar{\theta}_2 - \bar{\theta}_1} \ln\left|\frac{\bar{\theta}_1}{\bar{\theta}_2}\right|,$$

and

$$\tilde{A}(e^{\bar{\theta}_2 \tilde{x}_1} - e^{\bar{\theta}_1 \tilde{x}_1}) = 1.$$

Proof: See the appendix.

The properties of the value function corresponding to optimal fair insurance are summarized in the following figure:

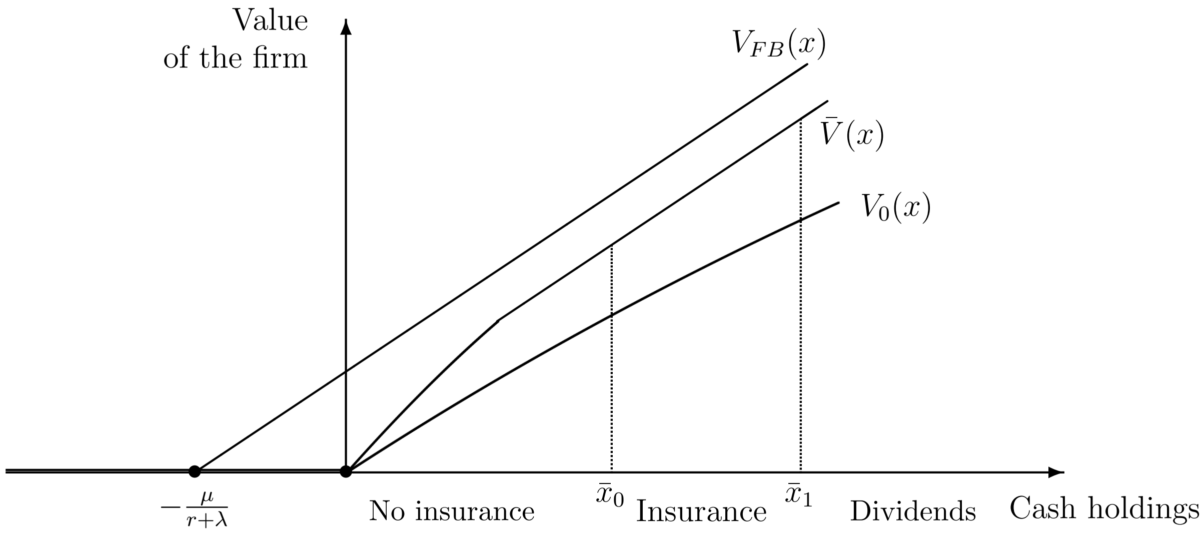

Figure 5: The gains from insurance: The value $\bar{V}(x)$ of the firm that insures optimally compared with the value $V_0(x)$ of the firm that does not insure, and the first best value $V_{FB}(x) = \max(0, x + \frac{\mu}{r+\lambda})$.

Like for hedging, the gains from insurance can be measured by the difference between the cost of financial frictions without and with insurance.

5.2 Positive loading factor

In the case of a positive loading factor ($\pi > 0$), we do not succeed to characterize the optimal policy. We will content ourselves to highlight the readers about the complexity of the study. In particular, the optimal policy for insurance may not exhibit the same pattern as in the case of fair insurance. As a first result, we will prove below that the optimal policy $i^*(x)$ must be equal to zero in the neighbourhood of the level of dividend payment x_1 as soon as the loss m is small.

Proposition 6 *Assume that $\pi > 0$. For m small enough, there is an open interval $\mathcal{O} = (x_1 - \varepsilon, x_1)$ where the optimal policy i^* equals zero .*

Proof: See the appendix.

As a second result, we will point out that there still exists situations where it is optimal to insure. Recall that,

$$D(1)\bar{V}(x) - D(0)\bar{V}(x) = \lambda \left[\bar{V}(x) - \bar{V}(x-m) - (m + \pi m^2)\bar{V}'(x)\right].$$

Therefore, for $x \leq m$, we assume optimally the risk if and only if

$$\varphi(x) \equiv \frac{e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}}{\bar{\theta}_2 e^{\bar{\theta}_2 x} - \bar{\theta}_1 e^{\bar{\theta}_1 x}}. \leq m(1 + \pi m).$$

Note that φ is a nondecreasing function with

$$\varphi(0) = 0, \quad \varphi'(0) = 1, \quad \varphi''(0) = 2\frac{\mu}{\sigma^2}.$$

this is equivalent to $\varphi(m) \leq m(1 + \pi m)$.
Conversely, when $\pi < \frac{\mu}{\sigma^2}$, $\varphi(m) \sim m + \frac{\mu}{\sigma^2} m^2$, and thus is greater than $m(1 + \pi m)$ for m small enough. Thus, it is optimal to buy insurance before hitting the level m. The last remark aggregates the previous ones. We claim that for $\pi < \frac{\mu}{\sigma^2}$ and m small enough the optimal policy should have four regimes: a non insurance regime in the neightbourhood of zero, an insurance regime around m and a non insurance regime in the neightbourhood of the dividend payment threshold and dividend payment.

6 Extensions

6.1 External Financing

So far, we have only considered the extreme case of a firm with no access to external financing. It is easy to see that introducing such possibilities has a dramatic impact on the gains from hedging. For example in the absence of financial frictions, the firm could attain its first best value by essentially transfering all its risk to a competitive bank (i.e. σdW_t, which has a zero NPV) and retaining the deterministic part (i.e. μdt, which has NPV $\frac{\mu}{r}$). Then there would be no need for the firm to retain cash and thus no gains from hedging or insurance.

Of course some form of financial frictions (due for example to imperfect verifiability of cash flows, moral hazard or agency problems) have to be introduced for hedging and insurance to become valuable. But then the financial structure and risk management policies would have to be endogenized simultaneously so as to limit the impact of those financial frictions. This is outside the scope of this paper. What we do instead is introduce exogenously two types of external financing:

a) a risky bond that pays a constant coupon flow cdt until the firm goes bust. This is easily captured by replacing μ by $(\mu - c)$ in all our formulas,

b) a credit line that allows the firm to incur an overdraft (i.e. a negative X_t) up to some limit $X_t = -l$, where the firm is liquidated.

If financial markets are competitive, the best credit line that can be obtained by the firm is characterized by two features:

- The credit limit l is equal to the liquidation value of the firm's assets for the bank. We assume that it is a fraction α of the expected present value $\frac{\mu}{r}$ of future cash flows. α measures the tangibility of the firm's assets.
- The interest rate charged on overdrafts is equal to r (the managers pays rX_t per unit of time if X_t becomes negative).

For simplicity, let us discuss the impact of these two financial instruments one after the other.

Leverage has a relatively straightforward effect, analogous to reduced profitability. This is because μ is replaced in all our formulas by $(\mu - c)$. In order to measure relative leverage, we use the variable $\frac{c}{\mu}$, the inverse of the interest coverage ratio. The impact of $\frac{c}{\mu}$ on hedging is discussed in the next section.

The impact of the credit line is more complex: equation (1) becomes

$$dX_t = (\mu - rX_t^-)dt + \sigma dW_t$$

and the value function becomes

$$V_{CL}(x) = \sup_Z E\left(\int_0^{T_{-l}} e^{-rs}\, dZ_s\right).$$

where $l = \frac{\alpha\mu}{r}$ and

$$T_{-l} = \inf\{t \geq 0,\, X_t \leq -l\}.$$

As before, one can show that V_{CL} is a concave nondecreasing function and there is a threshold x^*_{CL} above which dividends are distributed. Moreover, V_{CL} is characterized by:

$$\begin{cases} rV_{CL} & = (\mu - rx^-)V'_{CL} + \frac{1}{2}\sigma^2 V''_{CL} \\ V_{CL}(-l) & = 0 \quad V'_{CL}(x^*_{CL}) = 1 \quad V''_{CL}(x^*_{CL}) = 0 \end{cases}.$$

As established in next Proposition, the credit line contract decreases the cost of financial frictions (and therefore the gains from hedging). To illustrate this we have represented in Figure 6 the impact of the credit line on the value function in the absence of hedging (benchmark case). Gains from hedging, measured by the vertical distance to the first best value function V_{FB} are dramatically reduced, especially if α is large (more tangible assets).

Proposition 7 *The value function V_{CL} is a strictly increasing function of α. Consequently, $0 \leq x^*_{CL} \leq x^*$ where x^* is the optimal threshold in the benchmark model.*

The proof is reported to the Appendix. Note that if we define $V_{CL} = V_\alpha$ to illustrate the dependence in α, V_0 is the value function in the benchmark model and $V_1 = V_{FB}$ the unconstrained value function.

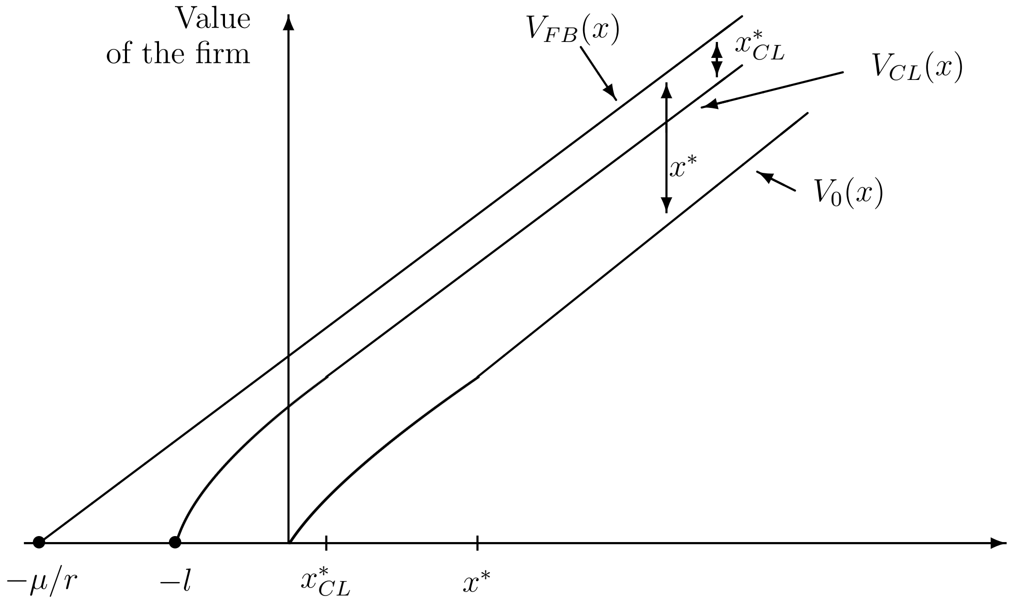

Figure 6: A credit line reduces the dividend threshold (and thus the potential gains from hedging from x^* to x^*_{CL}). The value function becomes V_{CL}.

To conclude, notice that even if credit line availability changes the extend of hedging, it does not alter our main, qualitative, findings:

- profitability and leverage have a non monotonic impact on hedging,
- insurance and hedging patterns are opposed.

6.2 Partial Hedging

As a second extension we discuss what happens when we allow the firm to choose a hedging ratio h in the interval $[0, 1]$. When hedging is costly, it will be optimal to do so in our model.[18] In fact the optimal hedging pattern will be consistent with our previous findings (see Figure 3). The main difference is that the value function of the firm is now

[18]This is consistent with empirical evidence. For example Allayanis and Ofek (2001) examine the decision to use foreign currency derivatives and the extent of currency hedging. They find that a firm's net exchange rate exposure is positively related with foreign sales, and negatively related with foreign currency derivatives use, which is consistent with a partial coverage of currency risk.

characterized by the non linear differential equation:

$$\begin{cases} \max\limits_{0\leq h\leq 1} L(h)V(x) = 0 \ \text{ for } x \leq x_1 \\ V(0) = 0\, V'(x_1) = 1\, V''(x_1) = 0 \end{cases} \tag{52}$$

with

$$L(h)V(x) = \frac{1}{2}(\sigma^2 + (1-h)^2\sigma_R^2)V''(x) + (\mu - h\frac{\sigma_R^2}{2}\pi)V'(x) - rV(x).$$

The main task is to construct a concave solution to equation (52). Unfortunately, we do not achieve this purpose. We content ourselves with providing qualitative results concerning the optimal hedging policy. Let us define by h^* the maximizer and assume the existence of an open interval $\mathcal{O}$ such that $0 < h^*(x) < 1$. Then,

$$h^*(x) = 1 + \frac{\pi}{2}\frac{V'(x)}{V''(x)},$$

and

$$L(h^*)V(x) = L(1)V(x) - \frac{\pi^2\sigma_R^2}{8}\frac{(V'(x))^2}{V''(x)}. \tag{53}$$

We start our analysis by looking forward assumptions ensuring that $h^*(x) = 0$. If it is optimal to assume the risk, the value function is given by the benchmark formula

$$V(x) = A(e^{\theta_2 x} - e^{\theta_1 x})$$

where $\theta_1 < 0 < \theta_2$ are the roots of the equation

$$\frac{1}{2}(\sigma^2 + \sigma_R^2)\theta^2 + \mu\theta = r.$$

Computing h^* near 0, we obtain

$$\pi \geq \frac{4\mu}{\sigma^2 + \sigma_R^2}$$

Therefore, we have

Proposition 8 *If $\pi \geq \frac{4\mu}{\sigma^2+\sigma_R^2}$, it is optimal to assume the risk. Moreover, the value function is given by the benchmark case with $\sigma^2 + \sigma_R^2$ in place of σ^2.*

From now, we assume that $\pi \leq \frac{4\mu}{\sigma^2+\sigma_R^2}$. Our next finding concerns the pattern of the maximizer h^*. Setting

$$a = \frac{\sigma^2}{2}, b = \mu - \frac{\sigma_R^2}{2}\pi \text{ and } \alpha = \frac{\pi^2\sigma_R^2}{8},$$

we can solve implicitly the nonlinear equation (53),

$$2aV''(x) = -bV'(x) + rV(x) - \sqrt{(bV'(x) - rV(x))^2 + 4a\alpha(V'(x))^2}. \tag{54}$$

Therefore,

$$L(h^*)V(0+) = L(1)V(0+) + \frac{\alpha}{2a}(b + \sqrt{b^2 + 4a\alpha})V'(0+).$$

Since V' is bounded below by one, we conclude that it is never optimal to totally hedge its position even in the neighbourhood of zero.

Conversely, we claim that there is a constant $x_0 < x_1$ such that $h^* = 0$ on (x_0, x_1). If not, putting in the equation (54) the conditions defining V on x_1, we obtain

$$0 = -b + rV(x_1) + \sqrt{(b - rV(x_1))^2 + 4a\alpha}$$

which yields to a contradiction.

7 Who Should Hedge?

We conclude this paper by deriving from our model several testable implications about which firms are more likely to hedge,[19] in the hope to shed light on the mixed findings of the empirical literature. We already found in Proposition 3 that, provided a firm has decided to use hedging instruments optimally, it will tend to buy hedging ($h = 1$) when it is cash poor ($x \leq x_0$) and to self-insure ($h = 0$) when it is cash rich ($x > x_0$). We now study the prior decision to create, within the firm, a risk management unit and to hire the personnel able to manage the hedging position of the firm according to the instructions given by top management. This decision is optimal if the gains from hedging exceeds the cost of creating this risk management unit.

Consider now an empirical economist who has collected data on the balance sheets of a large population of firms, and can therefore estimate the parameters of our model such as expected profitability μ, volatility of earnings, σ^2, leverage (measured here by the inverse interest coverage ratio $\frac{c}{\mu}$). Our model predicts that the probability that a firm has created a risk management unit is an increasing function of the gain from hedging, measured by the reduction in the costs of financial frictions obtained by hedging. When the cost of hedging is small ($\pi \sim 0$) we saw that this gain could be approximated by:

$$G_0 = x^*(\mu, \sigma^2 + \sigma_R^2) - x^*(\mu, \sigma^2)$$

when the firm is unleveraged ($c = 0$). As we have noticed in Section 5, this formula can be easily extended to the case where $c > 0$:

$$G_0 = x^*(\mu - c, \sigma^2 + \sigma_R^2) - x^*(\mu - c, \sigma^2). \tag{55}$$

[19]For simplicity, we focus on the hedging decision, since the formulas for the gains from insurance are more complex.

We already saw that x^* is an increasing, concave, function of σ^2. Thus we deduce immediately from formula (55) that:

$$\frac{\partial G_0}{\partial \sigma_R^2} > 0 \quad \text{and} \quad \frac{\partial G_0}{\partial \sigma^2} < 0.$$

This means that the gain from hedging increases with the volatility σ_R^2 of the hedgeable risk and decreases with the volatility σ^2 of the "operating" risk. More interestingly, the impact of μ and c (or indeed $\mu - c$) is non monotonic, as illustrated by Figure 7:

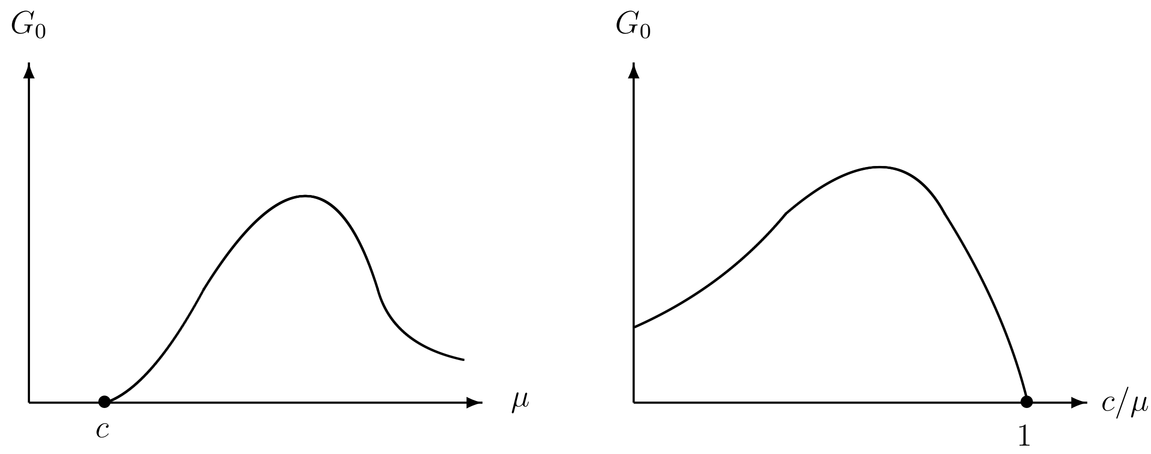

Figure 7: The gain from hedging as a function of profitability μ and leverage c/μ.

Thus profitability and leverage have a non monotonic (and highly non linear) impact on the gains from hedging. This may explain why empirical studies who use linear specifications have failed to derive a significant impact of profitability and leverage on the likelihood that a firm decides to hedge.

Appendix

Proof of Proposition 2: Let us recall that the cost of financial frictions is given by:

$$x^*(\mu,\sigma^2) = \frac{1}{\rho_2-\rho_1}\ln\frac{\rho_1^2}{\rho_2^2} = \frac{\sigma^2}{2\sqrt{\mu^2+2r\sigma^2}}\ln\left[\frac{\sqrt{\mu^2+2r\sigma^2}+\mu}{\sqrt{\mu^2+2r\sigma^2}-\mu}\right].$$

x^* is a continuous, positive function of μ satisfying

$$\lim_{\{\mu\to 0\}} x^*(\mu,\sigma^2) = \lim_{\{\mu\to\infty\}} x^*(\mu,\sigma^2) = 0.$$

A straightforward but tedious calculus gives

$$\frac{\partial x^*}{\partial\mu}(\mu,\sigma^2) = \sigma^2(\mu+2r\sigma^2)^{-\frac{3}{2}}\left[-\mu\text{Argth}\left(\frac{\mu}{\sqrt{\mu+2r\sigma^2}}\right)+\sqrt{\mu+2r\sigma^2}\right]$$

Thus $\frac{\partial x^*}{\partial\mu}$ has the sign of

$$g(\mu) = -\mu\text{Argth}\left(\frac{\mu}{\sqrt{\mu+2r\sigma^2}}\right)+\sqrt{\mu+2r\sigma^2}.$$

But, $g'(\mu) = -\text{Argth}\left(\frac{\mu}{\sqrt{\mu+2r\sigma^2}}\right)$ and $g(0)=\sqrt{2r\sigma^2}$ and $\lim_{\{\mu\to\infty\}} g(\mu) = -\infty$. Therefore, f changes sign once and x^* admits a unique maximum.

Moreover, setting $t=\frac{\mu}{\sigma^2}$ and $a=\frac{2r}{\mu}$, we get

$$x^*(\mu,\sigma^2)\equiv f(t) = \frac{1}{\sqrt{t^2+a}}\text{Argth}\left(\sqrt{\frac{t}{t+a}}\right).$$

We have $f'(0)=-1$ and $f''(t) = -2\text{Argth}\left(\sqrt{\frac{t}{t+a}}\right)$. Therefore,

$$\frac{\partial x^*}{\partial\sigma^2} = \frac{\partial t}{\partial\sigma^2}f'(t)\geq 0.$$

Proof of Theorem 2: The result is a consequence of the two following lemmas.

Lemma 1 *Let W satisfy the assumptions of Theorem 2. Then for any control (h_t, Z_t),*

$$W(x)\geq E\left(\int_0^{\tau_0} e^{-rs}\,dZ_s\right),$$

for all $x\geq 0$.

Proof of lemma 1 Fix a policy (h_t, Z_t) and write the process $Z_t = Z_t^c + Z_t^d$ where Z_t^c is the continuous part of Z_t and Z_t^d is the pure discontinuous part of Z_t. Let,

$$dX_t = (\mu - \frac{\sigma_R^2}{2} h_t)t + \sigma dW_t + \sigma_R(1-h_t)dW_t^R - dZ_t,$$

be the evolution of the cash under the policy (h_t, Z_t) and let us define

$$\tau_0 = \inf\{t \geq, \, X_t \leq 0\}.$$

Using the generalized Ito formula (see Dellacherie and Meyer Theorem VIII.27) and the equality $X_s - X_{s-} = -(Z_s - Z_{s-})$, we can write

$$\begin{aligned} e^{-r(t\wedge\tau_0)} W(X_{t\wedge\tau_0}) &= W(x) + \int_0^{t\wedge\tau_0} e^{-rs} L(h) W(X_s)\, ds \\ &+ \int_0^{t\wedge\tau_0} e^{-rs} W'(X_s)\left(\sigma dW_t + \sigma_R(1-h_t)dW_t^R\right) - \int_0^{t\wedge\tau_0} e^{-rs} W'(X_s)\, dZ_s^c \\ &+ \sum_{s\leq t\wedge\tau_0} e^{-rs}(W(X_s) - W(X_{s-})), \end{aligned}$$

where

$$L(h)W(x) = \left(\frac{\sigma^2 + \sigma_R^2(1-h)^2}{2}\right) W'' + (\mu - \frac{\sigma_R^2}{2} h) W' - rW.$$

Since W satisfies (21) and (22) the second term of the right hand side is negative. Since W is concave and increasing, $0 \leq W'(X_s) \leq W'(0)$ and thus the third term is a centered square integrable martingale. Taking expectations, we get

$$\begin{aligned} E\left(e^{-r(t\wedge\tau_0)} W(X_{t\wedge\tau_0})\right) &= W(x) - E\int_0^{t\wedge\tau_0} e^{-rs} W'(X_s)\, dZ_s^c \\ &+ E\sum_{s\leq t\wedge\tau_0} e^{-rs}(W(X_s) - W(X_{s-})). \end{aligned}$$

By concavity and since $W'(x) \geq 1$, we get $W(X_s) - W(X_{s-}) \leq -(Z_s - Z_{s-})$. Therefore,

$$W(x) \geq E\left(e^{-r(t\wedge\tau_0)} W(X_{t\wedge\tau_0})\right) + E\int_0^{t\wedge\tau_0} e^{-rs} W'(X_s)\, dZ_s.$$

By concavity, $W(x) \leq W'(0)x$ and thus

$$\liminf_{t\to\infty} E\left(e^{-r(t\wedge\tau_0)} W(X_{t\wedge\tau_0})\right) = 0.$$

We conclude by letting t tend to infinity.

Lemma 2 *Let W, h^* and $L_t(x_1)$ be given by Theorem 2. Then,*

$$W(x) = E\left(\int_0^{\tau_0} e^{-rs}\, dL_s(x_1)\right).$$

Proof of lemma 2 Assume that $x \leq x_1$. According to slight extension of Proposition 6.16 in Karatzas and Shreve (1991), $(X_t, L_t(x_1))$ is a solution to the following Skorohod problem:

$$\begin{cases} X_t \leq x_1 \\ \int_0^\infty \mathbb{1}_{\{X_s \neq x_1\}} dL_s(x_1) = 0 \end{cases}$$

Moreover, $L(h^*)W(x) = 0$ for all $x \leq x_1$. Applying Ito's formula in the same manner as in the proof of lemme 1, we get

$$E\left(e^{-r(t\wedge\tau_0)}W(X_{t\wedge\tau_0})\right) = W(x) - E\int_0^{t\wedge\tau_0} e^{-rs}W'(X_s)\, dL_s(x_1).$$

Since $W'(x_1) = 1$, the Skorohod problem gives

$$E\left(e^{-r(t\wedge\tau_0)}W(X_{t\wedge\tau_0})\right) = W(x) - E\int_0^{t\wedge\tau_0} e^{-rs}\, dL_s(x_1).$$

We conclude by letting t tend to infinity.

Proof of Proposition 3: When it is positive, x_0 is characterized by the condition that the firm is indifferent between hedging or not:

$$L(1)V(x_0) - L(0)V(x_0) = -\frac{1}{2}\sigma_R^2\left[\pi V'(x_0) + V''(x_0)\right] = 0$$

or

$$-\frac{V''(x_0)}{V'(x_0)} = \pi. \tag{A1}$$

Given the expression of V in the hedging region (formula (28) in Proposition 3), we deduce:

$$\theta_2^2 e^{\theta_2 x_0} - \theta_1^2 e^{\theta_1 x_0} = -\pi\left[\theta_2 e^{\theta_2 x_0} - \theta_1 e^{\theta_1 x_0}\right].$$

Thus

$$e^{(\theta_2-\theta_1)x_0} = \frac{\theta_1(\theta_1+\pi)}{\theta_2(\theta_2+\pi)},$$

which implies (30).

V being $\mathcal{C}^2$, we can also use the expression of V in the no-hedging region:

$$V(x) = Be^{\gamma_1 x} + Ce^{\gamma_2 x}.$$

The boundary conditions at x_1 give the values of B and C:

$$B = \frac{\gamma_2 e^{-\gamma_1 x_1}}{\gamma_1(\gamma_2-\gamma_1)}, \qquad C = -\frac{\gamma_1 e^{-\gamma_2 x_1}}{\gamma_2(\gamma_2-\gamma_1)}.$$

Condition (A1) then implies:

$$-\gamma_1\gamma_2 e^{\gamma_1(x_0-x_1)} + \gamma_1\gamma_2 e^{\gamma_2(x_0-x_1)} = \pi\left[\gamma_2 e^{\gamma_1(x_0-x_1)} - \gamma_1 e^{\gamma_2(x_0-x_1)}\right].$$

This gives:

$$e^{(\gamma_1-\gamma_2)(x_0-x_1)} = \frac{\gamma_1(\gamma_2+\pi)}{\gamma_2(\gamma_1+\pi)}, \tag{A2}$$

which implies formula (31). We just have to check that the right hand side of (A2) is positive. This comes from the fact that $\pi \leq \frac{2\mu}{\sigma^2+\sigma_R^2} < -\gamma_1$. ■

Proof of Theorem 4: The proof is very similar to that of Proposition 4. By definition, $\bar{x}_0$ is such that

$$D(1)\bar{V}(\bar{x}_0) = D(0)\bar{V}(\bar{x}_0), \text{ i.e.}$$

$$\bar{V}(\bar{x}_0) - \bar{V}(\bar{x}_0 - m) = m\bar{V}'(\bar{x}_0).$$

When $\bar{x}_0 < m$ (which will be checked ex post), $\bar{V}(\bar{x}_0 - m) = 0$, and the condition becomes

$$\bar{V}(\bar{x}_0) = m\bar{V}'(x_0).$$

Using the expression of $\bar{V}$ in the no-insurance regime (equation (50)), this gives:

$$e^{\bar{\theta}_2\bar{x}_0} - e^{\bar{\theta}_1\bar{x}_0} = m(1+\pi m)\left[\bar{\theta}_2 e^{\bar{\theta}_2\bar{x}_0} - \bar{\theta}_1 e^{\bar{\theta}_1\bar{x}_0}\right],$$

which implies (48). Similarly we can use the expression of $\bar{V}$ in the insurance regime (equation (51)), together with the values of $\bar{B}$ and $\bar{C}$ obtained as in the proof of Proposition 3:

$$\bar{B} = \frac{\bar{\gamma}_2 e^{-\bar{\gamma}_1\bar{x}_1}}{\bar{\gamma}_1(\bar{\gamma}_2-\bar{\gamma}_1)}, \qquad \bar{C} = -\frac{\bar{\gamma}_1 e^{-\bar{\gamma}_2\bar{x}_1}}{\bar{\gamma}_2(\bar{\gamma}_2-\bar{\gamma}_1)},$$

and

$$\bar{B}e^{\bar{\gamma}_1\bar{x}_0} + \bar{C}e^{\bar{\gamma}_2\bar{x}_0} = m\left[\bar{\gamma}_1\bar{B}e^{\bar{\gamma}_1\bar{x}_0} + \bar{\gamma}_2\bar{C}e^{\bar{\gamma}_2\bar{x}_0}\right].$$

After easy computations, we obtain formula (49). Then we have to check that $\bar{x}_0 \leq m$ and $\bar{x}_1 \geq \bar{x}_0$. The first condition is equivalent to prove that the function g defined by

$$g(m) = \ln(1-m\bar{\theta}_1) - \ln(1-m\bar{\theta}_2) - m(\bar{\theta}_2-\bar{\theta}_1).$$

is nonpositive. But, under the assumption $m \leq \dfrac{\mu}{r+\lambda}$, it is easy to check that $g(0) = 0$ and $g'(m) \leq 0$ for any $m \leq \frac{\mu}{r+\lambda}$. Therefore, g is nonpositive on $(0, \frac{\mu}{r+\lambda})$ which is the desired result.

The second condition is equivalent to

$$\bar{\gamma}_1^2(1-\bar{\gamma}_2 m) \geq \bar{\gamma}_2^2(1-\bar{\gamma}_1 m),$$

or:

$$m \leq \frac{\bar{\gamma}_1^2 - \bar{\gamma}_2^2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 - \bar{\gamma}_2)} = \frac{\bar{\gamma}_1 + \bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2} = \frac{\mu - \lambda m}{r},$$

that is $m \leq \frac{\mu}{r+\lambda}$.
Finally, we check that $\bar{V}$ is a concave solution to the variational inequalities (41), (42). ■

Proof of Proposition 5: Again, it is enough to check that $\bar{V}$ is a concave solution to the variational inequalities (41), (42).
For $x \leq \tilde{x}_1$, $D(1)\bar{V}(x)$ must be nonpositive. But,

$$D(1)\bar{V}(x) = \tilde{A}\lambda\left((1 - m\bar{\theta}_2)e^{\bar{\theta}_2 x} - (1 - m\bar{\theta}_1)e^{\bar{\theta}_1 x}\right).$$

If $(1 - m\bar{\theta}_2) \leq 0$ then $D(1)\bar{V}(x)$ is nonpositive, while if $(1 - m\bar{\theta}_2) \geq 0$ then $D(1)\bar{V}(x)$ is a nondecreasing function, nonpositive on the interval $[0, \frac{1}{\bar{\theta}_2 - \bar{\theta}_1} \ln\left(\frac{1-m\bar{\theta}_1}{1-m\bar{\theta}_2}\right)]$. Since, the condition $m \geq \frac{\mu}{r+\lambda}$ is equivalent to

$$\frac{2}{\bar{\theta}_2 - \bar{\theta}_1} \ln\left|\frac{\bar{\theta}_1}{\bar{\theta}_2}\right| \leq \frac{1}{\bar{\theta}_2 - \bar{\theta}_1} \ln\left(\frac{1 - m\bar{\theta}_1}{1 - m\bar{\theta}_2}\right),$$

$D(1)\bar{V}(x)$ is nonpositive on $[0, \tilde{x}_1]$.
For $x \geq \tilde{x}_1$, we have

$$D(1)\bar{V}(x) = -r(x - \tilde{x}_1) - \lambda(m - \frac{\mu}{r+\lambda}) \leq 0,$$

and

$$D(0)\bar{V}(x) = -r(x - \tilde{x}_1) - \lambda(\bar{V}(x) - \bar{V}(x-m) - \frac{\mu}{r+\lambda}).$$

Concavity of $\bar{V}$ yields the result. ■

Proof of Proposition 6: Using the equality,

$$D(1)\bar{V}(x) - D(0)\bar{V}(x) = \lambda\left[\bar{V}(x) - \bar{V}(x-m) - (m + \pi m^2)\bar{V}'(x)\right],$$

we give an expansion of $V(x-m)$ around x to obtain

$$D(1)\bar{V}(x) - D(0)\bar{V}(x) = \left[-\pi\bar{V}'(x) - \frac{1}{2}\bar{V}''(x)\right] m^2 + o(m^2).$$

Remembering that $\bar{V}'(x_1) = 1$ and $\bar{V}''(x_1) = 0$, we have

$$D(1)\bar{V}(x_1) - D(0)\bar{V}(x_1) = -\pi m^2 + o(m^2) < 0.$$

The conclusion follows from the continuity of the function $D(1)\bar{V}(x) - D(0)\bar{V}(x)$ ■

Proof of Proposition 7 Take $\alpha_1 \leq \alpha_2$, and define the associated liquidation thresholds $l_i, i = 1, 2$ and hitting times $T_{-l_i}, i = 1, 2$. We want to show that $V_{\alpha_1} \leq V_{\alpha_2}$. There is nothing to prove for $x \leq -l_1$. For $x \geq -l_1$, we have $T_{-l_1} \leq T_{-l_2}$ almost surely. Therefore, let us consider the dividend policy,

$$\hat{Z}_t = L_t^{x_1^*} \text{ if } t \leq T_{-l_1},$$

and

$$\hat{Z}_t = L_t^{x_2^*} \text{ if } T_{-l_1} \leq t \leq T_{-l_2},$$

we obtain

$$\begin{aligned} V_{\alpha_2}(x) &\geq E\left(\int_0^{T_{-l_2}} e^{-rs}\, d\hat{Z}_s\right) \\ &= E\left(\int_0^{T_{-l_1}} e^{-rs} dL_s^{x_1^*} + e^{-rT_{-l_1}} V_{\alpha_2}(-l_1)\right) \\ &= V_{\alpha_1}(x) + V_{\alpha_2}(-l_1) E(e^{-rT_{-l_1}}). \end{aligned}$$

Since V_{α_2} is a strictly increasing function with $V_{\alpha_2}(-l_2) = 0$, we have $V_{\alpha_2}(-l_1) > 0$ which implies the desired result. ∎

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