Public Contracting in Delegated Agency Games

(preliminary)

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Abstract: This paper studies games of public delegated common agency under asymmetric information. Using tools from non-smooth analysis and optimal control, we derive best responses and equilibria under weak conditions on schedules. At equilibrium, inefficiencies arise from two sources: inefficient contracting by a given coalition of active principals and inefficient participation (insufficient activity) by principals. Particular attention is given to the continuity of the equilibrium allocation and the characterization of the principals' activity sets. Existence of such equilibria is shown when principals have linear preferences. Our findings are illustrated by means of two examples of independent economic interest: a lobbying game between conflicting interest groups influencing the policy chosen by a political-decision maker and a game of voluntary contributions for a public good by congruent principals. Those examples illustrate that equilibrium distortions depend on whether principals have conflicting or congruent preferences.

Keywords: Delegated common agency, asymmetric information, public goods, lobbying games.

1 Introduction

Overview. This paper studies games of public delegated common agency under asymmetric information. Consider several principals offering contributions to an agent who produces a public good or takes a public decision on their behalf in a context where the

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agent has private information on his cost of doing so. Under *public common agency*, the agent's decision is observable and contractible for all principals. They offer contributions which stipulate how much the agent should be paid for any possible decision he may take. In a *delegated common agency* setting, the agent can select any subset of those offers and then choose accordingly which decision to implement.[1] We are interested in characterizing equilibria of those games.

Thanks to earlier effort,[2] much properties of public and delegated common agency are by now known when there is complete (or symmetric but incomplete) information on the agent's preferences. The existing literature pointed out two important lines of results. First, common agency games have efficient equilibria. Second, the principals' contributions in those equilibria are "truthful", i.e., they reflect the principals' preferences among alternatives. Intuitively, such a truthful schedule maximizes the payoff of any bilateral coalition between the agent and the corresponding principal because the former is made "residual claimant" for that coalition's payoff. A lump-sum payment can then be used to extract the agent's surplus and leave him just indifferent between accepting that principal's contract and contracting only with all other remaining principals. Whether the principals' have conflicting or congruent preferences over the agent's decision may have an impact on the redistribution of the efficient surplus but it does not affect the fact that "truthful" equilibrium outcomes are efficient.

Under asymmetric information, contribution schedules not only serve to "pass" the principals' preferences onto the agent but they are also used as screening devices. Principals elicit non-cooperatively the agent's private information. When designing his best response to others' contributions, a given principal trades off bilateral efficiency for the coalition he forms with the agent on the one hand and the information rent that the agent withdraws from his private information on the other hand. This trade-off is of course well-known from monopolistic screening environments.[3] However, under delegated common agency, it is modified in two important ways.

First, bilateral efficiency in a given principal-gent pair takes as given the other principals' contributions. Because of asymmetric information, those contributions are no longer truthful but also distorted for incentive reasons. Hence, bilateral efficiency in a given principal-agent pair is not enough to imply overall efficiency for the grand-coalition.

Second, the nature of the incentive distortion induced in any bilateral relationship depends also on other principals' offers. Indeed, when the agent refuses to deal with a

[1]This added possibility to select the set of principals whose offers are accepted distinguishes delegated from *intrinsic common agency*. Intrinsic common agency is studied with more care in Martimort and Stole (2009a). Public common agency should be distinguished from private common agency where different principals contract on different specific variables under the agent's control. See Martimort (2007) for a definition of private and public common agency.

[2]Bernheim and Whinston (1986), Grossman and Helpman (1994), Dixit and al. (1997), Laussel and Lebreton (1998, 2001), Chiesa and De Nicolo (2009) among others.

[3]Laffont and Martimort (2002).

given principal but still contracts with others, he obtains a type-dependent reservation payoff that affects contracting with that principal. Incentives whether to exaggerate or underestimate his own type in that relationship depend now not only on how difficult the agent finds it to please that principal but also on how such manipulations makes the agent look easier to buy or not for that principal.

The broad objective of this paper is to study equilibrium contracts in such competitive screening environments. Doing so, the analysis of public delegated agency games unveils a whole set of new issues that cannot be addressed in complete information models. How are the distortions induced by competing principals compounded at equilibrium? Given that the equilibrium outcome is inefficient, can we predict the directions of those distortions by simply looking at the principals' preferences and, in particular, how those distortions change when principals have either conflicting or congruent preferences? What is the activity set (i.e. the subset of types targeted with positive contributions) of a given principal? Under which circumstances do we have overlapping activity sets for different principals or instead activity sets that remain split apart?

Motivating examples. To motivate our theoretical analysis and give an overview of some of our findings, consider the following two archetypical examples.

Example 1: Lobbying competition. Two interest groups (principals) are willing to influence a decision-maker (the common agent) in a highly polarized environment. Principals have conflicting preferences and want to shift the agent's policy in opposite directions. The decision-maker has private information on his bliss point. Our analysis predicts that distortions move the equilibrium policy away from the status quo to favor the closest principal in the ideological space. The equilibrium reflects the conflicting forces of competing principals. A given interest group might secure exclusive influence on the decision-maker when their preferences are close to each other. Instead, types with ideal points too far away are too expensive to buy for an interest group. At equilibrium, interest groups may have overlapping areas of influence with an agent having "intermediate preferences" accepting both interest groups' contributions and making policy compromises.

Example 2: Voluntary contributions for a public good. An agent who is privately informed on his marginal cost of production produces a public good on behalf of two principals. Those principals have congruent preferences since they both like more public good being produced. They non-cooperatively offer contributions. Free-riding between those principals takes two forms in that context. First, the principals' marginal contributions are less than their marginal valuations to extract more of the agent's information rent. This induces excessive downward distortions in output compared to a cooperative contracting. Second, there might be limited participation with the weaker principal, i.e., the less eager to contribute, eschewing any contribution when the agent is too inefficient.

Results. These two examples illustrate several findings of our more general analysis.

Compounding inefficiencies. For a given coalition of active principals, their non-cooperative

behavior implies excessive distortions compared to the cooperative outcome. The equilibrium output compounds the incentive distortions that all principals induce. Whether distortions sum up or somewhat cancel each other depends on whether principals have congruent or conflicting preferences on the directions towards which the agent's decision should be tilted under asymmetric information. Overall, and in contrast with what arises with complete information "truthful equilibria", the nature of the principals' preferences has not only a redistributive impact but also an allocative one.
Non-truthful contributions. Contributions are no longer truthful under asymmetric information. More precisely, a principal concerned by the agent's incentives to exaggerate his type distorts downward his marginal contribution to make such manipulation less attractive. Instead, a principal who is more concerned by the agent's incentives to underestimate his type contributes more at the margin.
Activity sets. A given principal may find it too costly to offer contribution for certain types. Indeed, inducing a change in the agent's output requires giving him at least his reservation payoff obtained when contracting with all other active principals. This might be too costly compared with the corresponding efficiency gains that this principal may enjoy. Inefficient representation often occurs which, again, stands in contrast with the complete information environment. More generally, characterizing the principals' activity sets, i.e., the set of types who strictly gain from contracting with that principal, is a key aspect of the analysis. Formally, this amounts to finding where the agent's type-dependent participation constraint binds for each principal's best response. This is done by carefully looking at necessary conditions for such best response.[4] At a best response in a continuous equilibrium, a "smooth-pasting" condition holds on boundaries of the principals' activity sets: contribution schedules are zero both in value and at the margin.

A particular attention is given to the case where principals have preferences which are linear in the agent's decision. Activity sets are then connected intervals. Contributions are easily found and the equilibrium output satisfies simple and tractable modified Lindahl-Samuelson conditions. In passing, this characterization shows existence of equilibria for delegated common agency games for a large class of significant environments.
Technical contribution. To characterize best responses and equilibria, we build on the lessons of the existing literature on type-dependent participation constraints and broaden its lessons in a competitive screening environment.[5] This step is necessary to entertain the (a priori) possibility that the schedules offered by some principals may be discontinuous in an attempt to attract in a cheaper way the agent's services. Under weaker technical conditions than those assumed so far in the earlier literature and using tools from opti-

[4]This step is significantly more complex than in monopolistic screening environments. It is simplified only when the modeler can figure out a priori the shape of activity sets. This task is made easy when principals' surpluses are linear in output as we shall see below.

[5]Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Stole (1995), Jullien (2000).

mization in non-smooth analysis,[6] we derive conditions under which equilibrium outputs are continuous.

Organization of the paper. Section 2 reviews the literature. Section 3 presents our model of delegated common agency under asymmetric information. Section 4 describes the set of incentive feasible allocations for each principal (Section 4.1), discusses the notion of activity sets (Section 4.2) and finally sets up the best-response problem of a given principal as a generalized control problem in non-smooth analysis (Section 4.3). Necessary and sufficient conditions for a best response are then derived in Section 5. Strengthening those conditions, continuous equilibria are characterized in Section 6. Section 7 illustrates our findings by deriving continuous equilibria for our lobbying and public good examples where principals have monotonic preferences be they respectively conflicting or congruent. Section 8 concludes and highlights a few alleys for further research. Proofs are in an Appendix.

2 Literature Review

The study of public delegated common agency games under asymmetric information has been initiated in Martimort and Stole (2009b). Our focus there was on studying the convergence properties of equilibrium sets as one gets closer to complete information. With a sufficiently small uncertainty on types, all principals are always active on the whole (but tiny) type space and the value of a careful study of activity sets as developed below disappears. In other words, the only remaining inefficiency that arises for such delegated agency games is due to non-cooperative screening behavior and not to insufficient participation of some principals. Martimort and Stole (2009c) provided a general analysis of competition with nonlinear prices under both delegated and intrinsic common agency when manufacturers selling differentiated goods may choose to target only some subsets of consumers. This earlier paper focused on the case of private contracting with each principal having a specific screening variable (the quantity he sells to the buyer).[7] Here, we are instead interested in public agency environments where all principals use the same screening variable. A second difference is that, in Martimort and Stole (2009c), manufacturers rank the agent's types in the same way, with the agent having the highest valuation for both goods being the most attractive for both manufacturers.[8] Our analysis

[6] For introductions and recent developments in the mathematical tools used in non-smooth analysis, we refer to the seminal works by Clarke (1990), Loewen and Rockafellar (1997), Galbraith and Vinter (1997) and Vinter (2000) among others. This part of our analysis is of general interest beyond the characterization of best responses and may be of value for readers interested in principal-agent models with type-dependent participation constraints.

[7] Ivaldi and Martimort (1994) and Calzolari and Scarpa (2004) are earlier studies of delegated common agency games with private contracting but they focused a priori on cases where all types are served.

[8] Mezetti (1997) provided a model with conflicting and differentiated principals but his focus was on an intrinsic common agency setting, putting aside the complete characterization of the activity sets.

below is more general and allows for principals having conflicting preferences over whose agent's type is their most preferred one. This is exemplified by our lobbying game. Biais, Martimort and Rochet (2000) analyzed a model of competing market-makers on financial markets with traders privately informed on their willingness to buy or sell assets in a common value environment with private agency. Because of symmetry, all market-makers have similar activity sets with a hole where traders having a limited valuation for trading in either direction do not trade under asymmetric information.

3 The Model

Consider n principals P_i, indexed with the subscript $i \in \{1, .., n\} = \mathcal{N}$. Those principals offer contribution schedules to a common agent who chooses the level of a public good on their behalf.[9] The feasible set of possible outputs is a closed interval $\mathcal{Q} \subseteq \mathbb{R}$. Let $\mathcal{S}$ be an arbitrary coalition of principals in the power set $2^{\mathcal{N}}$ and let $|\mathcal{S}|$ be its cardinal.

3.1 Players and Preferences

Principal P_i's preferences are quasi-linear and defined over the level of public good $q \in \mathcal{Q}$ and the monetary payment made to the common agent $t_i \in \mathbb{R}$ as:[10]

$$V_i(q, t_i) = S_i(q) - t_i$$

where $S_i(\cdot)$ is some gross surplus function Lipschitz-continuous on $\mathcal{Q}$.

The agent A has also quasi-linear preferences given by:

$$U(q, \sum_{i=1}^{n} t_i, \theta) = \sum_{i=1}^{n} t_i - \theta q + S_0(q)$$

where again $S_0(\cdot)$ is some gross surplus function, Lipschitz-continuous on $\mathcal{Q}$. This function captures the possibility that the agent might also enjoy or dislike the public good.[11] The agent's marginal cost of producing the public good θ is, for simplicity, constant.[12]

[9] This public good can be viewed as a public infrastructure of variable size, or it might be given a more abstract interpretation in terms of a policy variable which would affect all principals' payoffs.

[10] Our model is general enough to encompass both the cases where a principal may dislike the public good over some range and where different principals may have conflicting preferences on which level of that public good should be chosen.

[11] At the cost of greater complexity, we could generalize preferences by introducing a type-dependent shift parameter as follows: $U(q, \sum_{i=1}^{n} t_i, \theta) = \sum_{i=1}^{n} t_i - \theta q + S_0(q) + u_0(\theta)$. The function $u_0(\theta)$ is the agent's type-dependent reservation payoff in the absence of any production of the public good.

[12] The quantity θq is a genuine cost function defined on $\mathcal{Q} \subseteq \mathbb{R}_+$ when the model applies in a public good context. It might also be viewed as an opportunity cost of moving a public policy away from the status quo in a lobbying model.

3.2 Information

The agent has private information on the efficiency parameter θ. This type is drawn from the set $\Theta = [\underline{\theta}, \bar{\theta}]$ according to a cumulative distribution $F(\theta)$ with an everywhere positive, atomless and bounded density $f(\theta) = F'(\theta)$. The hazard rates $R(\theta) = \frac{F(\theta)}{f(\theta)}$ and $T(\theta) = \frac{1-F(\theta)}{f(\theta)}$ satisfy the following standard assumptions.[13]

Assumption 1 *Monotone hazard rate properties:* $\dot{R}(\theta) > 0 > \dot{T}(\theta) \quad \forall \theta \in \Theta$.

3.3 Benchmarks

A first-best optimal level of public good $q^{FB}(\theta)$ is defined as:

$$q^{FB}(\theta) \in \arg\max_{q \in \mathcal{Q}} \sum_{i=0}^{n} S_i(q) - \theta q.$$

Assumption 2 $\sum_{i=0}^{n} S_i(q) - \theta q$ *is strictly concave in* q *on* $\mathcal{Q}$.

Strict concavity ensures that a solution to the complete information problem $q^{FB}(\theta)$ exists and is characterized, when interior, by means of first-order conditions.

$$\sum_{i=0}^{n} S_i'(q^{FB}(\theta)) = \theta. \tag{1}$$

For future reference, define the agent's *status quo* payoff $U_0(\theta)$ and optimal output $q_0(\theta)$ when principals do not contribute respectively as:[14]

$$U_0(\theta) = \max_{q \in \mathcal{Q}} S_0(q) - \theta q \text{ and } q_0(\theta) \in \arg\max_{q \in \mathcal{Q}} S_0(q) - \theta q.$$

When $0 \in \mathcal{Q}$ and $S_0(0) \geq 0$, we have $U_0(\theta) \geq 0$.

3.4 Strategy Spaces

The strategy space available to each principal is the set of non-negative contribution schedules which are defined over the domain $\mathcal{Q}$ and upper semi-continuous, namely $\mathcal{T} = \{t : \mathcal{Q} \to \mathbb{R}^+ \text{ and u.s.c}\}$.

We shall sometimes abuse slightly notations and denote the aggregate schedule offered by all principals except P_i as $t_{-i}(q) = \sum_{j \neq i} t_j(q)$.

[13] Bagnoli and Bergstrom (2005).

[14] Assuming strict concavity of $S_0(\cdot)$ ensures uniqueness of the maximizer. Otherwise, $q_0(\theta)$ is an arbitrary measurable selection in the correspondence.

3.5 Timing and Equilibrium

The delegated common agency game $\Gamma =< (V_i(\cdot))_{1\leq i\leq n}, U(\cdot), \Theta, F(\cdot) >$ unfolds as follows:

1. The agent learns his private information θ.

2. Principals offer non-cooperatively the contributions $t_i(q)$ to the agent. The agent can accept any subset of those contributions.

3. The agent chooses q and receives accordingly the corresponding payments from the contributing principals whose offers have been accepted.

Γ is a public delegated common agency game since all principals observe and contract on the publicly observable decision q and the agent can a priori choose any possible set of offers. We look for pure strategy Perfect Bayesian equilibria with deterministic mechanisms of Γ (in short equilibria) whose definition follows.

Definition 1 *An equilibrium of Γ is a vector of contribution schedules, a set and an output correspondences $\{(\bar{t}_i)_{1\leq i\leq n}, \mathcal{S}(\theta|t), q_0(\cdot|t)\}$[15] such that:*

1. *Given any profile of contributions $t \in \mathcal{T}^n$, $(\mathcal{S}(\theta|t), q_0(\theta|t))$ consists of the set of principals whose offers are accepted and an output which maximizes the agent's payoff:*
$$(\mathcal{S}(\theta|t), q_0(\theta|t)) \in \arg \max_{q\in\mathcal{Q}, \mathcal{S}\in 2^{\mathcal{N}}} \sum_{i\in\mathcal{S}} t_i(q) - \theta q + S_0(q);$$

2. *$\bar{t}_i$ maximizes principal P_i's expected payoff given the other principals' aggregate contribution schedules $\bar{t}_{-i}$:*
$$\bar{t}_i \in \arg\max_{t_i\in\mathcal{T}} \int_\Theta (S_i(q_0(\theta|t_i, \bar{t}_{-i})) - t_i(q_0(\theta|t_i, \bar{t}_{-i})))dF(\theta).$$

The definition above can be slightly simplified. Given that $\mathcal{T}$ contains only non-negative contributions, it is always weakly optimal for the agent to accept all contributions (eventually choosing an output that is not rewarded by a non-empty subset of inactive principals), i.e., $\mathcal{S}(\theta|t) \equiv \mathcal{N}$ in any continuation equilibrium.[16] To further simplify presentation, we will omit the dependence of the agent's output on the vector of contributions offered in continuation equilibria and denote by $q_0(\theta|t) \equiv \bar{q}(\theta)$ the equilibrium output.[17]

Remark 1 *The agent can always refuse all contributions, choose the status quo output $q_0(\theta)$ and get payoff $U_0(\theta)$ which represents his reservation payoff in any equilibrium.*

[15]Sometimes we refer below to those correspondences as a continuation equilibrium.

[16]Much of the interest of the analysis below comes from determining the set of principals who contribute a positive amount for a given type of the agent.

[17]It is worth noticing that existence of a measurable selector from the non-empty compact values correspondence $\arg\max_{q\in\mathcal{Q}} \sum_{i\in\mathcal{S}} t_i(q) - \theta q + S_0(q)$ follows from the Measurable Maximum (Aliprantis and Border, 1999, p. 570) when $\sum_{i\in\mathcal{S}} t_i(q)$ is continuous and will be assumed otherwise.

4 Setting the Stage

4.1 Incentive Feasible Set

We now characterize the set of incentive feasible allocations available to principal P_i.

Definition 2 *A rent/output profile* $(U(\theta), q(\theta))$ *is implementable by principal* P_i *under delegated common agency when the aggregate contribution offered by all other principals is* $\bar{t}_{-i} \in \mathcal{T}$ *if and only if there exists a contribution schedule* $t_i \in \mathcal{T}$ *such that:*

$$U(\theta) = \max_{q \in \mathcal{Q}} t_i(q) + \bar{t}_{-i}(q) - \theta q + S_0(q) \textit{ and } q(\theta) \in \arg\max_{q \in \mathcal{Q}} t_i(q) + \bar{t}_{-i}(q) - \theta q + S_0(q).$$

Let define the information rent $\bar{U}_{-i}(\theta)$ and the optimal output (or at least a selection within the best-response correspondence) $\bar{q}_{-i}(\theta)$ that an agent with type θ would choose when not taking P_i's contribution respectively as:

$$\bar{U}_{-i}(\theta) = \max_{q \in \mathcal{Q}} \bar{t}_{-i}(q) - \theta q + S_0(q) \text{ and } \bar{q}_{-i}(\theta) \in \arg\max_{q \in \mathcal{Q}} \bar{t}_{-i}(q) - \theta q + S_0(q).$$

The next Lemma follows from Definition 2:

Lemma 1 *Given the non-negative aggregate contribution offered by all other principals* $\bar{t}_{-i} \in \mathcal{T}$*, a rent/output profile* $(U(\theta), q(\theta))$ *is implementable by principal* P_i *through a contribution schedule* $t_i \in \mathcal{T}$ *(with an aggregate* $t_i + \bar{t}_{-i}$ *contribution) if and only if*

1. $U(\theta)$ *is convex,*

2. *The agent is as least weakly better off accepting* P_i*'s offer*

$$U(\theta) \geq \bar{U}_{-i}(\theta) \quad \forall \theta \in \Theta. \tag{2}$$

Item [1.] in Lemma 1 implies that $U(\theta)$ is absolutely continuous, with $q(\theta)$ decreasing. Moreover, both functions are a.e. differentiable with, at any differentiability point,

$$\dot{U}(\theta) = -q(\theta) \tag{3}$$

$$\dot{q}(\theta) \leq 0. \tag{4}$$

Item [1.] holds also for the rent/output profile $(\bar{U}_{-i}(\theta), \bar{q}_{-i}(\theta))$ since it is is itself implementable (when P_i offers a null contribution) and for the status quo profile $(U_0(\theta), q_0(\theta))$.[18]

Because contributions are non-negative, the agent is always at least weakly better off accepting all offers in the delegated common agency game under scrutiny and Item [2.] holds. One important aspect of our analysis is nevertheless to determine precisely the subset of types where the participation constraint (2) binds.

[18] This property is called "homogeneity" by Jullien (2000).

4.2 Activity Sets

In this respect, let define the *activity set* Ω_i of principal P_i as follows.

Definition 3 *Principal P_i's activity set is* $\Omega_i = \{\theta \in \Theta | \quad U(\theta) > \bar{U}_{-i}(\theta)\}$.

P_i's contribution is necessarily positive on his activity set. Because with this definition, activity sets are open it might be that, at a θ_i on the boundary of Ω_i, P_i's contribution is not necessarily zero but yet (2) is binding. This would be the case when P_i "becomes" an active principal with a positive transfer at some θ_i.

We refer to $\Omega_i^c = \Theta \setminus \Omega_i$ as the subset of types where the participation constraint (2) binds. Ω_i might be a finite collection of connected intervals Ω_i^j, namely $\Omega_i = \bigcup_{j\in\mathcal{J}} \Omega_i^j$, ($j \in \mathcal{J}$ where $\mathcal{J} \subset \mathbb{N}$). We denote also the set of active principals at a given type θ and its complement as respectively $\alpha(\theta) = \{i \in \mathcal{N}| \quad \theta \in \Omega_i\}$ and $\alpha^c(\theta) = \mathcal{N}/\alpha(\theta)$. Finally, let $|\alpha(\theta)|$ be the cardinal of this set.

4.3 A Generalized Control Problem

Lemma 1 allows us to restate principal P_i's optimization problem at a best response to the aggregate schedule $\bar{t}_{-i}$ offered by other principals as:

$$(\mathcal{P}_i): \max_{(U,q\in\mathcal{Q})} \int_\Theta (S_0(q(\theta))+S_i(q(\theta))+\bar{t}_{-i}(q(\theta))-\theta q(\theta)-U(\theta))f(\theta)d\theta \text{ subject to (2), (3), (4).}$$

Let denote now $(\mathcal{P}_i^r)$ the relaxed problem obtained from $(\mathcal{P}_i)$ by omitting the second-order condition (4). As standard in the screening literature, that latter condition will be checked ex post on the output profile obtained at equilibrium by imposing Assumption 1 on the type distribution.

Introducing the auxiliary variable $v(\theta) = -q(\theta)$, (3) can be written as:

$$\dot{U}(\theta) = v(\theta) \tag{5}$$

where $v(\theta) \in \mathcal{V} = -\mathcal{Q}$. In the sequel, it is useful to define the *extended-value Lagrangean* for $(\mathcal{P}_i^r)$, possibly taking values in $\mathbb{R}\bigcup\{+\infty\}$, as $L_i(\theta, u, v) = L^0(\theta, u) + L_i^1(\theta, v) + \psi_\mathcal{V}(v)$, where $L^0(\theta, u) = uf(\theta)$, $L_i^1(\theta, u, v) = -(S_0(-v) + S_i(-v) + \bar{t}_{-i}(-v) + \theta v)f(\theta)$ and $\psi_C(\cdot)$ denotes the indicator function of a given set C, namely $\psi_C(q) = \begin{cases} 0 & \text{if } q \in C \\ +\infty & \text{otherwise.} \end{cases}$

Using more compact notations borrowed from recent developments in non-smooth analysis,[19] $(\mathcal{P}_i^r)$ can be expressed as the following minimization of a *generalized Bolza problem* over the class of absolutely continuous arcs $U(\cdot)$:

$$(\mathcal{P}_i^r): \quad \min_{(U,v\in\mathcal{V})} \int_\Theta L_i(\theta, U(\theta), v(\theta))d\theta \quad \text{subject to (2) and (5).}$$

[19] Clarke (1990), Loewen and Rockafellar (1997), Galbraith and Vinter (1997), Vinter (2000), Vinter and Zheng (1998).

Definition 4 *A minimizing process is a solution to* $(\mathcal{P}_i^r)$ $(\bar{U}, \bar{v}) : \Theta \to \mathbb{R} \times \mathcal{V}$ *such that* $\bar{U}$ *is absolutely continuous.*

We shall be interested in non-trivial cases where such minimizer exists and denote accordingly by $V_i > -\infty$ the value of $(\mathcal{P}_i^r)$.

5 Optimality Conditions

We are now ready to characterize the solution to $(\mathcal{P}_i^r)$ by means of optimal control techniques. In the screening literature, the "state of the art" to solve problems like $(\mathcal{P}_i^r)$ is so far given by Jullien (2000). This paper studies screening models with type-dependent participation constraints which are particularly relevant to study public contracting under delegated agency. It characterizes solutions to such problems when the principal's objective is twice continuously differentiable, strictly concave in output, and participation constraints correspond to utility profiles that can themselves be implemented by a contract.[20] The second requirement certainly applies to our public delegated agency game. Indeed, when refusing the offer of a given principal, the agent optimally chooses output in the remaining set of contracts which provides an implementable profile. Under common agency, the first of this requirement is more problematic because it presumes a priori that equilibrium tariffs are twice differentiable. This rules out jumps in contribution schedules although such jumps could be found attractive by one principal to attract the agent's services and induce discontinuous changes in his output. In that respect, relaxing the degree of smoothness in the data is necessary to have a more complete view of competing screening environments. Theorems 1 and 2 below dispense from such smoothness properties but nevertheless provide necessary and sufficient conditions for optimality.

Necessary conditions. Those necessary conditions are actually quite similar to those found in smoother environments.

Theorem 1 *Let* $(\bar{U}, \bar{v})$ *be a minimizing process for* $(\mathcal{P}_i^r)$ *with* $V_i < +\infty$.*There exists a non-negative measure* $\mu_i(\cdot)$ *on* Θ *with supp* $\mu_i \subseteq \Omega_i^c$, *and a* μ_i- *integrable function* γ_i *on* Θ *such that the following necessary conditions are satisfied by the minimizer* $(\bar{U}(\theta), \bar{v}(\theta))$:

1. $r_i(\underline{\theta}) = r_i(\bar{\theta}) = 0$ *where*

$$r_i(\theta^-) = F(\theta) - \int_{[\underline{\theta}, \theta)} \gamma_i(s)\mu_i(ds) \quad \forall \theta \in (\underline{\theta}, \theta) \tag{6}$$

   and $\eta_i(\theta) = \int_{[\underline{\theta}, \theta)} \gamma_i(s)\mu_i(ds)$ *is piecewise absolutely continuous,*

2. $r_i(\theta^-)\bar{v}(\theta) - L_i^1(\theta, \bar{v}(\theta)) = \max_{v \in \mathcal{V}}\{r_i(\theta^-)v - L_i^1(\theta, v)\}$ *a.e.*,

[20] To be complete, Theorem 1 in Jullien (2000) assumes full separation of types (i.e., no bunching) also.

3. $\gamma(\theta) \in \{0,1\}$*, and* $\gamma(\theta)(\bar{U}(\theta) - \bar{U}_{-i}(\theta)) = 0$ *a.e..*

Items [1.] characterizes how the costate variable $r_i(\theta^-)$ for (3) evolves. Item [2.] tells us that $-\bar{q}(\theta) = \dot{\bar{U}}(\theta) = -\bar{v}(\theta)$ maximizes the Hamiltonian associated to $(\mathcal{P}_i^r)$. Alternatively, this optimality condition admits the following representation:

$$r_i(\theta^-) \in \partial_v co\left\{L_i^1\right\}(\theta, \bar{v}(\theta)). \tag{7}$$

which expresses the fact that the costate variable $r_i(\theta^-)$ belongs at any point on the optimal trajectory to the subdifferential of the convexification[21] $co\left\{L_i^1\right\}(\cdot)$ of the Lagrangean $L_i^1(\cdot)$. This convexification is needed to make our optimization problem, which may a priori entail jump discontinuities, more regular.

Item [3.] is a complementary slackness condition that establishes that $\eta_i(\theta)$ can only increase on inactivity sets, possibly with upward jumps at points where the participation constraint (2) starts being binding (the measure μ_i having then masses at those points). Inactivity sets contribute thus to reduce $r_i(\theta)$.

Altogether conditions [1.], [2.] and [3.] tell us that output distortions away from the bilaterally efficient ones in any given principal-agent pair are captured by means of a cumulative distribution function whose support is the inactivity set of a given principal at his best response. Under delegated agency, each principal's best response is thus characterized by such a distribution and the equilibrium allocation will compound the different distortions that different principals may induce. We shall investigate this issue in mode details below.

Application. To illustrate the power of Theorem 1, we now check whether a given allocation $(\tilde{U}(\theta), \tilde{q}(\theta))$ that could be expected in some "putative" equilibrium of a lobbying game fails to satisfy the necessary conditions for optimality.

Consider two competing interest groups (principals) with conflicting preferences $S_1(q) = -S_2(q) = q$. The decision-maker (agent) has some ideal policy he would like to pursue in the absence of any lobbying. To model this, we assume that $S_0(q) = -\frac{q^2}{2}$ where $q \in \mathcal{Q} = [-\bar{Q}, \bar{Q}]$ with $\bar{Q}$ being large enough to avoid corner solutions. Assume also that θ is uniformly distributed over $[-\delta, \delta]$ with $\delta < 1$. The agent's bliss point is located at $q_0(\theta) = -\theta$ with the corresponding payoff $U_0(\theta) = \frac{\theta^2}{2}$.[22]

The nature of the principals' conflicting preferences suggests that we may be able to construct a "putative" equilibrium with fierce "head-to-head" competition for the agent's services. Indeed principal P_1 enjoys higher policies and is ready to cajole types closer

[21] Take any function $f : \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$ and define the convex envelope of f as $co\{f\} : \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$: $co\{f\}(x) = \min_{\alpha_i \geq 0, \sum_{i\in\mathcal{I}} \alpha_i = 1, x_i \in \mathcal{R}^n} \left\{\sum_{i\in\mathcal{I}} \alpha_i f(x_i) | \quad x = \sum_{i\in\mathcal{I}} \alpha_i f(x_i)\right\} = \min_{\alpha \geq 0, (x_1,x_2)\in\mathcal{R}^n\times\mathcal{R}^n} \{\alpha f(x_1) + (1-\alpha) f(x_2)\}$ where the last equality follows from Caratheodory Theorem (Rockafellar and Wets, 2004, Theorem 2.29).

[22] Since principals are symmetrically biased in opposite directions, they would just agree on letting the agent choose his status quo policy had they cooperated.

to $-\delta$ since they find it more attractive to move up the policy. This is the reverse for principal P_2 who prefers types closer to δ. Let us construct such "candidate equilibrium".

Consider thus the following (symmetric) contribution schedules

$$\tilde{t}_1(q) = \begin{cases} \frac{5}{4}(1-\delta)^2 + \frac{1-\delta}{2}q + \frac{1}{4}q^2 & \text{if } q \geq 1-\delta \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \tilde{t}_2(q) = \bar{t}_1(-q). \tag{8}$$

Those contributions altogether implement a decreasing policy $\tilde{q}(\theta)$ such that:

$$\tilde{q}(\theta) = \begin{cases} \tilde{q}_1(\theta) = 1-\delta-2\theta & \text{if } \theta \in [-\delta, 0) \\ \tilde{q}_2(\theta) = -1+\delta-2\theta & \text{if } \theta \in (0, \delta]. \end{cases} \tag{9}$$

This policy is discontinuous at $\theta = 0$, with that type being indifferent between choosing $\tilde{q}(0^-) = 1-\delta > 0$ and $\tilde{q}(0^+) = -1+\delta < 0$. Finally, the agent's information rent is everywhere greater than his status quo payoff:

$$\tilde{U}(\theta) = \frac{3}{2}(1-\delta)^2 + (1-\delta)|\theta| + \theta^2 > U_0(\theta) \quad \forall \theta \in \Theta. \tag{10}$$

Had the agent taken only principal P_2's contribution, he would obtain a rent $\tilde{U}_2(\theta)$ and choose a policy $\tilde{q}_2(\theta)$) such that

$$\tilde{U}_2(\theta) = \max_q \left\{ \tilde{t}_2(q) - \theta q - \frac{q^2}{2} \right\} = \max \left\{ \frac{\theta^2}{2}, \max_{q \leq -1+\delta} \left\{ \frac{5}{4}(1-\delta)^2 - \frac{1-\delta+2\theta}{2}q - \frac{1}{4}q^2 \right\} \right\}$$

where the maximand on the right-hand side is achieved for $\tilde{q}_2(\theta) = -1+\delta-2\theta$. Hence, we find $\tilde{U}_2(\theta) = \frac{3}{2}(1-\delta)^2 + (1-\delta)\theta + \theta^2$ with $\dot{\tilde{U}}_2(\theta) = -\tilde{q}_2(\theta) = 1-\delta+2\theta \geq 0$.

With those contributions, the principals' activity sets are respectively $\Omega_1 = [-\delta, 0)$ and $\Omega_2 = (0, \delta]$ so that interest groups split the type space with the agent accepting only the contribution of one principal at once. On his own activity set, each principal behaves as a monopoly and the policy $\tilde{q}_i(\theta)$ are those that would be offered in the absence of a competing group. Schedules differ with respect to such exclusive contracting environment because principals bid up for the services of the marginal agent located at 0. The discontinuity in $\tilde{t}_2(q)$ at $\tilde{q}_2(0) = 1-\delta$ is indeed chosen so that principal P_1 is indifferent between inducing his "monopoly" policy $\tilde{q}_1(0) = -1+\delta$ from that type and compensating this agent for the extra payment he would get from P_2 when choosing $\tilde{q}_2(0)$.

We now check that the allocation $(\tilde{U}(\theta), \tilde{q}(\theta))$ in this "putative" equilibrium does not satisfy the necessary conditions in Theorem 1. This shows that such discontinuous jumps at 0 cannot arise.

Fix P_2's offer $\tilde{t}_2(q)$ and let us define the nonlinear part of P_1's Lagrangean $L_1^1(\theta, v)$ as

$$L_1^1(\theta, v) = \frac{1}{2\delta}\left(\frac{v^2}{2} + (1-\theta)v - \bar{t}_2(-v)\right) = \begin{cases} \frac{1}{2\delta}\left(\frac{v^2}{2} + (1-\theta)v\right) & \text{if } v < 1-\delta \\ \frac{1}{2\delta}\left(\frac{v^2}{4} + \frac{1+\delta-2\theta}{2}v - \frac{5}{4}(1-\delta)^2\right) & \text{if } v \geq 1-\delta. \end{cases}$$

[22] Note in particular that $\tilde{U}_2(\cdot)$ is increasing in the neighborhood of $\theta = 0$ since $1 > \delta$.

The discontinuity of $\tilde{t}_2(q)$ at $\tilde{q}_2(0)$ introduces a non-convexity in $L_1^1(\theta, v)$. (Indeed, note that $L_1^1(\theta,(1-\delta)^+) < L_1^1(\theta,(1-\delta)^-)$. The convexification $co\{L_1^1\}(\theta, v)$ of $L_1^1(\theta, v)$ with respect to v is made of three different pieces with a linear part joining two parabolas.[23] Hence, we get the following expression of $co\{L_1^1\}(\theta, v)$

$$co\{L_1^1\}(\theta, v) = \begin{cases} \frac{1}{2\delta}\left(\frac{v^2}{2} + (1-\theta)v\right) & \text{if } v \leq -1+\delta \\ \frac{\delta-\theta}{2\delta}(v-1+\delta) & \text{if } -1+\delta \leq v \leq 1-\delta \\ \frac{1}{2\delta}\left(\frac{v^2}{4} + \frac{1+\delta-2\theta}{2}v - \frac{5}{4}(1-\delta)^2\right) & \text{if } v \geq 1-\delta. \end{cases}$$

In particular, we have $\partial_v co\{L_1^1\}(\theta, 1-\delta) = \left[\frac{\delta-\theta}{2\delta}, \frac{1-\theta}{2\delta}\right] \neq \emptyset$.

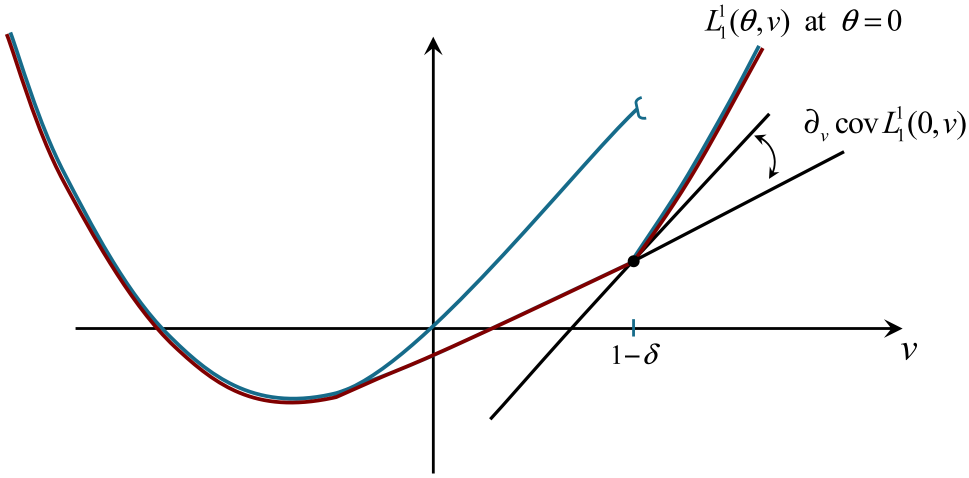

Figure 1: Convexification of $L_1^1(\cdot)$ and optimality.

Consider $r_1(\theta) = \frac{\theta+\delta}{2\delta}$ for $\theta \in [-\delta, \delta)$ with μ_1 putting mass 1 at δ only which corresponds to the case where the participation constraint (2) is binding only at δ. This r_1 is chosen at a best response by P_1 to the scheme $\bar{t}_2(q)$. First, notice that $r_1(0) = \frac{1}{2}$ so that, from Item [2.] in Theorem 1, principal P_1 is indifferent between $\tilde{q}(0^-)$ and $\tilde{q}(0^+)$, exactly as the agent with type 0 is.[24] More generally, we have $r_1(\theta^-) \in \partial_v co\{L_1^1\}(\theta, 1-\delta)$ for $\theta \in \left[0, \frac{1-\delta}{2}\right]$ (with equality only at $\theta = 0$) which indicates that, at a best response for P_1, $\bar{v}(\theta) = -\bar{q}(\theta) = 1-\delta$ for such $\theta \in \left[0, \min\left\{\frac{1-\delta}{2}, \delta\right\}\right]$. For $\delta \in \left[\frac{1}{3}, 1\right]$, the interval $\left[\frac{1-\delta}{2}, \delta\right]$ is non-empty and, for θ in that interval, we have $r_1(\theta^-) = \frac{\partial L_1^1}{\partial v}(\theta, \bar{v}(\theta))$, or put differently, $\bar{q}(\theta) = 1-\delta-4\theta$. For $\theta \in [-\delta, 0]$ instead, we have $r_1(\theta^-) = \partial_v co\{L_1^1\}(\theta, \tilde{v}_1(\theta))$, i.e., $\bar{q}(\theta) = \tilde{q}_1(\theta)$. The corresponding profile $(\bar{U}(\theta), \bar{q}(\theta))$ is obviously distinct from $(\tilde{U}(\theta), \tilde{q}(\theta))$. This shows that our "putative" equilibrium fails indeed to be an equilibrium.

Intuitively, P_1's best response to the offer $\tilde{t}_2(q)$ induces always a significant amount of bunching on the "corner" of $\tilde{t}_2(q)$. Indeed, with the "symmetric" contribution $\bar{t}_1(q)$ assumed in our "putative equilibrium", principal P_1 could not target types too far away

[23]Indeed, straightforward computations show that $L_1^1(\theta, 1-\delta) = L_1^1(\theta, -1+\delta) + \frac{\delta-\theta}{2\delta}(1-\delta-(-1+\delta))$.

[24]These indifferences are obviously necessary requirements for any equilibrium having an output discontinuity at type 0.

in the policy space and this is suboptimal. Instead, P_1 prefers to pay types slightly above zero a bit more to convince them to stick on a policy $\tilde{q}_2(0) = 1 - \delta$ which, although on the other side of the policy spectrum, is preferable from P_1's viewpoint than any policy $\tilde{q}_2(\theta)$ which is even further away.

Further properties. Theorem1 is of general interest for screening problems with type-dependent participation constraints but it does not exploit all the structure of our environment. Finer properties can be obtained once one realizes that reservation payoffs are themselves implementable.

Proposition 1 *$r_i(\theta)$ is continuous and μ_i has no mass point at any $\theta \in (\underline{\theta}, \bar{\theta})$.*

From Proposition 1, an output discontinuity, if any in the interior of the type space, cannot be due to jumps of the costate variable when reaching the boundary of an activity set. This limits the investigation of such discontinuity as coming from a lack of strict convexity of the Lagrangeans. Next section elaborates further on this issue and the Application above brings some complementary intuition. This example shows that a given principal cannot have a discontinuity at an interior boundary of an activity set at his best response; this principal would prefer to make a subset of the agent's types bunch on that discontinuity.

Sufficient conditions. We now adapt Arrow's sufficiency theorem for non-smooth problems like $(\mathcal{P}_i^r)$. This theorem relies heavily on the concavity (actually linearity) in U of the maximized Hamiltonian for $(\mathcal{P}_i^r)$.

Theorem 2 *Consider any $(\bar{U}, \bar{v})$ with a r_i which altogether satisfy the necessary conditions [1.] to [3.] in Theorem 1. Then, $(\bar{U}, \bar{v})$ solves $(\mathcal{P}_i^r)$.*

This Theorem is important because it allows us to check only a few conditions to assess whether a candidate allocation is a best response for a given principal, and beyond an equilibrium. This technique is illustrated in Section 7 below.

6 Continuous Equilibria

We provide below a condition, weaker than those in Jullien (2000), which ensures continuity of the equilibrium output. The features of the corresponding equilibria are developed in the specific context of Section 7. We first state the following condition:

Condition 1 Convexity. *$L_i^1(\theta, v)$ is finite valued, continuously differentiable in v, and there exists a constant $K > 0$ such that:*

$$|\nabla_v L_i^1(\theta, v_1) - \nabla_v L_i^1(\theta, v_2)| \geq K|v_1 - v_2| \quad \forall (v_1, v_2) \in \mathcal{V}^2.$$

By imposing a lower bound on the difference of Lagrangean gradients, Condition 1 imposes a minimal amount of convexity on Lagrangeans. This condition will be satisfied for the examples of continuous equilibria in Section 7 below as it can be easily checked. Contrary to the requirements in Jullien (2000), Condition 1 imposes only that the Lagrangean is continuously differentiable once. The degree of smoothness that is implicitly imposed on equilibrium schedules here is thus a priori less stringent.[25,26]

Theorem 3 *Assume that Condition 1 holds and* $(\bar{U}, \bar{v})$ *is a minimizing process for* $(\mathcal{P}_i^r)$ *with* $V_i < +\infty$ *that satisfies (2) and (5). Then,* $\bar{q}(\theta)$ *is continuous.*

To ensure continuity of the equilibrium output one must not only look carefully to principal P_i's objective function (Condition 1) but also avoid any mass point in the measure μ_i. Such singularity could induce discontinuity in the equilibrium output at a boundary point of this principal's activity set. This absence of singularity comes from the fact that $\bar{U}_{-i}(\theta)$ is itself an implementable profile as shown in Proposition 1.

Continuity of $\bar{q}(\theta)$ implies that whenever the type-participation constraint (2) starts being binding at an interior point of Θ, it does so in a smooth-pasting way. Both $\bar{U}(\theta) = \bar{U}_{-i}(\theta)$ and $\dot{\bar{U}}(\theta) = \dot{\bar{U}}_{-i}(\theta) \Leftrightarrow \bar{q}(\theta) = \bar{q}_{-i}(\theta)$ at any θ not only in the interior but also on the boundary of Ω_i^c if such boundary lies in the interior of Θ. As we shall see now, smooth-pasting has strong implications on the shape of equilibrium contributions close to the boundaries of activity sets.

We are now interested in deriving various properties of those continuous equilibria.

Output. Altogether Theorems 1 and 3 provide limited information on possible output distortions because they characterize only best responses and not yet equilibrium output. To be more explicit, we must account for how output distortions induced by different principals are compounded altogether. Next Theorem goes in that direction.

Theorem 4 *Suppose that conditions of Theorem 3 and Assumption 1 are satisfied.*

1. *The equilibrium output* $\bar{q}(\theta)$ *satisfies*

$$S_0'(\bar{q}(\theta)) + \sum_{i \in \alpha(\theta)} \vec{t}_i'(\bar{q}(\theta)) = \theta. \tag{11}$$

and

$$S_0'(\bar{q}(\theta)) + \sum_{i \in \alpha(\theta)} S_i'(\bar{q}(\theta)) = \theta + \frac{\sum_{i \in \alpha(\theta)} r_i(\theta)}{f(\theta)} \tag{12}$$

where $r_i(\theta)$ *defined in Theorem 3 is continuous and constant over any connected subset* $\Omega_i^j \subseteq \Omega_i$ $(j \in \mathcal{J})$ *of principal* P_i*'s activity set.*

[25] At equilibrium, the continuity of the solution will feed back on the property of the data of each principal's problem and ensure that those data are in fact continuously differentiable since equilibrium contributions turn out to be so. Schedules are not assumed a priori to be twice continuously differentiable.

[26] Note that Condition 1 needs to apply only to a principal to ensure continuity of his best response and thus of the equilibrium output if such exists.

2. The monotonicity condition (4) is satisfied a.e..

At a best-response, principal P_i implements an output which maximizes the virtual surplus of his bilateral coalition with the agent. In doing so, this principal takes into account that the agent's type θ must be replaced by his *virtual efficiency parameter* $\theta + \frac{r_i(\theta)}{f(\theta)}$ capturing how incentive and participation constraints interact under informational asymmetry. The presence of a type-dependent participation constraint affects this virtual type through the term $\eta_i(\theta) = \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds)$. This integral is a non-decreasing cumulative distribution function which is constant on any interval where (2) is slack. Its derivative, in the sense of distribution theory, is the costate variable for the type dependent participation constraint (2). For a continuous equilibrium, this derivative has no mass point in the interior of the type space.

Intuitively, when an agent with type θ behaves like a less (resp. more) efficient type $\theta + d\theta$ (resp. $\theta - d\theta$), he produces the same amount at a lower cost but he also takes into account how pretending being less (resp. more) efficient changes the payment that P_i offers to induce participation.The case $r_i(\theta) > 0$ corresponds to settings where P_i finds it more attractive to induce participation by the more efficient types. This might be either because P_i likes a lot the agent's output or because such efficient agent does not has a very attractive outside option when dealing with other principals only. Instead, $r_i(\theta) < 0$ when P_i finds it more attractive to induce participation by the least efficient types. Again, this might be either because P_i dislikes the agent's output or because such inefficient agent does not has a very attractive outside option.

At equilibrium, all the distortions induced by the active principals are compounded altogether (equation (12)). The point is that not all distortions may go simultaneously in the same direction and although $r_i(\theta)$ may be positive for some principals it may instead be negative with others. This will be illustrated by means of examples in Section 7 below.

Remark 2 *Observe that $\bar{q}(\theta)$ being strictly decreasing and continuous it admits an inverse function, denoted thereafter by $\bar{\theta}(q)$.*

Activity sets. Although Theorem 4 describes equilibrium outputs, it says little on where activity sets stop and start. The difficulty comes from finding a priori from the fundamental assumptions made on the principals' preferences and on the type distribution who are the active principals on a given set. This is generally a hard task towards which, guided by intuition, we devote some effort in Section 7. For the time being, let us simply investigate some general properties of the inactivity set Ω_i^c of a given principal P_i. Next proposition shows that the costate variable $r_i(\theta)$ is entirely defined on P_i's inactivity set.

Proposition 2 *Suppose that the conditions of Theorem 3 hold and that Ω_i^c has a non-empty interior $\overline{\Omega_i^c}$. For any $\theta \in \overline{\Omega_i^c}$, we have:*

$$S_i'(\bar{q}_{-i}(\theta)) = \frac{r_i(\theta)}{f(\theta)} \tag{13}$$

where $r_i(\theta)$ is defined in Theorem 1.

Condition (13) suggests an algorithm for finding out activity sets. Suppose that we have already computed an equilibrium $\bar{q}_{n-1}(\theta)$ with $n-1$ principals, say $i \in \{1,..,n-1\}$. To find out principal P_n's activity set it is enough to restrict to the subset of the type space where $\frac{d}{d\theta}\left(f(\theta)S_i'(\bar{q}_{n-1}(\theta))\right) - f(\theta) > 0$ since, from the monotonicity of $r_i(\cdot)$, $\frac{d}{d\theta}\left(f(\theta)S_i'(\bar{q}_{n-1}(\theta))\right) - f(\theta) \leq 0$ on any activity set. In a second step, one can use the smooth-pasting conditions to get the shape of P_n's contribution at any interior boundary point of Ω_i. Finally, we can reconstruct from there the rest of the schedule. This algorithm can be particularly efficient with only two principals as illustrated in Section 7.

Contribution schedules. We now derive some properties of the contribution $\bar{t}_i(\cdot)$ that principal P_i offers on any connected interval Ω_i^j ($j \in \mathcal{J}$) of his activity set Ω_i.

Proposition 3 *Suppose that the conditions of Theorem 3 hold.*

1. *The equilibrium schedule $\bar{t}_i(\cdot)$ is strictly differentiable at any equilibrium point $\bar{q}(\theta)$.*

2. *The following "smooth-pasting" conditions are satisfied by the best-response contribution offered by principal P_i at any boundary point θ of his inactivity set Ω_i^c:*

$$\bar{t}_i(\bar{q}(\theta)) = \bar{t}_i'(\bar{q}(\theta)) = 0. \tag{14}$$

3. *The equilibrium marginal contribution for principal P_i satisfies on each connected subset of activity Ω_i^j:*

$$\bar{t}_i'(q) = S_i'(q) - \frac{F(\bar{\theta}(q)) - M_i^j}{f(\bar{\theta}(q))} \tag{15}$$

   where $M_i^j = \int_{\underline{\theta}}^{\theta_i^j} \gamma_i(\theta)\mu_i(d\theta) \in [0,1]$ is constant on each connected subset of activity Ω_i^j. Moreover M_i^j increases with j.

Under complete information, Bernheim and Whinston (1986) showed that the so-called "truthful" schedules ($t_i(q) = \max\{0, S_i(q) - C_i\}$ for some C_i) sustain efficient outcomes as equilibria. Those schedules are not smooth at q_0 such that $t_i(q_0) = 0$ even though they are locally convex around that point like the equilibrium schedules under asymmetric information. With asymmetric information, the equilibrium schedule is much more constrained around any equilibrium point. Such schedule has now to go through equilibrium points corresponding to nearby types. This "extra information" implies smooth-pasting.

Failure of continuity. Failure of having Condition 1 hold might generate output discontinuities. Consider the following simple model of the provision of a discrete public good. Suppose for instance that $n = 2$ with $S_0(q) = 0$, $S_i(q) = s_i q$ with $s_i > 0$ for $i = 1, 2$ and that θ is distributed on $\Theta = [0, 1]$. The quantity of public good is now interpreted as a probability (i.e. $\mathcal{Q} = [0,1]$). We assume also that $s_1 + s_2 < 1$ so that even under complete information on cost, producing the public good is inefficient for the worst technologies.

Principals and their agent have linear surplus and cost functions. *De facto* the equilibrium has a "bang-bang" structure: the agent produces the public good for sure if and only if his cost is below some threshold.[27] Contributions are nevertheless continuous.

Proposition 4 *When Assumption 1 holds, there exists an equilibrium of the delegated common agency game with the following features.*

1. *Equilibrium schedules are continuous and given by:*

$$\bar{t}_i(q) = \max\left\{0, s_i - \frac{F(\theta^*)}{f(\theta^*)}\right\} q, \quad i = 1, 2 \tag{16}$$

where θ^* is the unique solution in $[0,1]$ to the equation

$$\sum_{i=1}^{2} \max\left\{0, s_i - \frac{F(\theta^*)}{f(\theta^*)}\right\} = \theta^*. \tag{17}$$

2. *The probability of producing the public good is:*

$$\bar{q}(\theta) = \begin{cases} 1 & \text{if } \theta \in [0, \theta^*] \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

7 Applications: Monotonic Equilibria

We now show how simple economic settings give rise to different patterns of equilibria and activity sets. Beyond, those examples also illustrate a few basic principles of delegated common agency models. Principles 1 and 2 below are indeed of interest in themselves as guides for readers interested in applying our methodology to other specific contexts.

7.1 Generalities

To classify the different patterns of activity sets that may arise in an equilibrium, it is useful to restrict attention to settings in which each principal's preference ordering over $\mathcal{Q}$ is monotonic: $S_i(\cdot)$ is either nondecreasing or nonincreasing. This allows us to categorize principals by the direction of their preferences. Between any pair of principals, $\{i, j\}$, whose orderings are either both nondecreasing or both nonincreasing, we say preferences

[27] A similar model with a bang-bang structure has been studied by Lebreton and Salanié (2003) in a lobbying context although the equilibrium is implemented with discontinuous payments as well: lobbyists (principals) contributing only if the decision-maker (their agent) is choosing the policy they prefer.

are *congruent.* If the preferences between any pair of principals are not congruent, we will denote them as *conflicting.*

With the assumption that principal preferences are monotonic, a natural focus is on equilibria in which each principal offers a monotone contract ordered in the same direction as the principal's preferences over $\mathcal{Q}$.

Definition 5 *An equilibrium is monotonic if each for each i, $t_i(\cdot)$ is nondecreasing (resp., nonincreasing) if $S_i(\cdot)$ is nondecreasing (resp., nonincreasing).*

It follows that in any monotone equilibrium, either $\Omega_i = [\underline{\theta}, \theta_i)$ or $\Omega_i = (\theta_i, \bar{\theta}]$ for all $i \in \mathcal{N}$. If Principals P_i and P_j have conflicting objectives, then $\Omega_i = (\theta_i, \bar{\theta}]$ and $\Omega_j = [\underline{\theta}, \theta_j)$;[28] if their preferences are congruent, then $\Omega_i = [\underline{\theta}, \theta_i)$ and $\Omega_j = [\underline{\theta}, \theta_j)$.[29]
Linear surplus functions. If one is ready to impose more structure on the principals' preferences, the shape of activity sets, contribution schedules and equilibrium output can be easily drawn from condition (13).

Proposition 5 *Suppose that principals have linear preferences, i.e., $S_i(q) = s_i q$ for some s_i. There exists a continuous equilibrium which is monotonic with the following features.*

1. *Principal P_i's activity set is $\Omega_i = [\underline{\theta}, \theta_i)$ (resp. $\Omega_i = (\theta_i, \bar{\theta}_i]$) where $s_i = R(\theta_i)$ (resp. $s_i = -T(\theta_i)$) and $r_i(\theta) = F(\theta)$ (resp. $r_i(\theta) = F(\theta) - 1$) if $s_i > 0$ (resp. $s_i < 0$).*

2. *The equilibrium output is continuous, montonically decreasing (with inverse $\bar{\theta}(q)$) and satisfies the modified Lindahl-Samuelson condition:*

$$S_0'(\bar{q}(\theta)) + \sum_{s_i \geq 0} \max\{0, s_i - R(\theta)\} + \sum_{s_i < 0} \min\{0, s_i + T(\theta)\} = \theta. \tag{19}$$

3. *Principal P_i's contribution schedule is:*

$$\bar{t}_i(q) = \begin{cases} \int_0^q \max\{0, s_i - R(\bar{\theta}(x))\}dx & \text{if } s_i \geq 0 \\ \int_0^q \min\{0, s_i + T(\bar{\theta}(x))\}dx & \text{if } s_i < 0. \end{cases} \tag{20}$$

Condition (20) shows that principals who enjoy the public good contribute at the margin a positive (or null) amount. However, this marginal contribution is less than their own marginal valuation with a discount being the hazard rate $R(\theta)$. Principals who dislike the public good contribute at the margin a negative (or null) amount but less (in absolute values) than their marginal valuations with a discount being now a hazard rate $T(\theta)$.

[28] This is without loss of generality given that we may permute subscripts for principals.

[29] Alternatively, principals may also be congruent when $\Omega_i = (\theta_i, \bar{\theta}]$ and $\Omega_j = (\theta_j, \bar{\theta}]$, i.e., both principals find it more attractive to contract with the least efficient types.

Summing those marginal contributions yields the modified Lindahl Samuelson condition in this common agency context.

Finally, Proposition 5 provides a simple proof of the existence of equilibria in public delegated common agency games which applies to large set of circumstances of economic interest as Sections 7.2 and 7.3 will illustrate.

Propositions 6 and 7 below illustrate further how the forces of competing principals can be combined to determine the overall equilibrium output distortion either with conflicting or congruent principals.

7.2 Lobbying for a Public Policy

Let us come back to the lobbying setting described in Section 5. Continuous equilibria exist in that framework as shown now.

Proposition 6 *Assume that $\delta < 1$. There exists an equilibrium of the lobbying game with the following features.*

1. *The equilibrium policy $\bar{q}(\theta)$ is continuous, decreasing in θ and satisfies*

$$\begin{cases} \bar{q}(\theta) = \begin{cases} q_1^*(\theta) = 1-\delta-2\theta & \text{for } \theta \in [-\delta, -1+\delta) \\ -3\theta & \text{for } \theta \in [-1+\delta, 1-\delta] \\ q_2^*(\theta) = -1+\delta-2\theta & \text{for } \theta \in (1-\delta, \delta] \end{cases} & \text{if } \delta \geq \frac{1}{2} \\ \bar{q}(\theta) = -3\theta & \text{if } \delta \leq \frac{1}{2} \end{cases}$$

with an inverse function defined as

$$\begin{cases} \bar{\theta}(q) = \begin{cases} \frac{1}{2}(1-\delta-q) & \text{if } 3(1-\delta) \leq q \\ -\frac{q}{3} & \text{if } 3(1-\delta) \geq q \geq 3(\delta-1) \\ \frac{1}{2}(-1+\delta-q) & \text{if } q \leq 3(\delta-1) \end{cases} & \text{if } \delta \geq \frac{1}{2} \\ \bar{\theta}(q) = -\frac{q}{3} & \text{if } \delta \leq \frac{1}{2}. \end{cases}$$

2. *Activity sets are $\Omega_1 = [-\delta, 1-\delta)$ and $\Omega_2 = (-1+\delta, \delta]$ when $\delta \geq \frac{1}{2}$ or there is full coverage by both principals when $\delta \leq \frac{1}{2}$.*

3. *The principals' contributions are given by*

$$\bar{t}_1(q) = \begin{cases} \int_{-3(1-\delta)}^{q} \max\{0, (1-\delta-\bar{\theta}(x))\}dx \text{ and } \bar{t}_2(q) = \bar{t}_1(-q) & \text{if } \delta \geq \frac{1}{2} \\ (1-\delta)q + \frac{q^2}{6} + \frac{3}{4} - \frac{3}{2}\delta^2; \quad \bar{t}_2(q) = \bar{t}_1(-q) & \text{if } \delta \leq \frac{1}{2}. \end{cases} \tag{21}$$

When there is enough uncertainty on the agent's type, i.e., $\delta \geq \frac{1}{2}$, each principal can secure himself an area of influence where he deals exclusively with the agent. This area is a neighborhood of the "most-preferred" type of the agent from that principal's viewpoint.

For instance, that area for principal P_1 contains type $-\delta$ which is the most eager to push policies up. Instead, in the middle of the type space, both principal do contribute and maintain overlapping areas of influence. Principals stop contributing for types which are too "far away" on the other side of the type space.

Figure 2 describes the equilibrium policy in that case.

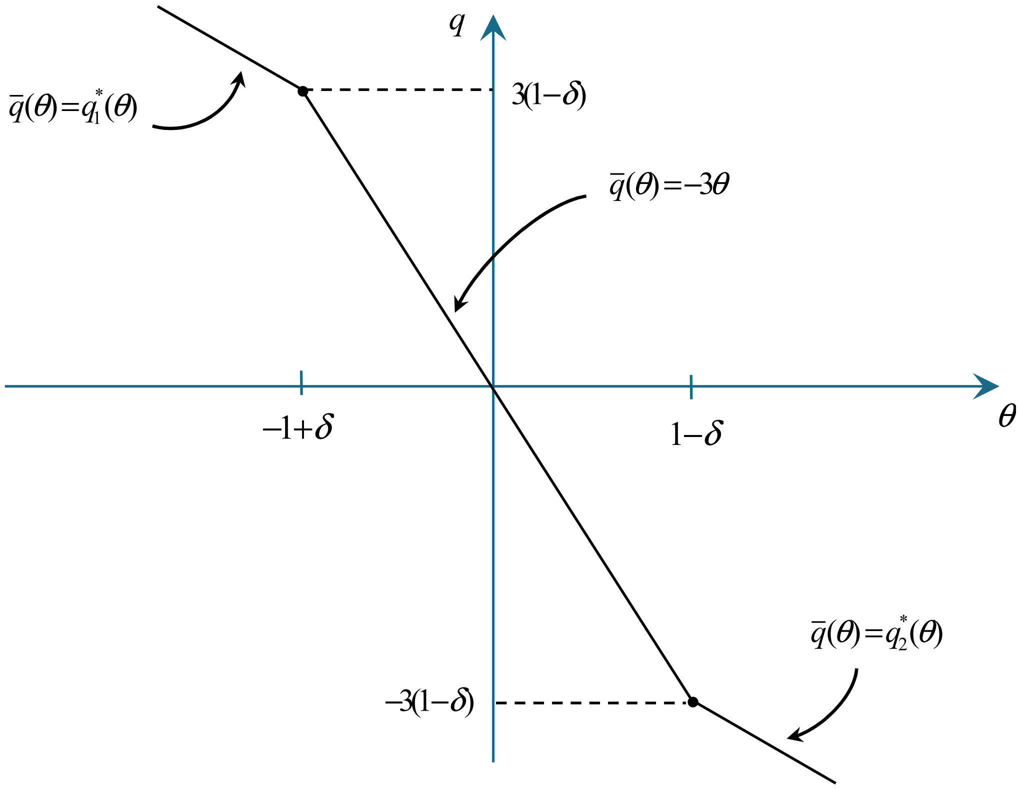

Figure 2: Policy in the continuous equilibrium of the lobbying game $\delta \geq \frac{1}{2}$.

Of course, as there is less uncertainty on the agent's type, i.e., $\delta \leq \frac{1}{2}$, the area where principals have overlapping influence covers now the whole type space. The equilibrium policy reflects then the preferences of both principals which each offering a positive contribution everywhere even though they want to push policies in opposite directions.[30,31]

[30]For more general results along those lines and further classifications of the patterns of contributions in a lobbying game, we refer to Martimort and Semenov (2008). That latter paper did not derive the continuity of equilibrium output as we do here and focused on different objective functions.

[31]As δ decreases to zero, the continuous equilibrium above converges towards an allocation where the agent located at 0 chooses $\bar{q}(0) = 0$, getting payoff $\bar{U}(0) = \frac{3}{2}$. Contributions converge towards $\bar{t}_1(q) = q + \frac{q^2}{6} + \frac{3}{4}$ and $\bar{t}_2(q) = \bar{t}_1(-q)$. Those contributions form an equilibrium of the complete information game. Clearly, those contributions are not "truthful" in the sense of Bernheim and Whinston (1986) although they implement the same allocation as the "truthful" ones that would be defined as $t_1^{tr}(q) = q$ and $t_2^{tr}(q) = t_1^{tr}(-q)$. See Martimort and Stole (2009b) for more results on the limits of equilibria of asymmetric information games as uncertainty converges towards zero.

Again this lobbying model shows a double failure of the Coase Theorem under asymmetric information. First, policy is inefficient even if all interest groups contribute. Second, some interest groups may fail to be represented.

Principle 1 *In a lobbying game under asymmetric information, common agency induces inefficient policy choices and inefficient representation of active interest groups with some groups possibly securing unchallenged influence for some subset of the type space.*

7.3 Voluntary Provision of a Public Good

Suppose that principals have valuations for a public good given by $S_i(q) = s_i q$ for all $q \in \mathcal{Q} = [0, \bar{Q}]$ with $s_i > 0$ $(i = 1, 2)$. We assume that $S_0(q) = -\frac{q^2}{2}$ so that $q_0(\theta) = U_0(\theta) \equiv 0$ and that θ is uniformly distributed on $\Theta = [0, \bar{\theta}]$. Finally, principals are ordered in terms of their marginal valuations for the public good and we refer to P_2 (resp. P_1) as the *strong principal* (resp. *weak principal*), with the extra technical assumptions $s_2 > 2s_1$ and $s_2 < 2\bar{\theta}$ that reduce the number of cases under scrutiny.

Had both principals cooperated in designing their contributions, the *optimal cooperative output* under asymmetric information $q^C(\theta)$ would be given by:[32]

$$q^C(\theta) = \max\left\{0, \left(\sum_{i=1}^{2} s_i\right) - 2\theta\right\}. \tag{22}$$

This decision corresponds to the modified Samuelson condition where the optimal level of public good (when positive) is such that the sum of marginal valuations $s_1 + s_2$ is just equal to the agent's virtual cost $\theta + \frac{F(\theta)}{f(\theta)} = 2\theta$.

Let define the *stand-alone output* that principal P_i would implement when contracting alone with the privately informed agent as:

$$q_i^*(\theta) = \max\{0, s_i - 2\theta\}. \tag{23}$$

Both $q^C(\theta)$ and $q_i^*(\theta)$ $(i = 1, 2)$ are non-increasing and $q_1^*(\theta) \le q_2^*(\theta) \quad \forall \theta \in \Theta$.

Denote also by $q^I(\theta)$ the non-increasing output schedule such that:

$$q^I(\theta) = \max\left\{0, \sum_{i=1}^{2} s_i - 3\theta\right\}. \tag{24}$$

This output schedule is the solution to the *intrinsic* common agency game where the agent has only the choice of accepting both offers or none, i.e., the participation constraint which is the same for both principals is $U(\theta) \ge 0$ for all $\theta \in \Theta$.[33] In particular, this output schedule entails a double distortion familiar from the intrinsic common agency literature

[32]The proof is standard and thus omitted.
[33]Martimort and Stole (2009a).

and we get $q^I(\theta) \leq q^C(\theta)$ with equality only at $\theta = 0$.[34] This hypothetical intrinsic setting allows to focus on one kind of distortions only, those that arise through non-cooperative contracting *taking as given* the set of active principals. The next proposition gives instead more attention to the inefficiency that comes from insufficient participation of principals. Those inefficiencies are specific to the delegated contracting setting.

Proposition 7 *There exists an equilibrium of the delegated common agency game with the following features.*

1. *The equilibrium output $\bar{q}(\theta)$ is continuous, decreasing in θ and satisfies*

$$\bar{q}(\theta) = \begin{cases} q_I(\theta) & \text{if } \theta \in [0, s_1] \\ q_2^*(\theta) & \text{otherwise} \end{cases}$$

   with an inverse function defined as

$$\bar{\theta}(q) = \begin{cases} \frac{1}{3}(\sum_{i=1}^{2} s_i - q) & \text{if } s_2 - 2s_1 \leq q \\ \max\left\{0, \frac{1}{2}(s_2 - q)\right\} & \text{otherwise.} \end{cases}$$

2. *Activity sets are $\Omega_1 = [\underline{\theta}, s_1)$ and $\Omega_2 = \left[\underline{\theta}, \frac{s_2}{2}\right)$ with $\Omega_1 \subset \Omega_2 \subset \Theta$.*

3. *Equilibrium contributions are piecewise quadratic, continuously differentiable with*

$$\bar{t}_1(q) = \begin{cases} \int_{s_2-2s_1}^{q} (s_1 - \bar{\theta}(x))dx & \text{if } s_2 - 2s_1 \leq q \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \bar{t}_2(q) = \int_0^q (s_2 - \bar{\theta}(x))dx. \quad (25)$$

Several remarks are in order. First, the equilibrium output in the delegated common agency game is always greater than in the intrinsic game. This captures the fact that the non-zero participation constraints under delegated agency force principals to give more rent than under intrinsic agency, raising thereby equilibrium output.

Second, the weak principal does not reward the agent for levels of output which are small enough. Indeed, only the strong principal does so. Not all principals contribute under asymmetric information, only those who are able to pay the corresponding agency cost do so. As a result, the equilibrium output reflects the existing set of active principals. More precisely, this equilibrium is obtained by piecing together the contribution that the strong principal would offer alone for the least efficient types with an intrinsic equilibrium allocation that would arise when both principals find it worth to contribute, i.e., for the most efficient types. Smooth-pasting at the threshold type s_1 allows to recover the analytical expressions of contributions beyond that point.

[34] Note of course that $q^I(0) > q_2^*(0) > q_1^*(0)$. The grand-coalition made of both principals and the agent produces always more than any simple bilateral coalition between one of those principals and the agent when the latter produces at zero cost.

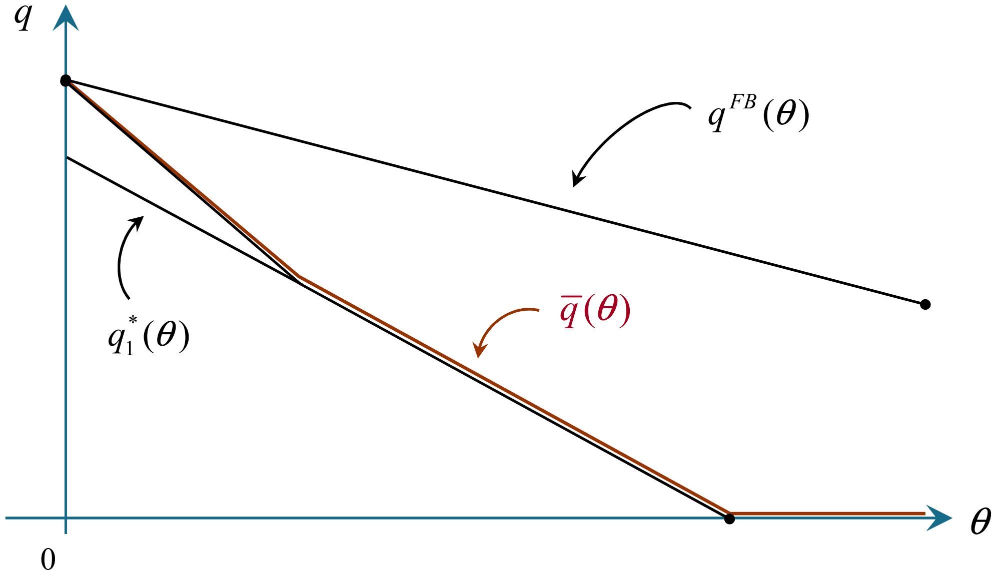

Figure 3: Output in the voluntary provisions of public good game.

Third, *free-riding* comes at two levels here. On the one hand, for the most efficient types, (i.e., if the marginal cost is small enough), both principals contribute but the equilibrium output is lower than if they were cooperating ($q^I(\theta) \leq q^C(\theta)$ at all θ). This is so because each principal reduces his marginal contribution below his marginal valuation for the public good without internalizing the fact that the other principal does so as well. On the other hand, under asymmetric information, the set of active contributors may be a strict subset of what it would be under complete information. This arises for the least efficient types for which the weak principal finds it optimal to stop contributing.

This double failure of the Coase Theorem under asymmetric information is a crucial finding that we state in a less formal way below:

Principle 2 *In a game of voluntary contributions to a public good under asymmetric information, decentralized bilateral contracting under the common agency institution induces inefficiently low output and inefficient representation of active principals compared with cooperative contracting.*

Remark 3 *The equilibrium output is not invariant with respect to redistributions of the principals' surplus such that the overall marginal surplus $\sum_{i=1}^{2} s_i$ is kept fixed. Such redistribution may indeed affect the set of active principals. This result is of course reminiscent of the literature on voluntary contributions (Bergstrom, Blume and Varian 1986, among others) although this literature obtains that result by making restriction in the set of instruments (fixed contributions instead of schedules) and assumes complete information.*

8 Conclusion

This paper has developed a methodology for solving public delegated common agency games under asymmetric information. In a nutshell, the basic lessons of this research is that the celebrated rent-efficiency trade-off of the standard agency literature must be significantly modified to account for the impact of competition among principals. First, compounded output distortions that arise at equilibrium reflect the intrinsic nature of the principals' preferences and, in particular, whether these preferences are conflicting or congruent. Moreover, those distortions can be summarized by a simple but modified Lindhal-Samuelson condition. Second, the principals' marginal contributions no longer reflect their marginal utility for the agent's decision as under complete information but must be modified to take into account agency costs. Third, a rich pattern of activity sets that reflects the pattern of principals' preferences may emerge at equilibrium with a rich set of implications in specific contexts.

On top of these results, this paper has also developed a methodological contribution that is needed to solve the principal-agent models with type-dependent participation constraints which are innocuous to those common agency environments. This contribution is of general interest and could certainly be applied in other contexts.

Beyond the application of this methodology to some specific settings of economic interest (trade, regulation, multi-unit auctions, common representation on retailer markets, etc...) that have already been deeply studied with the common agency methodology under complete information and would benefit from a serious consideration of agency problems, our paper unveils a few alleys for further theoretical works.

First, the techniques of "smooth-pasting" that we stressed in our study of continuous equilibria are likely to extend to private common agency games as well. Such extensions are particularly needed to analyze contexts where principals are congruent but asymmetric or even have conflicting preferences. This would be a valuable step beyond the symmetric-congruent model of competing nonlinear pricing studied in Martimort and Stole (2009c). From an applied perspective, this is particularly important since indeed students of market competition have generally found hard to reconcile existing models of competition in nonlinear prices which predict either exclusive contracting or restrict the analysis to the case where a buyer jointly buys from different sellers[35] with existing patterns of behavior on actual markets which are mostly characterized by overlapping activity sets.[36]

Second, some more applied settings may require to develop a framework where principals share some common screening devices but keep others private. For instance, one may think of specific games between competing manufacturers dealing with the same retailers and contracting on some commonly observed price downstream but keeping their sales of

[35]See for instance Stole (1995), Rochet and Stole (2002), Ivaldi and Martimort (1994) and the survey by Stole (2007).

[36]This remark benefitted from useful discussions with Eugenio Miravete.

intermediary goods secret. These settings lie somewhere in between the case of public delegated agency games and the case of private agency games. Extending our methodology to such environments may be important.

We plan to investigate such extensions in future research.

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Appendix

Proof of Lemma 1. The proof is standard and is thus omitted.[37] ■

Proof of Theorem 1. We apply the necessary conditions for optimality of a minimizing process $(\bar{U}, \bar{v})$ for the state-constrained Maximum Principle (Theorem 10.2.1.

[37] See for instance Champsaur and Rochet (1989) and Milgrom and Segal (2004) among others.

in Vinter (2000) or/and Theorem 3 in Vinter and Zheng (1998) when specialized to a quasi-linear generalized Lagrangean like ours) when $V_i < +\infty$. Checking that latter condition is immediate. Observe indeed that the null contribution $t_i(q) \equiv 0$ yields a payoff $\int_\Theta S_i(\bar{q}_{-i}(\theta))dF(\theta) = -\int_\Theta L_i(\theta, \bar{U}_{-i}(\theta), -\bar{q}_{-i}(\theta)) > -\infty$ to principal P_i.

Assumptions. Theorem 10.2.1. in Vinter (2000) applies when conditions [a.] to [c.] below are satisfied. We check those conditions.

[a.] *Measurability and lower semi-continuity in* (θ, v) *of* $L_i^1(\cdot,\cdot)$. lower semi-continuity in (θ, v) of $L_i^1(\cdot,\cdot)$ follows from the fact that $\bar{t}_{-i}$ is upper semi-continuous. Still by upper semi-continuity the sets $I_\alpha = \{v \in \mathcal{V} | \quad \bar{t}_{-i}(-v) < \alpha\}$ are open and thus measurable with respect to the $\sigma-$field $\mathcal{B}$ of Borel subsets of $\mathcal{V}$.[38]

[b.] *Lipschitz continuity in* u *of* $L_i(\theta, \cdot, v)$ *and boundedness of* $L_i(\theta, \bar{U}, v)$. There exists $k(\theta)$ such that:

$$|L_i(\theta, u', v) - L_i(\theta, u, v)| \le k(\theta)|u' - u|, \tag{A1}$$

$$L_i(\theta, \bar{U}(\theta), v) \ge -k(\theta). \tag{A2}$$

Observe that we have $|L_i(\theta, u', v) - L_i(\theta, u, v)| = |f(\theta)(u' - u)| \le (\max_{\theta\in\Theta} f(\theta))|u' - u|$ so that $L_i(\theta, \cdot, v)$ is Lipshitz continuous for any (θ, v) and (A1) holds for $k(\theta) \ge (\max_{\theta\in\Theta} f(\theta))$.

The requirement that $L_i(\theta, \bar{U}(\theta), v) \ge -k(\theta)$ specializes in our context to $k(\theta) \ge -f(\theta)\bar{U}(\theta) - L_i^1(\theta, v)$ for all $v \in \mathcal{V}$ and a.e. $\theta \in \Theta$. Because $f(\theta)$ is bounded and $L_i^1(\theta, \cdot)$ is bounded below, the right-hand side is bounded for every $\theta \in \Theta$ and an acceptable $k \in L^1$ exists that satisfies both (A1) and (A2). Take for instance

$$k(\theta) = \max\left\{\max_{\theta\in\Theta} f(\theta); \left(\max_{\theta\in\Theta} -f(\theta)\bar{U}(\theta) - \min_{v\in\mathcal{V}} L_i^1(\theta, v)\right)\right\}.$$

[c.] *Lipschitz condition on participation:* The function $u \to h_i(\theta, u) = \bar{U}_{-i}(\theta) - u$ is Lipschitz continuous in u and lower semi-continuous in (θ, u). This is obviously the case since $\bar{U}_{-i}(\theta)$ is absolutely continuous (see comments following Lemma 1).

Necessary conditions. Theorem 10.2.1 in Vinter (2000) shows then that there exist an arc p_i which is absolutely continuous on Θ, a non-negative real number $\lambda_i \ge 0$, a non-negative measure μ_i and a μ_i-integrable function γ_i on Θ such that the following conditions hold:[39]

$$\lambda_i + \|p_i\|_{L^\infty} + \int_\Theta \mu_i(d\theta) > 0. \tag{A3}$$

[38]Theorem 12, p. 61, in Royden (1988).

[39]Let us briefly add some useful notations. For $f: \quad \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$, the epigraph of f is the set $epif \equiv \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | \quad \alpha \ge f(x)\}$. For a given set $C \subseteq \mathbb{R}^n$, a vector v is normal to C at $\bar{x}$ in the regular sense if and only if $\langle v, x - \bar{x}\rangle \le o(|x - \bar{x}|) \quad \forall x \in C$. Let denote $\hat{N}_C(\bar{x})$ that normal regular cone and let $N_C(\bar{x}) = \limsup_{x \to_C \bar{x}} \hat{N}_C(x)$ be the normal cone at $\bar{x}$ obtained by taking limits. The following definitions of normal subgradients, subgradients and limiting subgradients are also used in the sequel $\hat{\partial} f(\bar{x}) = \{v | (v, -1) \in \hat{N}_{epif}(\bar{x}, f(\bar{x}))\}$, $\partial f(\bar{x}) = \{v | (v, -1) \in N_{epif}(\bar{x}, f(\bar{x}))\}$, $\partial^\infty f(\bar{x}) = \{v | (v, 0) \in N_{epif}(\bar{x}, f(\bar{x}))\}$, $\partial_x^> h(\theta, \bar{x}) = co\{\xi : \exists(\theta_i, x_i) \to (\theta, \bar{x}) \text{ s.t. } h(\theta_i, x_i) > 0 \, \forall i, \, \nabla_x h(\theta_i, x_i) \to \xi\}$.

$$\dot{p}_i(\theta) \in co\left\{\eta | \left(\eta, p_i(\theta) + \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds), -\lambda_i\right) \in N_{\mathrm{epi}(L(t,\cdot,\cdot))}\left(\bar{U}, \bar{v}, L(\theta, \bar{U}, \bar{v})\right)\right\} \text{ a.e.} \tag{A4}$$

$$p_i(\underline{\theta}) = p_i(\bar{\theta}) - \int_{\Theta} \gamma_i(s)\mu_i(ds) = 0. \tag{A5}$$

$$\left(p_i(\theta) - \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds)\right)\bar{v}(\theta) - \lambda_i L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))$$

$$= \max_{v\in\mathbb{R}} \left(p_i(\theta) - \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds)\right) v - \lambda_i L_i(\theta, \bar{U}(\theta), v). \tag{A6}$$

$$\gamma_i(\theta) = -\partial^{>} h_i(\theta, \bar{U}(\theta) = \begin{cases} 0 & \text{if } h_i(\theta, \bar{U}(\theta)) < 0 \\ 1 & \text{if } h_i(\theta, \bar{U}(\theta)) = 0. \end{cases} \tag{A7}$$

Transformation of those necessary conditions. Several remarks help us to rewrite those necessary conditions and recover Items [1.] to [3.] in Theorem 1.

- Let $\partial L_1(\theta, \dot{U})$ be the limiting subdifferential and let $\partial^\infty L_1(\theta, \dot{U})$ be the asymptotic limiting subdifferential. Following Rockafellar and Wets (Proposition 10.5), the limiting normal cone of the epigraph of $L(t, \cdot, \cdot)$ can be expressed as

  $$N_{\mathrm{epi}(L(\theta,\cdot,\cdot))}\left(U, \dot{U}, L(\theta, U, \dot{U})\right) =$$
  $$\{(\xi f(\theta), \xi\tau, -\xi) \in \mathbb{R}^3 \mid \tau \in \partial L_1(\theta, \dot{U}(\theta)), \xi \geq 0\} \bigcup \{(0, \tau, 0) \in \mathbb{R}^3 \mid \tau \in \partial^\infty L_1(\theta, \dot{U}(\theta))\}.$$

  Suppose that $\lambda_i > 0$, then

  $$\text{co}\left\{\xi \mid \left(\xi, p(\theta) - \int_{[\underline{\theta},\theta)} \gamma(s)\mu(ds), -\lambda\right) \in N_{\mathrm{epi}(L(\theta,\cdot,\cdot))}\left(\bar{U}, \bar{v}, L(\theta, \bar{U}, \bar{v})\right)\right\} = \{\lambda_i f(\theta)\}.$$

  If instead $\lambda = 0$, then

  $$\text{co}\left\{\xi \mid \left(\xi, p(\theta) - \int_{[\underline{\theta},\theta)} \gamma(s)\mu(ds), 0\right) \in N_{\mathrm{epi}(L(\theta,\cdot,\cdot))}\left(\bar{U}, \bar{v}, L(\theta, \bar{U}, \bar{v})\right)\right\} = \{0\}.$$

  Together, we may conclude that $\dot{p}(\theta) \in \{\lambda_i f(\theta)\}$ almost everywhere or, more simply,

  $$\dot{p}_i(\theta) = \lambda_i f(\theta) \text{ a.e.} \tag{A8}$$

  Let us state the following definition:

  Definition 6 *The minimizing process $(\bar{U}, \bar{v})$ is a normal extremal when conditions (A3) to (A7) above are obtained for $\lambda_i > 0$.*

We are now ready to establish:

Lemma 2 *A minimizing process* $(\bar{U}, \bar{v})$ *is a normal extremal.*

Proof: From (A8) and (A5), we get:

$$p_i(\theta) = \lambda_i F(\theta). \tag{A9}$$

Then define

$$r_i(\theta^-) = \lambda_i F(\theta) - \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds). \tag{A10}$$

Suppose that $\lambda_i = 0$, then $p_i(\theta) = 0$ (thus $\|p_i\|_{L^\infty} = 0$) and $r_i(\theta^-) = -\int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds)$. Using (A5) yields

$$\int_\Theta \gamma_i(s)\mu_i(ds) = 0 \tag{A11}$$

where $\gamma_i(s) = 1$ on *supp* μ_i. Since μ_i is a measure, (A11) implies that this is a zero measure (and thus $\int_\Theta \mu_i(ds) = 0$). Inserting the corresponding values of λ_i, $\|p_i\|_{L^\infty}$ and $\int_\Theta \mu_i(ds) = 0$ yields a contradiction with (A3). ■

From Lemma 2, we may as well use the normalization that let $r_i(\theta^-)$ be defined as in (6) in which case (A9) becomes

$$p_i(\theta) = F(\theta). \tag{A12}$$

- Taking into account the expression of the extended-value Lagrangean $L_i(\cdot)$, the right-hand side of (A6) can be rewritten as:

$$\max_{v\in\mathbb{R}} \left(p_i(\theta) - \int_{[\underline{\theta},\theta)} \gamma_i(s)\mu_i(ds)\right) v - \lambda_i L_i^1(\theta, v) - L^0(\theta, \bar{U}(\theta))$$
$$= \max_{v\in\mathcal{V}} r_i(\theta)v - L_i^1(\theta, v) - L^0(\theta, \bar{U}(\theta)). \tag{A13}$$

  Simplifying yields Item [2.] when taking the normalization $\lambda_i = 1$.

- From condition (A7), we deduce also that *supp* $\mu_i \subseteq \Omega_i^c$. Moreover, this condition implies the complementary slackness condition in Item [3.].

This ends the proof of the Theorem.

Further notations and properties. For further references used in the proofs below, define first the Hamiltonian as:

$$\hat{H}_i(\theta, u, v, r) \equiv rv - L_i(\theta, u, v).$$

Define also the optimized or "pure" Hamiltonian $H_i(\theta,u,\cdot):\mathbb{R}\to\mathbb{R}\bigcup\{+\infty\}$, or Legendre-Fenchel dual transform of $L_i(\theta,u,\cdot)$, as:

$$H_i(\theta,u,r)\equiv\sup_{v\in\mathbb{R}}\hat{H}_i(\theta,u,v,r)=\max_{v\in\mathcal{V}}\{rv-L_i^1(\theta,v)\}-L^0(u). \tag{A14}$$

That the maximum above is achieved follows from the compactness of $\mathcal{V}$ and the lower semi-continuity of $L_i^1(\theta,\cdot)$. Note also that $H_i(\theta,u,r)$ is convex in r and concave (actually linear) in u, closed and proper. Note that, by the Legendre-Fenchel Transform Theorem,[40] the biconjugate $co\left\{L_i^1\right\}(\theta,v)$ of $L_i^1(\theta,v)$ is convex, closed and proper and $epi\left\{co\left\{L_i^1\right\}\right\}(\theta,\cdot)=co\{epiL_i^1(\theta,\cdot)\}$. Hence, we get:

$$co\left\{L_i^1\right\}(\theta,v)+L^0(\theta,u)\equiv\max_{r\in\mathbb{R}}\{rv-H_i(\theta,u,r)\}$$

and

$$H_i(\theta,u,r)\equiv\max_{v\in\mathbb{R}}\{rv-co\left\{L_i^1\right\}(\theta,r)-L^0(\theta,u)\}\equiv\max_{v\in\mathbb{R}}\{rv-L_i^1(\theta,v)-L^0(\theta,u)\}$$

where the first and second equality follows from dualization and the last one is just the definition of $H_i(\theta,u,r)$.

Item [2.] in Theorem 1 implies by the rules governing subdifferentials of convex functions first condition (7) and

$$\dot{\bar{U}}(\theta)=\bar{v}(\theta)=-\bar{q}(\theta)\in\partial_r H_i(\theta,\bar{U}(\theta),r_i(\theta^-)). \tag{A15}$$

■

Proof of Proposition 1. Fix $\theta\in\Theta$ and observe that $\mu_i(\{\theta\})=0$ if $h_i(\theta,\bar{U}(\theta))=\bar{U}_{-i}(\theta)-\bar{U}(\theta)<0$. Suppose on the other hand that $h_i(\theta,\bar{U}(\theta))=0$. Then, we get:

$$\frac{1}{\epsilon}(h_i(\theta+\epsilon,\bar{U}(\theta+\epsilon))-h_i(\theta,\bar{U}(\theta)))\le 0\le\frac{1}{\epsilon}(h_i(\theta,\bar{U}(\theta))-h_i(\theta-\epsilon,\bar{U}(\theta-\epsilon)).$$

Passing to the limit as $\epsilon\downarrow 0$ and taking into account the right- and left-hand side differentiability of $\bar{U}_{-i}(\cdot)$ at any point θ yields:

$$\dot{\bar{U}}_{-i}(\theta^+)-\bar{v}(\theta^+)\le 0\le\dot{\bar{U}}_{-i}(\theta^-)-\bar{v}(\theta^-)\text{ or }\bar{v}(\theta^-)\le\dot{\bar{U}}_{-i}(\theta^-)\le\dot{\bar{U}}_{-i}(\theta^+)\le\bar{v}(\theta^+) \tag{A16}$$

where the inequality $\dot{\bar{U}}_{-i}(\theta^-)\le\dot{\bar{U}}_{-i}(\theta^+)$ follows from the convexity of the implementable profile $\bar{U}_{-i}(\cdot)$.

From (A15) and (A16), we get $\bar{v}(\theta^-)\in\partial_p H_i(\theta,\bar{U}(\theta),r_i(\theta^-))\le\bar{v}(\theta^+)\in\partial_p H_i(\theta,\bar{U}(\theta),r_i(\theta^+))$. Because, $H_i(\theta,\bar{U}(\theta),\cdot)$ is convex in r, we get $r_i(\theta^-)\le r_i(\theta^+)$. Using (6), we obtain:

$$r_i(\theta^-)-r_i(\theta^+)=\gamma_i(\theta)\mu_i(\{\theta\})\le 0. \tag{A17}$$

[40]See Rockafellar and Wets (2004, p.474).

Given that $\gamma_i(\theta) = 0$ for $\theta \in \Omega_i$, it follows that $r_i(\cdot)$ is continuous at any such $\theta \in \Omega_i$.

Consider now $\theta \in \Omega_i^c$. We know then that $\gamma_i(\theta) = 1$. Henceforth, (A17) implies $\mu_i(\{\theta\}) = 0$, i.e., μ_i has no atom $(\underline{\theta}, \bar{\theta})$, and $r_i(\cdot)$ is continuous. ■

Proof of Theorem 2. We adapt the argument of Arrow's sufficiency theorem using the basic approach of Seierestad and Sydsaeter (1987) but relaxing their continuity and smoothness assumptions. Let U be any admissible arc, i.e., absolutely continuous on Θ and such that (2) and (5) hold. Define

$$\Delta_i = \int_\Theta \Big(L_i(\theta, U(\theta), \dot{U}(t)) - L_i(\theta, \bar{U}(\theta), \dot{\bar{U}}(\theta)) \Big)\, d\theta.$$

We will demonstrate that $\Delta_i \geq 0$.

Take $r_i(\theta^-)$ as defined in (6). It follows that

$$\Delta_i = \int_\Theta \Big(\hat{H}_i(\theta, \bar{U}(\theta), \dot{\bar{U}}(\theta), r_i(\theta^-)) - \hat{H}_i(\theta, U(\theta), \dot{U}(\theta), r_i(\theta^-)) \Big)\, dt + \int_\Theta r_i(\theta^-) \cdot \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta.$$

By Item [2.] in Theorem 1, we have for any admissible arc U

$$H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) = \hat{H}_i(\theta, \bar{U}(\theta), \dot{\bar{U}}(\theta), r_i(\theta^-)) \geq \hat{H}_i(\theta, U(\theta), \dot{U}(\theta), r_i(\theta^-)).$$

From which we deduce

$$\begin{aligned} \hat{H}_i(\theta, \bar{U}(\theta), \dot{\bar{U}}(\theta), r_i(\theta^-)) - \hat{H}_i(\theta, U(\theta), \dot{U}(\theta), r_i(\theta^-)) &\geq H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) - H_i(\theta, U(\theta), r_i(\theta^-)) \\ &= f(\theta)(U(\theta) - \bar{U}(\theta)) \end{aligned}$$

where the last equality follows from (A14).

Using (A12), we get:

$$\Delta \geq \int_\Theta \dot{p}_i(\theta) \left(U(\theta) - \bar{U}(\theta) \right) d\theta + \int_\Theta r_i(\theta^-) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta.$$

Decomposing $r_i(\theta^-)$ back into its absolutely-continuous component, $p_i(\theta)$, and its possibly discontinuous component, $\eta_i(\theta) \equiv \int_{[S,\theta)} \gamma_i(s)\mu_i(ds)$, we can write

$$\begin{aligned} \Delta &\geq \int_\Theta \dot{p}_i(\theta) \left(U(\theta) - \bar{U}(\theta) \right) d\theta + \int_\Theta p(\theta) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta - \int_\Theta \eta_i(\theta) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta \\ &= \int_\Theta \left[\frac{d}{d\theta} \left(p_i(\theta)(U(\theta) - \bar{U}(\theta) \right) \right] d\theta - \int_\Theta \eta_i(\theta) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta \\ &= p_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})) - p_i(\underline{\theta})(U(\underline{\theta}) - \bar{U}(\underline{\theta})) - \int_\Theta \eta_i(\theta) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta \\ &= p_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})) - \int_\Theta \eta_i(\theta) \Big(\dot{U}(\theta) - \dot{\bar{U}}(\theta) \Big)\, d\theta \qquad \text{(A18)} \end{aligned}$$

where the last equality follows from (A5).

We want to simplify the second term. Note that $\eta_i(\theta)$ may be discontinuous at a finite number of points where (2) starts being binding (boundaries of activity sets). Denote such interior points by $\tau_j \in (\underline{\theta}, \bar{\theta})$, $j = 1, \ldots, k-1$, and set $\tau_k = \bar{\theta}$ to allow for jumps on the boundary at $\bar{\theta}$. Set $\tau_0 = \underline{\theta}$ for notational ease, although no jump may occur at $\underline{\theta}$. We denote the size of these upward jumps as

$$\delta_j = \int_{[\underline{\theta}, \tau_j]} \gamma_i(s)\mu_i(ds) - \int_{[\underline{\theta}, \tau_j)} \gamma_i(s)\mu_i(ds) = \mu_i(\{\tau_j\}) > 0,$$

where the inequality follows from the fact that μ_i is a non-negative measure and charges a positive mass at such τ_j with also $\gamma_i(\tau_j) = 1$ at such point from (A7). It follows that we may write

$$\int_\Theta \eta_i(\theta) \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta \;=\; \sum_{j=0}^{k-1} \int_{[\tau_j, \tau_{j+1})} \eta_i(\theta) \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta + \sum_{j=1}^{k} \delta_j (U(\tau_j) - \bar{U}(\tau_j)).$$

Because $\eta_i(\theta)$ is continuously differentiable on each open set (τ_j, τ_{j+1}), we can integrate by parts to obtain

$$\begin{aligned}\int_\Theta \eta_i(\theta) \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta = &\sum_{j=0}^{k-1} \left\{ \left[\eta_i(\theta)(U(\theta) - \bar{U}(\theta))\right]_{\tau_j}^{\tau_{j+1}} + \delta_{j+1}(U(\tau_{j+1}) - \bar{U}(\tau_{j+1})) \right\} \\ &- \sum_{j=0}^{k-1} \int_{[\tau_j, \tau_{j+1})} \gamma_i(\theta) \left(U(\theta) - \bar{U}(\theta)\right) \mu_i(d\theta).\end{aligned}$$

Simplifying, we get

$$\begin{aligned}\int_\Theta \eta_i(\theta) \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta = &\,\eta_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})) + \sum_{j=0}^{k-1} \delta_{j+1}(U(\tau_{j+1}) - \bar{U}(\tau_{j+1})) \\ &- \sum_{j=0}^{k-1} \int_{[\tau_j, \tau_{j+1})} \gamma_i(\theta) \left(U(\theta) - \bar{U}(\theta)\right) \mu_i(d\theta).\end{aligned}$$

Item [3.] in Theorem 1 implies that $\gamma_i(\theta)(U(\theta) - \bar{U}(\theta)) = \gamma_i(\theta)(U(\theta) - \bar{U}_{-i}(\theta) + \bar{U}_{-i}(\theta) - \bar{U}(\theta)) \leq 0$ for all θ. Thus,

$$\int_\Theta \eta_i(\theta) \cdot \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta \geq \eta_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})) + \sum_{j=0}^{k-1} \delta_{j+1}(U(\tau_{j+1}) - \bar{U}(\tau_{j+1})).$$

The necessary condition (A5) implies that $p_i(\bar{\theta}) + \eta_i(\bar{\theta}) + \delta_{\bar{\theta}} = 0$. Thus,

$$\int_\Theta \eta_i(\theta) \left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta \geq -p_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})) + \sum_{j=1}^{k-1} \delta_j (U(\tau_j) - \bar{U}(\tau_j)).$$

Because complementary slackness (Item [3.] in Theorem 1) requires $\gamma_i(\theta)(\bar{U}(\theta)-\bar{U}_{-i}(\theta)) = 0$ for θ a.e., it follows that at jump points (points at which μ_i is an atom with positive mass), we must have $\bar{U}(\tau_j) = \bar{U}_{-i}(\tau_j)$. From the fact that U is admissible and satisfies (2), it follows that $U(\tau_j) - \bar{U}(\tau_j) \geq 0$. Given $\delta_j < 0$, we have $\delta_{\tau_j}(U(\tau_j) - \bar{U}(\tau_j)) \geq 0$. Thus,

$$\int_\Theta \eta_i(\theta)\left(\dot{U}(\theta) - \dot{\bar{U}}(\theta)\right) d\theta \geq -p_i(\bar{\theta})(U(\bar{\theta}) - \bar{U}(\bar{\theta})).$$

Combining this result with the lower bound for Δ in (A18) yields $\Delta \geq 0$ as desired. ■

Proof of Theorem 3. To prove continuity of $\bar{q}(\theta)$, we adapt the arguments in Galbraith and Vinter (2004) to our strategic setting.

Lemma 3 *The following properties hold:*
[1.] For each $(\theta, u, r) \in \Theta \times \mathbb{R}^2$, $\partial_r H_i(\theta, u, r)$ is single-valued, continuously differentiable and $\partial_r H_i(\theta, u, r) = \nabla_r H_i(\theta, u, r)$;
[2.] Fix (θ, u), $r \to \nabla_r H_i(\theta, u, r)$ is locally Lipschitz continuous.

Proof. By the representation of the subdifferential we have:

$$\partial_r H_i(\theta, u, r) = \left\{\xi : r\xi - L_i(\theta, u, \xi) = \max_{v\in\mathcal{V}} rv - L_i^1(\theta, v) - L^0(\theta, u)\right\}.$$

The max above is achieved because $\mathcal{V}$ is compact. Hence $\partial_r H_i(\theta, u, r)$ is non-empty.

Take now two triplets (θ, u, r) and (θ, u, r') both in $\Theta\times\mathbb{R}^2$ and choose v and v' such that $v \in \partial_r H_i(\theta, u, r)$ and $v' \in \partial_r H_i(\theta, u, r')$. By the fundamental property of convex subdifferentials (Rockafellar and Wets, 2004, p. 511), $r \in \partial_v L_i^1(\theta, v)$ and $r' \in \partial_v L_i^1(\theta, v')$. By Condition 1, $\partial_v L_i^1(\theta, v) = \nabla_v L_i^1(\theta, v)$ is continuously differentiable, we have: $\partial_v L_i(\theta, u, v) = \nabla_v L_i^1(\theta, v) + N_{\mathcal{V}}(v)$ and $\partial_v L_i(\theta, v') = \nabla_v L_i^1(\theta, v') + N_{\mathcal{V}}(v')$ where $N_{\mathcal{V}}(v) = \{x|(v, x - \bar{x}) \leq 0\}$ (resp. $N_{\mathcal{V}}(v')$) is the normal cone at $v \in \mathcal{V}$ (resp. v'). Therefore, we have $r = \nabla_v L_i^1(\theta, v) + e$ and $r' = \nabla_v L_i^1(\theta, v') + e'$ where $e \in N_{\mathcal{V}}(v)$ and $e' \in N_{\mathcal{V}}(v')$. From this, we deduce:

$$|r - r'||v - v'| = |\nabla_v L_i^1(\theta, v) - \nabla_v L_i^1(\theta, v') + e - e'||v - v'|$$

$$= (\nabla_v L_i^1(\theta, v) - \nabla_v L_i^1(\theta, v'))(v - v') + e(v - v') + e'(v' - v). \quad \text{(A19)}$$

Because $e \in N_{\mathcal{V}}(v)$ and $e' \in N_{\mathcal{V}}(v')$, we have $e(v - v') \geq 0$ and $e'(v' - v) \geq 0$. Inserting into (A19), we get:

$$|r - r'||v - v'| \geq (\nabla_v L_i^1(\theta, v) - \nabla_v L_i^1(\theta, v'))(v - v') \geq K|v - v'|^2$$

and

$$|v - v'| = |\partial_r H_i(\theta, u, r) - \partial_r H_i(\theta, u, r')| \leq \frac{1}{K}|r - r'|. \quad \text{(A20)}$$

where the last but one equality follows from Condition 1.

From this, we immediately deduce that $\partial_r H_i(\theta, u, r)$ is single-valued (take $r = r'$ in the above inequality and observe that any pair $(v, v') \in \partial_p H_i(\theta, u, r)^2$ is such that $v = v'$), we thus denote $\partial_p H_i(\theta, u, r) = \nabla_r H_i(\theta, u, r)$. Inequality (A20) tells us also that $p \to \nabla_r H_i(\theta, u, r)$ is locally Lipschitz continuous with Lipschitz modulus $\frac{1}{K}$. ■

From Lemma 3, the representation (A15), and condition (A5), we deduce that $\bar{v}(\theta)$ has left- and right-hand limits at all points $\theta \in \Theta$ and one-sided limits at the end-points. Finally, we deduce that $\bar{U}(\theta)$ is Lipschitz continuous.

Lemma 4 *$\bar{v}(\theta)$ is continuous on Θ.*

Proof. From the proof of Proposition 1, we know that $\bar{v}(\theta^-) = \partial_r H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) = \nabla_r H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) \leq \bar{v}(\theta^+) \in \partial_r H_i(\theta, \bar{U}(\theta), r_i(\theta^+)) = \nabla_r H_i(\theta, \bar{U}(\theta), r_i(\theta^+))$ where $\partial_r H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) = \nabla_r H_i(\theta, \bar{U}(\theta), r_i(\theta^-))$ (resp. $\partial_r H_i(\theta, \bar{U}(\theta), r_i(\theta^+)) = \nabla_r H_i(\theta, \bar{U}(\theta), r_i(\theta^+))$) by Lemma 3 (Item [1.]). Because $r_i(\cdot)$ is continuous (Proposition 1),

$$\bar{v}(\theta^-) = \nabla_r H_i(\theta, \bar{U}(\theta), r_i(\theta)) = \bar{v}(\theta^+)$$

so that $\bar{v}(\cdot)$ is continuous as well. ■

This ends the proof of Theorem 3. ■

Proof of Theorem 4. We first prove the following preliminary result.

Proposition 8 *Under the assumptions of Theorem 3, in any pure strategy equilibrium with deterministic mechanisms, the equilibrium output $\bar{q}(\theta)$ satisfies $\forall i \in \mathcal{N}$:*

$$0 \in \partial_q co \left\{ -\widetilde{W}_i \right\} (\bar{q}(\theta), \theta), \tag{A21}$$

and

$$-co \left\{ -\widetilde{W}_i \right\} (\bar{q}(\theta), \theta) = \widetilde{W}_i(\bar{q}(\theta), \theta) \tag{A22}$$

where $\widetilde{W}_i(q, \theta) = S_0(q) + S_i(q) + \bar{t}_{-i}(q) - \left(\theta + \frac{r_i(\theta)}{f(\theta)}\right) q$ and $r_i(\theta)$ is defined in Theorem 3.

Proof. Let first define *a stochastic mechanism* for principal P_i as a pair $\{t_i(m_i), \mu_i(\cdot|m_i)\}_{m_i \in \mathcal{M}_i}$ where $\mathcal{M}_i$ is an arbitrary message space for the agent to communicate with P_i. Conditionally on a message m_i, such stochastic mechanism stipulates a payment $t_i(m_i)$ to the agent and recommends him to choose outputs according to the distribution function $\mu_i(\cdot|m_i)$ whose support is included in $\mathcal{Q}$.[41] When $\mathcal{M}_i \equiv \Theta$, the stochastic mechanism is *direct*. Denote by $\Delta\mathcal{T}$ the corresponding strategy space of such direct stochastic mechanisms.[42]

[41]Because of quasi-linearity of the agent's utility function, there is no value of using stochastic payments for incentive reasons.

[42]Considering deviations within such larger strategy space might increase principal P_i's payoff. We will find below conditions so that such deviations are suboptimal in any equilibrium.

Observe that, in any pure-strategy equilibrium with deterministic mechanisms, we must have a.e. $co\{L_i^1\}(\theta,\bar v(\theta)) = L_i^1(\theta,\bar v(\theta))$. Suppose the contrary, then, by definition, we have $co\{L_i^1\}(\theta,\bar v(\theta)) < L_i^1(\theta,\bar v(\theta))$ for a set I of non-zero measure of types θ. From Caratheodory Theorem, there exists $(v_1(\theta),v_2(\theta)) \in \mathcal{V}^2$ and $\alpha(\theta)\in(0,1)$ such that $\bar v(\theta) = \alpha(\theta)v_1(\theta)+(1-\alpha(\theta))v_2(\theta)$. Consider now the new mechanism obtained by replacing for any type $\theta \in I$ the deterministic mechanism implementing $(\bar U(\theta),\bar v(\theta))$ by a direct stochastic mechanism that recommends to the agent to randomize between $v_1(\theta)$ and $v_2(\theta)$ with probabilities $\alpha(\theta)$ and $1-\alpha(\theta)$. This stochastic mechanism entails a payment

$$t_i(\theta) = \bar U(\theta) - \theta\bar v(\theta) - \alpha(\theta)(\bar t_{-i}(-v_1(\theta)) + S_0(-v_1(\theta)) - (1-\alpha(\theta))(\bar t_{-i}(-v_2(\theta)) + S_0(-v_2(\theta)))).$$

On the complementary set I^c, the mechanism is unchanged and remains deterministic. Incentive compatibility is preserved by definition since we have still $\dot U(\theta) = \bar v(\theta)$ a.e. both on I and I^c. This direct stochastic mechanism allows principal P_i to reach a payoff

$$\int_\Theta \left(co\{L_i^1\}(\theta,\bar v(\theta)) + L^0(\bar U(\theta))\right) d\theta < \int_\Theta \left(L_i^1(\theta,\bar v(\theta)) + L^0(\bar U(\theta))\right) d\theta$$

since $\int_I co\{L_i^1\}(\theta,\bar v(\theta))d\theta < \int_I L_i^1(\theta,\bar v(\theta))d\theta$ and $\int_{I^c} co\{L_i^1\}(\theta,\bar v(\theta))d\theta = \int_{I^c} L_i^1(\theta,\bar v(\theta))d\theta$. This new mechanism would be a valuable deviation with respect to what that principal can get with a deterministic mechanism. Therefore, this cannot arise in any pure equilibrium with deterministic mechanisms and we get (A22).

Finally, (A21) follows from (7). ■

Strict differentiability of $\bar t_i(\cdot)$ at equilibrium point.[43] Because $L_i^1(\theta,\cdot)$ is strictly convex in v and continuously differentiable, $-\widetilde{W}_i(q,\theta)$ is also strictly convex in q, so that $co\{-\widetilde{W}_i\} = -\widetilde{W}_i$ and $\partial_q\{-\widetilde{W}_i\} = \nabla_q\{-\widetilde{W}_i\}$. Condition (A21) can thus be rewritten as:

$$\begin{aligned}
0 &= \partial_q\left\{-\left(S_0(q)+S_i(q)+\bar t_{-i}(q) - \left(\theta + \frac{r_i(\theta)}{f(\theta)}\right)q\right\}\right\}|_{q=\bar q(\theta)}\\
&= \nabla_q\left\{-\left(S_0(q)+S_i(q)+\bar t_{-i}(q) - \left(\theta + \frac{r_i(\theta)}{f(\theta)}\right)q\right\}\right\}|_{q=\bar q(\theta)}\\
&\Leftrightarrow 0 = -S_0'(\bar q(\theta)) - S_i'(\bar q(\theta)) + \theta + \frac{r_i(\theta)}{f(\theta)} + \partial_q(-\bar t_{-i})(\bar q(\theta))
\end{aligned}$$

where the last equality comes from observing that $h(\theta,q) = -S_0(q) - S_i(q) + \left(\theta + \frac{r_i(\theta)}{f(\theta)}\right)q$ is strictly differentiable in q and $\partial_q(h+g)(\theta,q) = \partial_q h(\theta,q) + \partial_q g(q)$ when h is strictly differentiable in q and $g = -\bar t_{-i}$ is lower semi-continuous since $\bar t_{-i}\in\mathcal{T}$.[44]

[43] A function $f:\ \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$ is strictly differentiable at a point $\bar x$ if $f(\bar x)$ is finite and there exists a vector v, which is the gradient $\nabla f(\bar x)$ such that $f(x') = f(\bar x) + \langle v, x'-\bar x\rangle + o(|x'-\bar x|)$.

[44] Rockafellar and Wets (2004, Exercice 10.10).

The fact that $L_i^1(\theta,\cdot)$ is finite, convex and continuously differentiable implies that

$$0=\partial_q^\infty\left\{-\left(S_0(q)+S_i(q)+\bar{t}_{-i}(q)-\left(\theta+\frac{r_i(\theta)}{f(\theta)}\right)q\right\}\right\}|_{q=\bar{q}(\theta)}$$

which implies also that $\partial_q^\infty(-\bar{t}_{-i})(\bar{q}(\theta))=0$.[45] Since $-\bar{t}_{-i}$ is lower semi-continuous, $\partial_q(-\bar{t}_{-i})(\bar{q}(\theta))$ is a singleton and $\partial_q^\infty(-\bar{t}_{-i})(\bar{q}(\theta))=0$, $-\bar{t}_{-i}$ is also strictly differentiable at $\bar{q}(\theta)$ and $\partial_q(-\bar{t}_{-i})(\bar{q}(\theta))=\nabla_q(-\bar{t}_{-i})(\bar{q}(\theta))$.[46]

Turning now to the optimality condition of the agent's problem, we must have:

$$0=\partial_q\left\{-\left(S_0(q)-\theta q+\bar{t}_i(q)+\bar{t}_{-i}(q)\right)\right\}|_{q=\bar{q}(\theta)}$$

which, by an argument similar to one used above, implies also that $\partial_q^\infty(-\bar{t}_i)(\bar{q}(\theta))=0$.[47] Since $-\bar{t}_i$ is lower semi-continuous, $\partial_q(-\bar{t}_i)(\bar{q}(\theta))$ is a singleton and $\partial_q^\infty(-\bar{t}_i)(\bar{q}(\theta))=0$, $-\bar{t}_i$ is also strictly differentiable at $\bar{q}(\theta)$ and $\partial_q(-\bar{t}_i)(\bar{q}(\theta))=\nabla_q(-\bar{t}_i)(\bar{q}(\theta))$.

Conditions on equilibrium output. Condition (A21) can finally be rewritten as:

$$S_0'(\bar{q}(\theta))+S_i'(\bar{q}(\theta))+\bar{t}_{-i}'(\bar{q}(\theta))=\theta+\frac{r_i(\theta)}{f(\theta)}. \tag{A23}$$

Summing those equalities over i yields:

$$S_0'(\bar{q}(\theta))+\sum_{i=1}^n S_i'(\bar{q}(\theta))+(n-1)\left(S_0'(\bar{q}(\theta))+\sum_{i=1}^n \bar{t}_i'(\bar{q}(\theta)-\theta\right)=\theta+\frac{\sum_{i=1}^n r_i(\theta)}{f(\theta)}. \tag{A24}$$

Observe that $\bar{t}_i(q)$ being strictly differentiable at any equilibrium point $\bar{q}(\theta)$ implies

$$0\in\partial_q\{-S_0(q)-\sum_{i=1}^n\bar{t}_i(q)+\theta q\}|_{q=\bar{q}(\theta)}$$

which itself implies

$$S_0'(\bar{q}(\theta))+\sum_{i=1}^n \bar{t}_i'(\bar{q}(\theta))=\theta. \tag{A25}$$

Now, note that at any $\theta\in\overline{\Omega_i^c}$, it must be that

$$\bar{t}_i(\bar{q}_{-i}(\theta))=\bar{t}_i'(\bar{q}_{-i}(\theta))=0. \tag{A26}$$

So that, finally, (11) holds. Inserting those findings into (A24) yields

$$S_0'(\bar{q}(\theta))+\sum_{i=1}^n S_i'(\bar{q}(\theta))=\theta+\frac{\sum_{i=1}^n r_i(\theta)}{f(\theta)} \tag{A27}$$

[45] Rockafellar and Wets (2004, Exercice 8.8).
[46] Rockafellar and Wets (2004, Theorem 9.18.c).
[47] Rockafellar and Wets (2004, Exercice 8.8).

which, taking again (A26) into account yields (12).

Monotonicity of $\bar{q}(\theta)$. We already know that $\bar{q}(\cdot)$ is continuous on $(\underline{\theta}, \bar{\theta})$. This implies that it is enough to show monotonicity on each interval I where $\alpha(\theta)$ is fixed (denote $\bar{I}$ the interior of such interval). Note that on any such interval $\sum_{i\in\alpha(\theta)} r_i(\theta)$ is constant and such that $\sum_{i\in\alpha(\theta)} r_i(\theta) \le |\alpha(\theta)|$. Differentiating (12) w.r.t. θ on such interval I yields:

$$\dot{\bar{q}}(\theta)\left(\sum_{i\in\alpha(\theta)\bigcup\{0\}} S_i''(\bar{q}(\theta))\right) = 1 + \frac{d}{d\theta}\left(\frac{|\alpha(\theta)|F(\theta) - \sum_{i\in\alpha(\theta)} r_i(\theta)}{f(\theta)}\right). \quad \text{(A28)}$$

The derivative of the right-hand side becomes:

$$\frac{d}{d\theta}\left(\frac{\left(|\alpha(\theta)| - \sum_{i\in\alpha(\theta)} r_i(\theta)\right)F(\theta) - \left(\sum_{i\in\alpha(\theta)} r_i(\theta)\right)(1-F(\theta))}{f(\theta)}\right)$$

$$= \left(|\alpha(\theta)| - \sum_{i\in\alpha(\theta)} r_i(\theta)\right)\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) - \left(\sum_{i\in\alpha(\theta)} r_i(\theta)\right)\frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) \;>\; 0 \quad \text{(A29)}$$

where the last inequality follows from Assumption 1. Inserting into (A28) yields the requested monotonicity $\dot{\bar{q}}(\theta) < 0 \quad \forall\theta \in \bar{I}$. ■

Proof of Proposition 2. On any connected interval included in Ω_i^c, equation (A26) holds both in the interior but also on the boundaries from Proposition 3. Taking into account (A23) and the agent's first-order condition for optimality (11) yields (13). ■

Proof of Proposition 3. We prove each point in turn.

Strict differentiability of equilibrium schedules. We have shown in passing in the proof of of Theorem 4 that $\bar{t}_i(q)$ is strictly differentiable at any equilibrium point $\bar{q}(\theta)$.

Smooth-pasting. Consider θ on the boundary of Ω_i^c, i.e., such that $\bar{U}(\theta) = \bar{U}_{-i}(\theta)$, or

$$\bar{t}_i(\bar{q}(\theta)) + \bar{t}_{-i}(\bar{q}(\theta)) - \theta\bar{q}(\theta) + S_0(\bar{q}(\theta)) = \bar{t}_i(\bar{q}_{-i}(\theta)) + \bar{t}_{-i}(\bar{q}_{-i}(\theta)) - \theta\bar{q}_{-i}(\theta) + S_0(\bar{q}_{-i}(\theta))$$

where this equality follows from the fact that $\bar{q}(\theta)$ is continuous at such θ and $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$ on Ω_i^c. This yields $\bar{t}_i(\bar{q}(\theta)) = 0$.

Finally, (A25) above tells us that any $\theta \in \Omega_i^c$ is such that:

$$\bar{t}_i'(\bar{q}(\theta)) + \bar{t}_{-i}'(\bar{q}(\theta)) + S_0'(\bar{q}(\theta)) = \theta = \bar{t}_{-i}'(\bar{q}_{-i}(\theta)) + S_0'(\bar{q}_{-i}(\theta)).$$

Together with the continuity property $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$, we get $\bar{t}_i'(\bar{q}(\theta)) = 0$.

Marginal contributions. First, observe that (A23) can be rewritten as

$$S_0'(\bar{q}(\theta)) + S_i'(\bar{q}(\theta)) + \sum_{j\in\alpha(\theta)/i} \bar{t}_j(\bar{q}(\theta)) = \theta + \frac{r_i(\theta)}{f(\theta)} \quad \text{(A30)}$$

since inactive principals contribute zero both at the margin and in value. Using then (11), we get the expression of the following marginal contribution:

$$\bar{t}_i'(\bar{q}(\theta)) = S_i'(\bar{q}(\theta)) - \frac{r_i(\theta)}{f(\theta)}. \tag{A31}$$

Now observe that, on a connected subset Ω_i^j of the activity set Ω_i, we have $\int_{\underline{\theta}}^{\theta} \mu_i(d\theta) = \int_{\underline{\theta}}^{\theta_i^j} \mu_i(d\theta) = M_i^j \in [0,1]$. ■

Proof of Proposition 4. Type θ^* is just indifferent between producing or not given the contributions defined in (16). Types below (resp. above) that threshold produce (resp. do not produce) the public good for sure which gives us condition (18).

Contributions defined in(16) are non-negative and linear in the probability of producing the public good so that the agent always accept those contributions.

To check that those schedules are best responses to each other, observe that one can rewrite $(\mathcal{P}_i)$ as:

$$(\mathcal{P}_i): \quad \max_{(U,q)} \int_{\Theta} \left(\left(s_i + \max\left\{0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)}\right\} - \theta \right) q(\theta) - U(\theta) \right) f(\theta) d\theta$$

subject to $q \in [0,1]$, (3), (4) and (2).

Again, the second-order condition (4) is suppressed and only checked ex post.

Assuming first that (2) binds at $\bar{\theta}$ (and possibly on an non-empty interval including that boundary), we get

$$U(\theta) = \bar{U}_{-i}(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} q(x)dx.$$

Inserting this expression into the integrand above yields a more compact expression of $(\mathcal{P}_i)$ as:

$$(\mathcal{P}_i): \quad \max_{q\in[0,1]} \int_{\Theta} \left(\left(s_i + \max\left\{0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)}\right\} - \theta \right) q(\theta) - U(\theta) \right) f(\theta) d\theta.$$

Optimizing pointwise, the solution is $\bar{q}(\theta) = \begin{cases} 1 & \text{if } \theta \in [0,\hat{\theta}] \\ 0 & \text{otherwise} \end{cases}$ where $\hat{\theta}$ is uniquely defined when Assumption 1 holds as the solution to:

$$s_i + \max\left\{0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)}\right\} = \hat{\theta} + \frac{F(\hat{\theta})}{f(\hat{\theta})}. \tag{A32}$$

Two cases might arise. If θ^* that solves (17) is such that $s_i \geq \frac{F(\theta^*)}{f(\theta^*)}$, then the solution to (A32) is $\hat{\theta} = \theta^*$. If instead θ^* that solves (17) is such that $s_i < \frac{F(\theta^*)}{f(\theta^*)}$, then $\hat{\theta} < \theta^*$, which implies that $\bar{q}(\theta) = 0 < \bar{q}_{-i}(\theta) = 1$ on $(\hat{\theta}, \theta^*)$ (with also $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$ for all $\theta \in \Theta/(\hat{\theta}, \theta^*)$. Hence, we get that the slope of $U(\theta)$ is less than that of $\bar{U}_{-i}(\theta)$. A contradiction with our starting assumption that (2) binds at $\bar{\theta}$. In that case, the best-strategy for principal P_i is to offer no contribution at all and (17) still holds.

Gathering everything, the contribution schedules that implement this equilibrium outcomes are given by (16). ■

Proof of Proposition 5. We follow the steps of the general analysis in characterizing such an equilibrium through its activity sets, contributions and output.

Activity sets. When $S_i(q) = s_i q$ for some $s_i > 0$ (the proof for $s_i < 0$ is similar and omitted), (13) amounts to

$$F(\theta) - f(\theta)s_i = \int_{[\underline{\theta},\theta)} \gamma_i(\theta)\mu_i(d\theta) \quad \forall \theta \in \overline{\Omega_i^c}. \tag{A33}$$

We want to prove that the inactivity set Ω_i^c is of the form $[\theta_i, \bar{\theta}]$ where $s_i = R(\theta_i)$. Take μ_i absolutely continuous with respect to the Lebesgue measure, i.e., $\mu_i(d\theta) = m_i(\theta)d\theta$ with $m_i(\theta) = f(\theta) - \dot{f}(\theta)s_i$ for $[\theta_i, \bar{\theta}]$ being a positive density. Observe that:

$$\frac{m_i(\theta)}{f(\theta)} = 1 - \frac{\dot{f}(\theta)}{f(\theta)}s_i > 1 - \frac{f(\theta)}{F(\theta)}s_i \geq 0$$

where the first inequality follows from $s_i > 0$ and Assumption 1 (since then $1 \geq \frac{F(\theta)\dot{f}(\theta)}{f^2(\theta)}$) and the second inequality follows from Assumption 1 and the definition of θ_i.

This μ_i allows us to satisfy the necessary Conditions [1.] to [3.] in Theorem 1 which are also sufficient from Theorem 3. It is also non-singular which shows that the equilibrium output is continuous.

Contributions. Using the specification of the activity set $\Omega_i = [\underline{\theta}, \theta_i^*)$ given in Item [2.] of the proposition and the smooth-pasting condition (14) yields immediately (20).

Outputs. (19) follows from summing the expressions of marginal transfers coming from (20). ■

Proof of Proposition 6. We follow the steps of the general analysis above in characterizing such an equilibrium through its activity sets, output and contributions.

Activity sets. First, observe that $-3\theta \geq \bar{q}_1(\theta)$ if and only if $\theta \leq -1 + \delta$. We now prove that $\Omega_1 = [-\delta, 1-\delta)$ when $\delta \geq \frac{1}{2}$ and $\Omega_1 = [-\delta, \delta)$ when $\delta \leq \frac{1}{2}$.

Let us begin with $\delta \geq \frac{1}{2}$. We use (13) to derive the non-singular measure μ_1 on $\Omega_1^c = [1-\delta, \delta]$. We have on that interval :

$$r_1(\theta) = \frac{\theta + \delta}{2\delta} - \int_{1-\delta}^{\theta} \mu_1(d\theta) = \frac{S_1'(\bar{q}_2(\theta))}{2\delta} = \frac{1}{2\delta}.$$

Reminding that μ_1 is non-singular for a continuous equilibrium and denoting $\mu_1(d\theta) = \eta_1(\theta)\frac{d\theta}{2\delta}$, we compute $\eta_1(\theta) = 1$ if $\theta \in \Omega_1^c$. The non-singular measure μ_2 on $\Omega_2^c = [-\delta, -1+\delta]$ is obtained by symmetry.

Consider now the case $\delta \le \frac{1}{2}$. We have then full coverage with $\mu_1(\{\delta\}) = 1$ and by symmetry $\mu_2(\{\delta\}) = 1$.

Outputs. The formula for computing the equilibrium output follows immediately from using (12) and the definition of the activity sets above.

Contributions. Principals have conflicting preferences and thus marginal contributions on their activity sets are of the form

$$\bar{t}_1'(q) = S_1'(q) - R(\bar{\theta}(q)) = \frac{\delta - \bar{\theta}(q)}{2\delta} \text{ and } \bar{t}_2'(q) = S_2'(q) + T(\bar{\theta}(q)) = -\frac{\delta + \bar{\theta}(q)}{2\delta}.$$

When $\delta \ge \frac{1}{2}$, the binding participation constraint for principal P_1, namely $\bar{U}(1-\delta) = \bar{U}_2(1-\delta)$, determines completely $\bar{t}_1(q)$ as in (21). In particular, note that $\bar{t}_1(\bar{q}(1-\delta)) = \bar{t}_1'(\bar{q}(1-\delta)) = 0$, i.e., the smooth-pasting condition holds. By symmetry, we obtain $\bar{t}_2(q)$.

When $\delta \le \frac{1}{2}$, there is full coverage and the binding participation constraint for principal P_1 becomes $\bar{U}(\delta) = \bar{U}_2(\delta)$. It again determines completely $\bar{t}_1(q)$ as in (21). Note in particular that $\bar{t}_1(\bar{q}(\delta)) > 0$ because of full coverage. A symmetry argument gives $\bar{t}_2(q)$.

Rent. For completeness, note that the agent's information rent is given by the following expressions $\bar{U}(\theta) = \begin{cases} \frac{3}{2}(1-\theta^2) - 3\delta^2 & \text{if } \delta \le \frac{1}{2} \\ 3(1-\delta)^2 - \int_0^\theta \bar{q}(x)dx & \text{if } \delta \ge \frac{1}{2}. \end{cases}$ ■

Proof of Proposition 7.

Activity sets. First, observe that $q^I(\theta) \ge q_2^*(\theta)$ if and only if $\theta \le s_1$. We now prove that $\Omega_1 = [0, s_1)$. To do so, we use (13) to derive the non-singular measure μ_1 on $\Omega_1^c = [s_1, \bar{\theta}]$. We have:

$$r_1(\theta) = \frac{\theta}{\bar{\theta}} - \int_0^\theta \mu_1(d\theta) = \frac{S_1'(\bar{q}_2(\theta))}{\bar{\theta}} = \frac{s_1}{\bar{\theta}}$$

with $\bar{q}_2(\theta) = q_2^*(\theta)$. Reminding that μ_1 is non-singular for a continuous equilibrium and denoting $\mu_1(d\theta) = \eta_1(\theta)\frac{d\theta}{\bar{\theta}}$, we compute immediately $\eta_1(\theta) = 1$ if $\theta \in [s_1, \bar{\theta}]$.

We now prove that $\Omega_2 = [0, s_2)$. We use again (13) to derive the non-singular measure μ_2 on $\Omega_2^c = [s_2, \bar{\theta}]$. We have:

$$r_2(\theta) = \frac{\theta}{\bar{\theta}} - \int_0^\theta \mu_2(d\theta) = \frac{S_2'(0)}{\bar{\theta}} = \frac{s_2}{\bar{\theta}}.$$

Reminding that μ_2 is non-singular and denoting $\mu_2(d\theta) = \eta_2(\theta)\frac{d\theta}{\bar{\theta}}$, we get $\eta_2(\theta) = 1$ if $\theta \in [s_2, \bar{\theta}]$.

Outputs. The formula for computing the equilibrium output follows immediately from using (12) and the definition of the activity sets above.

Contributions. Principals have congruent preferences and thus marginal contributions, on their activity respective sets, are of the form

$$\bar{t}_i'(q) = S_i'(q) - R(\bar{\theta}(q)) = s_i - \frac{\bar{\theta}(q)}{\bar{\theta}}.$$

Moreover, the binding participation constraint for principal P_2, namely $\bar{U}(s_2) = \bar{U}_1(s_2) = 0$ determines completely $\bar{t}_2(q)$ as in (25). In turn, the binding participation constraint for principal P_1, namely $\bar{U}(s_1) = \bar{U}_2(s_1) = \max_q \left\{\bar{t}_2(q) - \theta q - \frac{q^2}{2}\right\} = \int_{\frac{s_2}{2}}^{s_1}(s_2 - 2\theta)d\theta > 0$ determines completely $\bar{t}_1(q)$ as in (25).

Rent. For completeness, the agent's information rent is $\bar{U}(\theta) = \int_{\theta}^{\bar{\theta}} \bar{q}(x)dx$. ■