

INDUSTRY EQUILIBRIUM WITH OPEN SOURCE AND PROPRIETARY FIRMS

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ABSTRACT. We present a model of industry equilibrium to study the coexistence of Open Source (OS) and Proprietary (P) firms. Two novel aspects of the model are: (1) participation in OS arises as the optimal decision of profit-maximizing firms, and (2) OS and P firms may (or may not) coexist in equilibrium. Firms decide their type and investment in R&D, and sell packages composed of a primary good (like software) and a complementary private good. The only difference between both kinds of firms is that OS share their technological advances on the primary good, while P keep their innovations private. The main contribution of the paper is to determine conditions under which OS and P coexist in equilibrium. Interestingly, this equilibrium is characterized by an asymmetric market structure, with a few large P firms and many small OS firms.

KEYWORDS: Industry Equilibrium, Open Source, Innovation, Complementarity, Technology Sharing, Cooperation in R&D (JEL O31, L17, D43).

1. INTRODUCTION

Collaboration in research enhances the chances of discovery and creation. This is true not only for scientific discoveries, but also for commercial innovations. However, innovators face incentives to limit the

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access of competitors to their innovations. According to the traditional view in the economics of innovation, innovators innovate because they obtain a monopolistic advantage over their competitors. Therefore, innovators should prevent others from gaining access to their discoveries, either by keeping them secret or by protecting them with patents.

This view contrasts with the Open Source (OS) development model, which has been intensively used in the software industry and in other industries at various points in time, as documented in the next section. In OS, developers voluntarily choose to disclose their technological improvements so that they can be copied, used and improved by other innovators free of charge. But if everybody has access to the same technologies, then how do developers benefit from their collaborations? What do they receive in exchange for renouncing their monopolistic advantage? The answer is that OS producers profit by selling goods and services which are complementary to the OS good.

The case of the Linux operating system is a good example. Linux receives substantial contributions of commercial firms like IBM, HP and Red Hat, among others, which benefit from selling complementary goods and services. For example, IBM sells consulting services and complementary proprietary software, HP sells personal computers and computer servers, and Red Hat sells training and support services.

Still, this leaves open the questions of why OS and Proprietary (P) firms coexist in the same markets, and what are the implications of such coexistence on market structure and investments in R&D. Existing literature has yet to address these questions, which are instead the main focus of this paper.

The importance of the topic is best seen by looking at the software industry, where OS and P firms coexist in almost all market segments. In the market of relational database management systems (databases), for example, OS products like MySQL, Apache Derby and PostgreSQL compete against P alternatives like Oracle 11g, IBM DB2 and Microsoft SQL Server (see Table 1 in next section for more examples).

We present a model of industry equilibrium with endogenous technology sharing. Firms decide whether to become OS or P, how much to invest in product development, and the price of their products. For firms electing the OS regime, a contractual arrangement (such as the General Public License) forces them to share their improvements to the main product if they want to benefit from the contributions of other OS firms. P firms, on the other hand, develop their products on their own. Both kinds of firms sell a complementary good, the quality of which depends on the individual investment in the development of the primary good. Consumers value the quality of both goods (vertical

differentiation) but also have idiosyncratic tastes for the products of different firms (horizontal differentiation).

Depending on parameter values, there are equilibria with both kinds of firms and equilibria with only OS firms. When the consumer valuation of the complementary good is low in comparison with the valuation of the primary good, the equilibrium has both kinds of firms. In this case, the market structure is asymmetric with few large P firms and many small OS firms.

This finding is consistent with the observations of recent surveys. Seppä (2006) compares both kinds of firms, and finds that OS firms tend to be younger and generally smaller than P firms. Bonaccorsi and Rossi (2004) show that the most important motive for firms to participate in OS projects is that it allows small firms to innovate.

The intuition behind this result is the following. When firms invest in R&D, they increase the quality of both primary and complementary goods. For OS firms, the primary good is non-rival, so they can appropriate only a fraction of the quality increase. The complementary good, on the other hand, is a private good so firms fully appropriate the increase in quality. The key parameter in the model measures the relative valuation of the primary good in comparison with the complementary good, so it can also be interpreted as the degree of public good of the investment in R&D.

When the relative valuation of the complementary good is high, the public good problem is less important, and OS and P tend to have similar investments and market shares. In this case, all firms decide to be OS to benefit from lower development costs. When the relative valuation of the primary good is high, on the other hand, the public good problem becomes more important and free riding implies lower investment and market shares for OS firms. Nevertheless, OS firms still benefit from lower development costs. In equilibrium, higher market shares and prices for P firms are compensated with higher development costs and no firm finds it profitable to deviate to become the other kind (notice that we are talking about individual market shares, the total market share of the OS project may be higher than the sum of market shares of P firms).

A second result of the paper is the characterization of product quality under the OS and P regimes. Individual investment in R&D may be small for OS firms because of free-riding. However, P firms do not share their technological advances, generating a duplication of effort. As a consequence, either model may yield higher product quality in equilibrium.

We find that when OS and P coexist the products of P firms are of higher quality than those of OS firms. On the other hand, when all firms are OS two things are possible: OS may prevent the entry of a higher quality good or it may result in a product of higher quality than that of a potential P firm. The latter is the case when the consumers' valuation of the complementary good is high enough relative to the valuation of the primary good.

Welfare will be suboptimal because of the public good problem in OS and the duplication of effort of P firms. In Section 4, we show that a subsidy to OS development can improve welfare not only because it increases the investment in R&D, but also because it encourages commercial firms to participate in OS, enhancing collaboration as a result.

The equilibrium with OS and P firms is characterized by an asymmetric market structure, even though all firms are ex-ante symmetric. In Section 5 we argue that this result is even stronger if there are initial asymmetries in firm size. Larger firms ex-ante have more incentives to remain P, and the difference in market shares between OS and P will tend to increase.

The baseline model assumes symmetric consumer preferences for OS and P products. However, given that OS firms sell the same primary good, their products are likely to be more similar than those of P firms. In Section 6 we modify the baseline model to allow for a higher cross-price elasticity between OS products. We find that the main result of the paper still holds: when OS and P firms coexist, the market share of P firms is higher than that of OS firms. However, in this case, we also find that if the substitutability between OS products is high enough, there are equilibria with only P firms, and also multiple equilibria.

In the baseline model, we focus on the analysis of the investment in the primary good, and we assume that the quality of the complementary good is determined by individual contributions on the primary good, through a learning effect. In Section 7 we analyze what happens when firms can invest directly in the complementary good. We show that as the importance of the direct investment increases relative to the learning effect, the number of firms in OS decreases. If the effect of direct investment is high enough, the equilibrium has both kinds of firms for all parameter values. Therefore, coexistence becomes more likely when firms can invest in the complementary good without affecting the quality of the primary good.

Finally, in Section 8 we study the equilibrium effects of partial and full compatibility between the primary and complementary goods of

different firms. In particular, P firms can sell the primary and complementary goods for a positive price, while the price of the OS primary good is zero. We find that as the degree of compatibility increases, the market share and profits of P firms increase relative to those of OS firms. This suggests that OS will be more successful when the complementary good is more specific to the primary good, like in the case of support and training services, customizations, platform-specific software, and mobile devices (like MP3 players, PDAs or cell phones).

The model and the results are interesting for a variety of reasons. First, endogenizing the participation decision is crucial for understanding the motivations of commercial firms to participate in OS projects. Second, to the best of our knowledge, this is the first analysis of direct competition between for-profit OS and P firms. Third, we show there are forces leading to an asymmetric market structure, even when all firms are ex-ante symmetric. Fourth, we obtain conditions under which OS can overcome free-riding and produce a good of high quality, even without coordination of individual efforts. Finally, the model allows an analysis of welfare and optimal policy.

It is important to remark that even though the model is specially designed to analyze OS, it has wider applicability. In particular, it can be used to analyze industries where firms cooperating in R&D coexist with firms developing technologies on their own (read the literature review for more details on the relation of this paper with the literature of cooperation in R&D).

The main contribution of this paper is to present the first tractable model of competition between profit maximizing OS and P firms. As such, the model captures the main ingredient shaping the decision to share technologies with rivals or not: the trade-off between appropriability and collaboration. We believe our paper is an important first step in the analysis of the behavior of profit maximizing OS firms. In Section 9 we discuss interesting directions for further research.

All proofs are relegated to the Appendix.

1.1. Open source in detail. There are clear antecedents of OS in the history of technological change and innovation. Well documented examples are the iron industry in Cleveland, UK (Allen 1983); the Cornish pumping engine (Nuvolari 2004); the silk industry in Lyon (Foray and Perez 2006); the Japanese cotton textile industry (Saxonhouse 1974); the paper industry in Berkshire, US (McGaw 1987); and the case of the Viennese chair (Kyriazidou and Pesendorfer 1999). In all these episodes, inventors shared their improvements with other inventors, which led to a fast technical advance.

One of the characteristics in common with OS is the presence of complementarities. For example, in the case of the iron industry in Cleveland, entrepreneurs were also owners or had mining rights of the mines in the Cleveland district. Improvements in the efficiency of blast furnaces lead to an increase in the value of the iron ore deposit. In the case of the Cornish engine, technical advances were publicized by mine managers, stimulated to do so by the owners of these mines.

OS has been used to develop software since the early years of computer science, but gained special relevance in the 1990s, with the success of Linux, Apache and Sendmail, among other programs. Software programmers started to develop software as OS to avoid the restrictions imposed by P firms on the access to the source code.

The participation of individual developers in OS is still very important, but the same is true for commercial firms. In the case of embedded Linux, for example, 73.5% of developers work for commercial firms and contribute 90% of the total investment in code (Henkel 2006). Lakhani and Wolf (2005) show that 55% of OS developers contribute code at work, and these programmers contribute 50% more hours than the rest. Lerner, Pathak, and Tirole (2006) show that around 30% of OS contributors work for commercial firms (however, they cannot identify non-US commercial contributors). Moreover, they show that commercial firms are associated with larger and more dynamic OS projects (commercial contributors have four times more sensitivity to the growth of the project).

The coexistence of OS and P in software markets is pervasive, as can be seen in Table 1. The server operating system market is a good example. According to IDC (2008), the market shares of server operating systems installed in new computer servers in 2008 were: Microsoft 38%, Unix 32.3%, Linux 13.7%, and other 16.1%. This shows that Linux has a significant market share in the market for server operating systems. However, there are reasons to think that Linux's market share is underestimated by IDC. First, the measurement is a flow, not a stock. Second, the operating system is very often changed by users in the years following the acquisition of a computer server and Linux is considered to run better on old computers. It is also interesting to notice that most Unix systems nowadays are also OS. If we sum the shares for Unix-like systems (Unix plus Linux), we get that OS operating systems have the largest share in the server operating systems market.

The decision to become OS is affected by dynamic factors. For example, the decision to open Netscape's source code was in part due to the loss of market share to Internet Explorer. However, it is important

Table 1: Coexistence of OS and P software.

Software	Open Source	Proprietary
Operating Systems	Linux, OpenSolaris	Windows
Web browsers	Mozilla/Netscape	Internet Explorer
Web servers	Apache	MS Internet Information Server
Mail servers	Sendmail	IBM Lotus Domino MS Exchange Server
Databases	MySQL, PostgreSQL	Oracle 11g, MS SQL Server
Content management	Plone	MS Sharepoint, Vignette
Application servers	JBoss, Zope	IBM WebSphere, MS .net
Blog publishing	WordPress	Windows Live Writer

to remark that in many opportunities, OS products were the first to be introduced in the market and then P products appeared. Moreover, OS and P firms coexist even in newly developed software markets, like application servers, blog publishing applications and content management systems.

We think static models like ours can be used to study the *equilibrium* industry structure in this kind of markets. In particular, there are several factors affecting the decision to become OS which can be explained in the context of a simple static model, like the way in which commercial OS firms profit from their collaborations, and the exact role of free-riding and duplication of effort in determining equilibrium market shares and cost of innovation.

Commercial firms participate in OS projects because they sell goods and services complementary to the software. For example, IBM provides support for over 500 software products running on Linux, and has more than 15,000 Linux-related customers worldwide.¹

The presence of complementarities in OS has been documented in recent empirical work. Henkel (2006) presents results from a survey of embedded Linux developers and show that 51.1% of developers work for manufacturers of devices, chips or boards and 22.4% work for specialized software companies. Dahlander (2005) finds that the dominant trend for appropriating the returns of innovation in OS is the sale of a complementary service. Fosfuri, Giarratana, and Luzzi (2008) show

¹www.ibm.com/linux/ (accessed May 12, 2009).

that firms with larger stock of hardware patents and hardware trademarks are more likely to participate in OS.

The sale of a complementary service can indeed be profitable. The case of Red Hat is illustrative. According to its financial statements, in fiscal year 2009 Red Hat invested \$130 million in R&D, and obtained \$652 million in revenues for its subscription and training services.

Many firms develop OS and P software at the same time. For example, IBM contributes code to Linux, but makes most of its software revenue in the middleware segment, where most of its programs are P. Even Microsoft is becoming increasingly open, with its participation in Cloud computing, for example. Our model can be used to address this issue, by noticing that the complementary good sold by OS firms may be a complementary P software. Another interesting issue would be to analyze multiproduct software firms, and to determine which software should be OS and which should be kept P. For the purposes of this paper, however, we concentrate in the analysis of a particular software segment, abstracting from the interactions with other segments.

OS licenses are the instruments guaranteeing the access of developers to the source code. Some licenses allow further modification of the source code without imposing any restriction on developers. Restrictive OS licenses, on the other hand, require the disclosure of further improvements to the source code when programs are distributed (programmers are still allowed to keep their innovations private if the program is for personal use). The most popular OS license is the General Public License (GPL), which is a restrictive license. The GPL is used by Linux, MySQL, Perl and Java, for example. It is true that some OS contributors disclose improvements to the source code even when these modifications are for personal use. However, restrictive licenses are the most important means for the success of OS projects. For example, the survey of embedded Linux developers finds that the main reason why developers disclose their contributions to the code is because they are forced to do so by the GPL (Henkel 2006).

1.2. Related literature. The first papers on OS were mainly concerned with explaining why individual developers contribute to OS projects, apparently for free (read Lerner and Tirole 2005, von Krogh and von Hippel 2006, for good surveys). The initial answers were altruism, personal gratification, peer recognition and career concerns. The motivations of commercial OS firms, on the other hand, have been studied less intensively.

Lerner and Tirole (2001, 2002, 2005) present a description of OS and identify directions for further research. Some of the questions related

with the present paper are: (i) what are the incentives of for-profit firms to participate in OS, (ii) what development model provides higher quality and welfare, and (iii) what is the influence of the competitive environment in OS. More importantly, these authors remark that direct competition between P and OS firms has received little attention.

Existing papers addressing competition between the two paradigms are duopoly models of a profit maximizing P firm and a community of not-for-profit OS developers, selling at marginal cost (Mustonen 2003, Bitzer 2004, Gaudeul 2005, Casadesus-Masanell and Ghemawat 2006, Economides and Katsamakas 2006). Introducing profit-seeking OS firms is important because it allows us to analyze the incentives to invest in R&D and the decision to become OS or P. Other papers introduce profit maximizing OS firms (Henkel 2004, Bessen 2006, Schmidtke 2006, Haruvy, Sethi, and Zhou 2008), but do not deal with direct competition between the two paradigms. Moreover, all these papers are exogenously assuming the market structure. The contributions of our paper are: (i) to present an analysis of direct competition between for-profit OS and P firms, (ii) when the decision to become OS or P is endogenous, and (iii) the market structure is determined endogenously as a result of firms' decisions.

Finally, this paper is also related to the literature of cooperation in R&D in Research Joint Ventures. A first strand of papers analyzed the effects of sharing R&D on the incentives to perform such investments (D'Aspremont and Jacquemin 1988, Kamien, Muller, and Zang 1992, Suzumura 1992). In particular, Kamien, Muller, and Zang show that free-riding incentives are so strong that a joint venture where firms share R&D but do not coordinate their R&D levels has a lower total investment than the individual investment of each one of these firms when there is no cooperation in R&D. We show that this result can be reversed if firms sell a complementary private good and the strength of the complementarity is high enough.

A second strand of papers analyzed the endogenous formation of research coalitions. Bloch (1995) presents a model in which firms decide sequentially whether to join the association or not, and compete in quantities after associations are formed. In equilibrium, two associations are formed. However, firms do not decide their optimal investments in R&D, so this model cannot be used to analyze the free-riding incentives created by association. Poyago-Theotoky (1995) and Yi and Shin (2000) assume that firms set their R&D levels cooperatively after associating. In this case, they show that firms in the joint venture invest more in R&D, and have higher profits than outsiders. We show this result is reversed when firms do not coordinate their R&D levels.

A third strand of papers analyzed the endogenous determination of spillovers among firms conducting R&D. Katsoulacos and Ulph (1998) show that firms selling complementary goods may choose maximal spillovers (i.e. decide to be OS), even when they take their decisions non-cooperatively. However, firms are not competing in the same industry. In our model, firms are direct competitors in the markets for the primary and complementary goods. Amir, Evstigneev, and Wooders (2003) present a duopoly model in which firms set cooperatively their R&D levels and the strength of the spillover. In their model, firms choose maximal spillovers, but this is due to the fact that they take their decisions cooperatively.

As can be seen, the literature of cooperation in R&D is an important precedent for our paper. Nevertheless, to the best of our knowledge, previous papers have not analyzed the case of endogenous formation of a coalition cooperating in R&D, when R&D levels are determined non-cooperatively. In particular, our contribution to this literature is the result that the equilibrium in which some firms decide to cooperate and others do not is characterized by an asymmetric market structure, where firms cooperating in R&D have smaller market shares.

2. THE MODEL

2.1. Technology. There are n firms selling packages composed of a primary good (which is potentially OS) and a complementary private good. Firms may improve the quality of both goods by investing in a single R&D technology. Let x_i be the investment in R&D of firm i . The cost of the investment is $c x_i$, which is a fixed cost, and the marginal cost of producing packages is zero.

The quality of the primary good depends on the investment of all firms in the project. For P firms, quality is simply $a_i = \ln(x_i)$. For OS firms, quality is $a_{os} = \ln(\sum_{i \in os} x_i)$.

The quality of the complementary good is $b_i = \ln(x_i)$ for all firms. There is a learning effect: firms improve the quality of their complementary good when they participate more in the development of the primary good. For example, if a software firm participates more in an OS project, it gains valuable knowledge and expertise and then can offer a better support service.

2.2. Preferences. There is a continuum of consumers. Each consumer has income y and buys only one package. Consumer j 's indirect utility from consuming package i is:

$$(1) \quad v_{ij} = \alpha a_i + \beta b_i + y - p_i + \varepsilon_{ij},$$

where α is the valuation of the quality of the primary good, β is the valuation of the quality of the complementary good, p_i is price, and ε_{ij} is an idiosyncratic shock (unobservable by firms) representing the heterogeneity in tastes between consumers. This specification for preferences allows for vertical (a_i and b_i) and horizontal (ε_{ij}) product differentiation.

Each consumer *observes* prices and qualities and then chooses the package that yields the highest indirect utility. The total mass of consumers is 1, so aggregate demands are equivalent to market shares. To obtain closed-form solutions for the demands we make the following assumption, which corresponds to the multinomial logit model (McFadden 1974, Anderson, De Palma, and Thisse 1992):

Assumption 1. *The idiosyncratic taste shocks ε_{ij} are i.i.d. according to the double exponential distribution:*

$$\Pr(\varepsilon_{ij} \leq z) = \exp(-\exp(-\nu - z/\mu))$$

where ν is Euler's constant ($\nu \approx 0.5772$) and μ is a positive constant.

Under Assumption 1, the market share (demand) of firm i is:

$$(2) \quad s_i = \frac{\exp\left(\frac{\alpha a_i + \beta b_i - p_i}{\mu}\right)}{\sum \exp\left(\frac{\alpha a_i + \beta b_i - p_i}{\mu}\right)}.$$

The ε_{ij} 's have zero mean and variance $\mu^2\pi^2/6$, hence μ measures the degree of heterogeneity between consumers. We will show that the equilibrium depends on two important relations:

$$\delta = \frac{\alpha + \beta}{\mu}, \quad \gamma = \frac{\alpha}{\alpha + \beta}.$$

δ measures the relative importance of vertical vs. horizontal product differentiation and γ represents the relative importance of the primary good vs. the complementary good (γ can also be interpreted as the degree of public good of the investment in R&D).

To guarantee the existence of a symmetric equilibrium we need enough horizontal differentiation relative to vertical differentiation. Let $\mu \geq \alpha + \beta$, which is a sufficient condition. Thus $\delta \in [0, 1]$.

2.3. Game and equilibrium concept. The model is a two-stage non-cooperative game. The players are the n firms. In the first stage firms decide their type (OS or P), and in the second stage they make their investment and price decisions (x_i, p_i) .

Given investments (quality) and prices, each consumer chooses her optimal package. These decisions are summarized by consumer demands (s_i) and embedded into the firms' payoffs: $\pi_i = s_i p_i - c x_i$.

The equilibrium concept is Subgame Perfect Equilibrium. We will only analyze symmetric equilibria, i.e. all firms deciding to be of the same type in the first stage will play the same equilibrium strategy in the second stage.

2.4. Modeling assumptions. In this section we discuss the main assumptions of the model, and the consequences of relaxing them.

Bundling and compatibility. We have assumed that firms sell packages (bundles) composed of one unit of the primary good and one unit of the complementary good. Under this assumption, the two goods become effectively one and each firm sets only one price, which greatly simplifies the model and allows us to focus on the effects of technology sharing on the decision to be P or OS.

Implicitly, we are assuming that (i) primary and complementary goods are perfect complements, and (ii) complementary goods designed for one primary good are incompatible with other primary goods. Under these assumptions, each consumer must choose a primary good and a complementary good from the same firm.

Industry examples and economic theory indicate that the incompatibility assumption may be a good description for several markets in which OS is important.

On the theory side, Matutes and Regibeau (1992) present a duopoly model to study compatibility and bundling decisions. Each firm sells two perfectly complementary components (consumers need one component of each kind), and components may be compatible or incompatible across firms. Matutes and Regibeau show there are equilibria where firms choose to make their components incompatible in order to commit to pure bundling.

There are many examples of incompatible components in the software industry. For example, Red Hat specializes in providing support services for Linux, and this support service has little value for Windows users (likewise, Microsoft's support service has little value for Linux users). Also, many applications that run in Mac OS X cannot run in Windows, (and many applications for the iPhone do not run on other mobile devices).

Clearly, the case of compatible components is also pervasive. For example, MS Office can be used in Macs, and Sun servers can run a variety of operating systems (although they are designed to run better on Sun's OpenSolaris). For this reason, in Section 8 we modify

the model to analyze the partial compatibility and full compatibility cases. Firms become multiproduct firms, and set separate prices for the primary and complementary goods. P firms obtain revenues from selling the primary and complementary goods, whereas OS firms only obtain revenues from selling the complementary good (the price of the primary good is zero).

For tractability, we assume there is only one P firm competing against several OS firms,² and we focus on the analysis of the equilibrium prices, investments and market shares as a function of the degree of compatibility.

We find that as goods become more compatible, the market share of the OS complementary goods falls. Also, the profit of the P firm increases relative to the profit of an OS firm. Therefore, OS will tend to perform better when the complementary good is more specific to the primary good, like in the case of support and training services, customizations, platform-specific software, and mobile devices (like MP3 players, PDAs or cell phones).

Investment in the complementary good. In the baseline model, firms cannot invest directly to increase the quality of the complementary good. Instead, the quality of the complementary good increases with individual investments in the primary good. This may happen because there is a learning effect (the support service of a firm which contributes more code to Linux is likely to be better than the support service of a firm that contributes less), or because firms contribute in areas of the primary good that are more relevant for their own complementary goods (a firm selling financial software will be more interested in developing Linux's mathematical capabilities, and will benefit more from these contributions than the rest).

As we will see, the assumption of a learning effect is *not essential* for our key result –the existence of an equilibrium with both kinds of firms, in which P firms have a larger market share and quality than OS– which obtains when the learning effect is small (even zero). Instead, the learning effect allows for the possibility that OS firms have higher quality and market share than P firms, and for the existence of equilibria with only OS firms, which will happen if the learning effect is large enough.

However, it is interesting to ask what would happen if firms could invest in the complementary good without contributing to the primary

²In general, models with multiproduct firms are usually very difficult to solve. They usually consist of a duopoly game, and only symmetric equilibria are analyzed, which limits applicability for the present case (n firms of two different types).

good. In Section 7 we extend the basic model to allow for this possibility. Now, the quality of the complementary good is a weighted average of the individual investment in the primary good (learning effect) and the direct investment in the complementary good. We show that as the importance of the direct investment increases relative to the learning effect, the incentive to be OS decreases, and the model converges to the equilibrium with both kinds of firms.

Demand specification. Logit demands follow from our assumption of the double exponential distribution for the idiosyncratic taste term. This distribution is similar to the normal distribution (which would yield the probit model when applied to equation (1)), but has the advantage of providing an analytically tractable demand system whereas the normal does not.³

We have assumed a specific demand structure due to the difficulty of analyzing asymmetric equilibria. While we are confident that our results will continue to hold with other models of product differentiation, the formal extension of our analysis to a general demand system is by no means straightforward. As a first step, it therefore seems reasonable to study a simple case, such as the logit, which provides explicit expressions for the demand functions.⁴

Given that taste shocks are distributed in the real line, every firm will have some consumers with strong preferences for its products, and therefore even firms with very low quality products will end up with some positive demand. It could be argued that this is the reason why OS firms may subsist in equilibrium, even when they have lower quality than P firms. However, this argument would be only relevant if OS firms were selling an extremely low quality product, which is never a optimal decision whenever: (i) Inada conditions hold for the quality

³The logit is a common model in discrete choice theory (see, for example Ben-Akiva and Lerman 1985), and has been widely used in econometric applications (see Train, McFadden, and Ben-Akiva 1987, and references therein), in marketing (McFadden 1986), and in theoretical work (Anderson, De Palma, and Thisse 1992, Besanko, Perry, and Spady 1990, Anderson and de Palma 1992, Anderson and Leruth 1993).

⁴Other alternatives for modelling an oligopoly with vertically and horizontally differentiated goods are the linear demand model and Salop's circular city with exogenous locations. We have studied the linear demands case, and have found that results still hold under this alternative specification. However, the analysis becomes much more complex. An appendix with the analysis of the linear demand case may be obtained from the authors upon request. The circular city model would add additional complications because the equilibrium would depend on whether OS and P firms were selling neighboring goods or not.

improving technologies, (ii) firms can appropriate some fraction of their investment.⁵

Finally, the assumption that the taste shocks are i.i.d. across packages implies that the differentiation between OS and P firms is symmetric. However, OS packages share the same primary good, so they are likely to be more similar than P packages. In Section 6 we introduce a nested logit model to introduce a difference in the substitutability between OS and P firms.

We find that as OS packages become more similar, the equilibrium number of firms in OS decreases. Depending on parameter values, the equilibrium may have only OS firms, only P firms, or both kinds of firms. If the difference in substitutability is high enough, all firms will decide to be P in equilibrium. This extension provides an important result: we should expect to see a higher proportion of OS firms in industries where firms have more possibilities to differentiate their complementary goods.

3. SOLUTION OF THE MODEL

3.1. Second stage. Let n_{os} be the number of firms deciding to be OS in the first stage. In the second stage, firms choose p_i and x_i to maximize $\pi_i = s_i p_i - c x_i$, taking as given the demands and the decisions of other firms.

Working with the first order conditions and imposing symmetry we get the optimal price:

$$(3) \quad p_i = \frac{\mu}{1 - s_i},$$

and the optimal investment in R&D for OS and P firms:

$$(4) \quad x_{os} = \frac{\alpha + \beta}{c} s_{os} \left(1 - \gamma \frac{n_{os} - 1}{n_{os}(1 - s_{os})} \right),$$

$$(5) \quad x_p = \frac{\alpha + \beta}{c} s_p.$$

The term inside the parenthesis of (4) represents free-riding: assuming $s_{os} = s_p$, OS have less incentives to invest than P because they can appropriate a smaller fraction of their investment. In other words, there is a public good problem given that OS are sharing their technological advances.

⁵This condition holds if there is at least one P firm or $\beta > 0$. There is zero appropriability only if all firms are OS and $\beta = 0$, but in this case the comparison between OS and P quality is irrelevant.

From (2), we can get the ratio of market shares s_{os}/s_p . Introducing equations (3) to (5), taking logs and rearranging terms we get:

$$(6) \quad (1 - \delta) \ln \left(\frac{s_{os}}{s_p} \right) + \frac{1}{1 - s_{os}} - \frac{1}{1 - s_p} \\ = \delta \ln \left(1 - \gamma \frac{n_{os} - 1}{n_{os}(1 - s_{os})} \right) + \delta \gamma \ln(n_{os}).$$

This equation says that the difference in market shares depends on the resolution of the conflict between *free-riding* and *duplication of effort*. To see this, notice that the left hand side is increasing in s_{os} and decreasing in s_p , so the difference in market shares will increase if the right hand side does. The first term on the right hand side is just the difference between x_{os} and x_p (free-riding). The second term is a multiplicative effect due to the elimination of the duplication of effort in OS (collaboration effect).

The second-stage equilibrium is completely characterized by (6) and the condition that the sum of the market shares is equal to 1:

$$(7) \quad n_{os} s_{os} + (n - n_{os}) s_p = 1.$$

Proposition 1. *A second-stage equilibrium exists and is unique. Given n_{os} , the equilibrium market shares solve (6) and (7).*

In what follows we study the comparative statics of the second-stage equilibrium. In Lemma 1 we present a simple condition to determine which kind of firm will have higher market share (quality and price).

Lemma 1. *$s_p > s_{os}$ if $\gamma > \hat{\gamma}(n_{os}, n)$, and $s_p < s_{os}$ in the opposite case, where $\hat{\gamma}(n_{os}, n)$ is increasing in n_{os} and n and solves:*

$$\gamma \frac{n_{os}^\gamma}{n_{os}^\gamma - 1} \frac{n_{os} - 1}{n_{os}} = \frac{n - 1}{n}.$$

The comparison of prices and quality is equivalent to the comparison of market shares: if $s_{os} > s_p$, then $p_{os} > p_p$ and $\alpha a_{os} + \beta b_{os} > \alpha a_p + \beta b_p$, and vice versa. Lemma 1 provides an important result: as n_{os} or n increase, it is more likely that OS firms will have higher market share (quality and price) than P firms.

Lemmas 2 and 3 analyze the effects of changes in δ and γ on s_{os} . The effects on s_p have the opposite sign.

Lemma 2. *s_{os} is increasing in δ if $\gamma < \hat{\gamma}(n_{os}, n)$, and decreasing in δ in the opposite case.*

Lemma 2 has a clear interpretation. When δ increases, vertical differentiation gets more important relative to horizontal differentiation. This means that investing in R&D has a larger effect on demand, which benefits firms with higher quality products. If $\gamma < \hat{\gamma}$, then the firms with a higher quality product are the OS firms, and therefore, their market share increases relative to the market share of the P firms. The opposite happens when $\gamma > \hat{\gamma}$.

Lemma 3. *There exists $\gamma_d \in (0, \hat{\gamma})$ such that s_{os} is increasing in γ for $\gamma < \gamma_d$, and decreasing in γ for $\gamma > \gamma_d$.*

Lemma 3 implies that the graph of s_{os} with respect to γ (the degree of public good of the investment) is hump-shaped. For low values of γ , collaboration dominates free-riding (investment is mostly private), so s_{os} is increasing in γ . For high values of γ , free-riding dominates collaboration and s_{os} is decreasing in γ .

3.2. First stage. In the first stage of the game, firms decide whether to be OS or P, taking as given the decisions of the rest of firms and forecasting their equilibrium payoffs in the second stage. Let $\pi(n_{os})$ be the second stage equilibrium payoffs when n_{os} firms decide to be OS. Replacing the second stage equilibrium values of prices and investments for both kinds of firms we get:

$$(8) \quad \pi_{os}(n_{os}) = \mu \frac{s_{os}}{1 - s_{os}} \left(1 - \delta(1 - s_{os}) + \delta\gamma \frac{n_{os} - 1}{n_{os}} \right),$$

$$(9) \quad \pi_p(n_{os}) = \mu \frac{s_p}{1 - s_p} (1 - \delta(1 - s_p)),$$

where $s_{os} = s_{os}(n_{os})$ and $s_p = s_p(n_{os})$ are the second stage equilibrium market shares. Equilibrium profits are always positive, given that δ , γ , s_{os} and s_p are all between 0 and 1. Comparing equations (8) and (9), we can see the direct effect of collaboration in profits, which is the saving in the investment cost of OS firms (third term inside the parenthesis of the first equation).

A number n_{os} of firms in OS is an equilibrium if and only if $\pi_{os}(n_{os}) \geq \pi_p(n_{os} - 1)$ and $\pi_p(n_{os}) \geq \pi_{os}(n_{os} + 1)$. These conditions are what D'Aspremont, Jacquemin, Gabszewicz, and Weymark (1983) called internally stable and externally stable coalition conditions. The first inequality says that firms deciding to be OS cannot gain by deviating and becoming P. The second inequality is a similar condition on the decision of being P. The equilibrium conditions can be summarized by

the function $f(n_{os}) = \pi_{os}(n_{os}) - \pi_p(n_{os}-1)$:

$$f(n_{os}) = \mu \frac{s_{os}}{1-s_{os}} \left(1 - \delta(1-s_{os}) + \delta\gamma \frac{n_{os}-1}{n_{os}} \right) - \mu \frac{\tilde{s}_p}{1-\tilde{s}_p} (1 - \delta(1-\tilde{s}_p)),$$

where $s_{os} = s_{os}(n_{os})$ and $\tilde{s}_p = s_p(n_{os}-1)$. Using this function, the equilibrium conditions can be restated as $f(n_{os}) \geq 0$ and $f(n_{os}+1) \leq 0$.

The equilibrium may be such that both kinds of firms coexist (interior equilibrium) or all firms choose to be of the same kind. $n_{os} = 0$ is always an equilibrium. For $n_{os} = 1$ to be an equilibrium we need $f(2) \leq 0$. Likewise, for $n_{os} = n$ to be an equilibrium we need $f(n) \geq 0$.

Proposition 2. *A Subgame Perfect Equilibrium for the two-stage game exists and is unique.*

Figure 1 shows an example of the $f(n_{os})$ schedule for $\gamma = 1$, $\delta = 0.9$, $\mu = 1$ and $n = 10$. In this case, the equilibrium has 6 firms in OS.

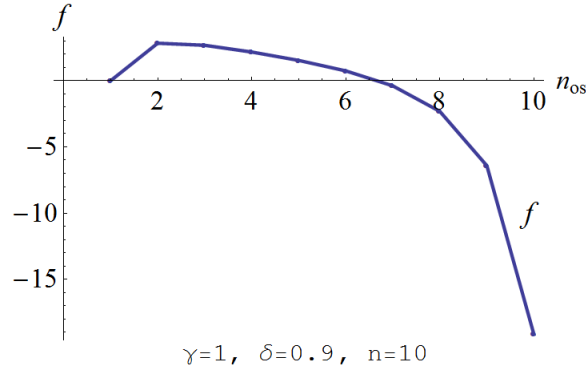


Figure 1: Equilibrium number of firms in OS.

When firms choose between OS or P, they compare the relative benefits of collaboration and secrecy. There are two elements associated with this trade-off. On one hand, free-riding and collaboration affect the equilibrium market shares, as analyzed in Section 3.1. On the other hand, OS firms have a lower investment cost. Being P will be more profitable than being OS only if free-riding is sufficiently strong as to overcome the positive effects of collaboration.

The following proposition characterizes the subgame perfect equilibrium of the two-stage game, depending on the value of γ .

Proposition 3. *Given $n > 3$ and δ , there exist $0 < \bar{\gamma} < \hat{\gamma} < 1$, such that in equilibrium:*

- i. *If $\gamma > \hat{\gamma}$, both kinds of firms co-exist and P have higher quality and market share than OS.*
- ii. *If $\bar{\gamma} < \gamma \leq \hat{\gamma}$, all firms decide to be OS, but a P firm would have higher quality and market share.*
- iii. *If $\gamma \leq \bar{\gamma}$, all firms decide to be OS, and a P firm would have lower quality and market share.*

Proposition 3 is the most important result of the paper. It shows there are three kinds of equilibria. For high γ the degree of public good of the investment in R&D is high. In this case, there is an interior equilibrium with both kinds of firms, where the quality of P goods is higher than that of OS goods. For intermediate values of γ , all firms decide to be OS. However, if one of the firms would become P, it would produce a good of higher quality than the OS firms. This means that OS is preventing the entry of a product of better quality. Finally, for low γ the public good problem is not very important, so all firms decide to be OS, but OS quality is higher than that of a potential deviator.

Figure 2 shows the regions corresponding to the three equilibria for different values of n and γ , and for $\delta = 1$. The area corresponding to equilibria with coexistence first increases but then decreases with n . This means that large numbers favor cooperation, even without coordination of individual investments.

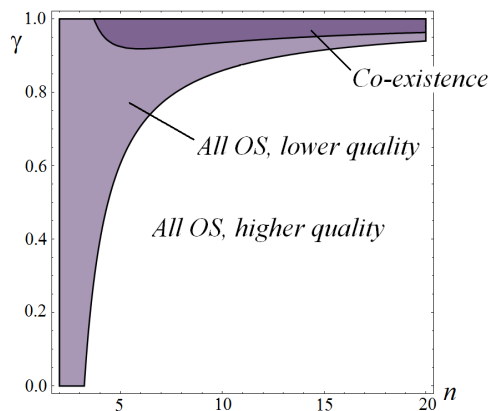


Figure 2: Equilibrium regions.

3.3. Open membership and OS licenses. In this section we analyze the effects of changes in n_{os} on profits. Figure 3 shows the profit

schedules of P and OS firms for $\gamma = 1$, $\delta = 1$ and $n = 10$. We can see that π_{os} increases, keeps approximately constant for some values of n_{os} and finally increases again. Interestingly, at the equilibrium ($n_{os} = 6$), the profits of OS firms are increasing in n_{os} , which means that OS firms would not find it profitable to limit access to the OS project. This gives a rationale for OS licenses, such as the GPL, guaranteeing open membership in OS projects (as we explained above, the only restriction in the GPL is that whenever modifications to the program are distributed, they have to be made available to the rest of developers in the project).

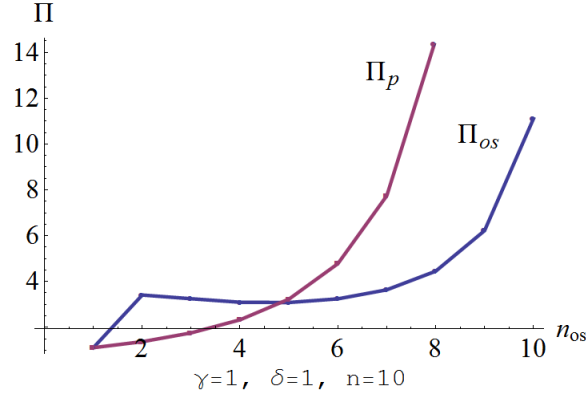


Figure 3: Firm profits as functions of n_{os} .

Simulations show that the profit of OS firms is increasing in n_{os} at the equilibrium for any γ and δ . Interestingly, the profit of P firms is also increasing when the equilibrium has both kinds of firms. This means that both OS and P prefer to compete against OS firms rather than P firms.

Notice that the result holds even though OS firms are direct competitors in the markets for the primary and complementary goods. If the firms were not direct competitors but were benefiting from the development of the OS good, the result would be even stronger. However, the result also depends on the fact that the number of firms in the industry is fixed. If free-entry into the project would stimulate the entry of new firms in the industry the result could be reversed.

4. WELFARE ANALYSIS

One of the advantages of the logit model is that it can be used to construct a representative consumer whose utility embodies the aggregate behavior of the continuum of users (Anderson, De Palma, and Thisse 1992).

Let s_i be the quantities of each variety consumed by the representative consumer, and let $\sum s_i = 1$. Total income is y and s_0 represents consumption of the numeraire. The utility of the representative consumer is

$$U = \sum (\alpha a_i + \beta b_i) s_i - \mu \sum s_i \ln(s_i) + s_0.$$

This utility embodies two different effects. The first term represents the direct effect from consumption of the n varieties, in the absence of interactions. The second term introduces an entropy-effect, which expresses the preference for variety of the representative consumer.

The utility function is quasilinear, which implies transferable utility. Thus, social welfare is the sum of consumer utility and firm profits:

$$(10) \quad W = \sum (\alpha a_i + \beta b_i) s_i - \mu \sum s_i \ln(s_i) + y - \sum c x_i.$$

The Social Planner's problem is to maximize (10) subject to $\sum s_i = 1$. It is obvious that the Social Planner would have all the firms sharing their improvements to the primary good. Also, given the concavity and symmetry of the utility function, the social planner will set $s_i = 1/n$ for all i . To determine the optimal investment, the Social Planner maximizes

$$W = \alpha \ln(n x^*) + \beta \ln(x^*) + \mu \ln(n) + y - n c x^*,$$

which leads to an optimal investment equal to $x^* = (\alpha + \beta)/cn$.

Product quality is suboptimal regardless of the number of OS and P firms: OS is subject to free-riding, which leads to a suboptimal investment in R&D, but P firms do not share their improvements on the primary good, generating an inefficient duplication of effort.

4.1. Government policy. Now we turn to an analysis of government policy. We will show that the first best can be achieved by using a tax-subsidy scheme. The cost of R&D for OS is $c_{os} = (1 - \kappa) c$, where κ is a proportional subsidy on the investment of OS firms. This subsidy, in turn, is financed by proportional or lump-sum taxes paid by consumers.

The ratio of investments of OS and P firms is

$$\frac{x_{os}}{x_p} = (1 - \kappa)^{-1} \frac{s_{os}}{s_p} \left(1 - \gamma \frac{n_{os} - 1}{n_{os}} \frac{1}{1 - s_{os}} \right),$$

and the equation characterizing the equilibrium market shares becomes

$$\begin{aligned} (1 - \delta) \ln \left(\frac{s_{os}}{s_p} \right) + \frac{1}{1 - s_{os}} - \frac{1}{1 - s_p} \\ = \delta \ln \left(1 - \gamma \frac{n_{os} - 1}{n_{os}(1 - s_{os})} \right) + \delta \gamma \ln(n_{os}) - \delta \ln(1 - \kappa). \end{aligned}$$

An increase in the subsidy increases the difference in investments and market shares between OS and P, and also decreases the cost of investment for OS, so firms are more tempted to become OS. The following lemma is equivalent to Lemma 1 and shows that if the subsidy is high enough, OS firms will have a higher market share than P in a second stage equilibrium.

Lemma 4. *$s_{os} > s_p$ in a second-stage equilibrium if*

$$\kappa > 1 - \left(1 - \gamma \frac{n}{n-1} \frac{n_{os} - 1}{n_{os}} \right) n_{os}^\gamma,$$

and $s_p > s_{os}$ in the opposite case.

In particular, if $\kappa > 1 - (1 - \gamma n(n-2)(n-1)^{-2})(n-1)^\gamma$, then $s_{os} > s_p$ for $n_{os} = n-1$, and therefore all firms want to be OS.

The following proposition shows the optimal policy.

Proposition 4. *The optimal subsidy is $\kappa^* = \gamma$, which attains the first best levels of investment. In equilibrium, all firms decide to be OS.*

The subsidy has a double effect: it increases the investment of OS firms, and it encourages P firms to become OS (to share R&D). The optimal subsidy is increasing in the degree of public good of the investment in R&D. In other words, the subsidy should be higher for projects where the valuation of the complementary good is not very high.

Note that in our model, lump-sum or proportional taxes are equivalent. This is because each consumer buys one product, and therefore proportional taxes do not affect quantities sold. This is why financing the subsidy with proportional taxes does not cause a deadweight loss and the policy-maker can achieve the first best.

5. INITIAL ASYMMETRIES

We have assumed firms are ex-ante symmetric, which allowed us to concentrate on ex-post differences arising endogenously in the model. However, it would be interesting to analyze what happens when there are initial asymmetries, which could be due to initial differences in the stock of R&D or installed base.

Specifically, suppose all firms are initially P and have different stocks of R&D. Firms can increase quality by investing in R&D, and have to choose to become OS or remain P. Firms deciding to become OS will have to share not only their current investment in R&D, but also their initial stocks.

For P firms, initial differences will persist ex-post. Larger firms may invest more or less than smaller firms, but will finish with a larger stock of R&D. Firms deciding to be OS, on the other hand, will tend to be more similar because they have to share their initial R&D stocks. This means that larger firms will have less incentives to become OS. In equilibrium, firms deciding to be OS will be smaller ex-ante and ex-post. The reason is twofold: larger firms ex-ante have more incentives to remain P, and P firms have more incentives to maintain a larger stock of R&D ex-post.

6. LOWER DIFFERENTIATION FOR OS PRODUCTS

Given that OS packages share the same primary good, they are likely to be more similar than P packages. To introduce this difference in the degree of substitutability, we use a nested logit model (Ben-Akiva 1973). This adds an element of *endogenous* horizontal differentiation to the trade-off between collaboration and secrecy. By becoming P, firms get their product more differentiated in comparison with OS firms.

The main consequences are that (i) the equilibrium number of firms in OS will be smaller than in the previous model, (ii) there are equilibria with only P firms, and (iii) there are parameter values leading to multiple equilibria.

Consumers are heterogeneous in two different dimensions: they have idiosyncratic tastes for the primary good and idiosyncratic tastes for the complementary good. Differences in substitutability will be driven by the relative strength of these two forces. Following the nested logit representation of Cardell (1997), consumer j 's indirect utility from consuming package i , based on primary good k is:

$$v_{ij} = \alpha a_k + \beta b_i + y - p_i + \sigma \eta_{kj} + (1 - \sigma) \varepsilon_{ij},$$

where η_{kj} is a primary good idiosyncratic component, and $\sigma \in [0, 1]$ weighs the different idiosyncratic components. Assumption 2 replaces Assumption 1 for the standard logit case.

Assumption 2. *The idiosyncratic components ε_{ij} , corresponding to complementary good i , are i.i.d. according to the double exponential distribution with scale parameter μ . The idiosyncratic components η_{kj} , corresponding to primary good k , are i.i.d. according to a distribution*

such that $\sigma \eta_{kj} + (1 - \sigma) \varepsilon_{ij}$ is distributed double exponential with scale parameter μ .

Assumption 2 implies that the horizontal differentiation term $\sigma \eta_{kj} + (1 - \sigma) \varepsilon_{ij}$ has the same distribution than ε_{ij} in the previous model. Cardell shows there is a unique distribution for η_{kj} such that Assumption 2 holds.

The parameter σ determines the relative strength of the horizontal differentiation forces. As σ increases, consumers get more differentiated in their tastes for the primary good, and less differentiated in their tastes for the complementary good. When $\sigma = 0$ consumers only have idiosyncratic preferences for the complementary good, and the model becomes the standard logit model of previous sections. When $\sigma = 1$ consumers only have idiosyncratic preferences for the primary good, and all OS firms sell a homogeneous good.

The proportion of consumers choosing OS variant i can be decomposed in the following way:

$$(11) \quad s_i = s_{i|os} S_{os},$$

where S_{os} is the aggregate market share of the OS primary good, and $s_{i|os}$ is the share of OS variant i within the OS project.

Under Assumption 2, i 's market share within the OS project depends only on the relative differences in the quality of the complementary good:

$$s_{i|os} = \frac{\exp\left(\frac{\beta b_i - p_i}{(1 - \sigma)\mu}\right)}{\sum_{i \in OS} \exp\left(\frac{\beta b_i - p_i}{(1 - \sigma)\mu}\right)}.$$

The aggregate market share S_{os} depends on the average value of the OS varieties (the expected value of the maximum of the utilities), V_{os} :

$$S_{os} = \frac{\exp(V_{os}/\mu)}{\exp(V_{os}/\mu) + \sum_{i \in P} \exp((\alpha + \beta b_i - p_i)/\mu)},$$

$$V_{os} = (1 - \sigma)\mu \ln\left(\sum_{i \in OS} \exp\left(\frac{\alpha a_i + \beta b_i - p_i}{(1 - \sigma)\mu}\right)\right).$$

The market share of a P firm is simply:

$$(12) \quad s_i = \frac{\exp\left(\frac{\alpha a_i + \beta b_i - p_i}{\mu}\right)}{\exp\left(\frac{V_{os}}{\mu}\right) + \sum_{i \in P} \exp\left(\frac{\alpha a_i + \beta b_i - p_i}{\mu}\right)},$$

This is because P nests are composed only by one P product (i.e. the average value of the nest is the value of that product).

The optimal price and investment of P firms have the same functional forms as before. The optimal price and investment of OS firms become:

$$(13) \quad p_{os} = \frac{\mu}{1 - s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}},$$

$$(14) \quad x_{os} = \frac{\alpha + \beta}{c} s_{os} \left(1 - \frac{\gamma}{1 - \sigma} \frac{(n_{os}-1)/n_{os}}{\left(1 - s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}\right)} \right).$$

From (12) and (11), we obtain the ratio of market shares s_{os}/s_p . Introducing prices and investments, taking logs and rearranging terms we get:

$$(15) \quad (1-\delta) \ln \left(\frac{s_{os}}{s_p} \right) + \frac{1}{1 - s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}} - \frac{1}{1 - s_p} \\ = \delta \ln \left(1 - \frac{\gamma}{1 - \sigma} \frac{(n_{os}-1)/n_{os}}{\left(1 - s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}\right)} \right) + (\delta \gamma - \sigma) \ln (n_{os}).$$

As in the standard logit case, to guarantee the existence of a symmetric equilibrium we need enough horizontal differentiation relative to vertical differentiation. We assume $\mu(1-\sigma) \leq \alpha + \beta$, which is a sufficient condition. This implies that $\sigma \leq 1 - \delta$. Proposition 5 summarizes the equilibrium of the second-stage of the game.

Proposition 5. *A second-stage equilibrium for the nested model exists and is unique. Given n_{os} , the equilibrium market shares solve (15) and (7).*

Comparing equations (6) and (15), we can see that the higher substitutability between OS varieties has three effects on equilibrium market shares. First, there is a lower investment of OS due to the lower return to investment (first term on the right hand side of (15)). Second, there is a direct negative effect on the average value of the complementary good (second term on the right hand side of (15)). Consumers care for variety, and therefore the value of choosing an OS package decreases when the complementary good becomes less differentiated. Third, OS firms will set a lower price in equilibrium because of higher substitutability (second term on the left hand side of (15)). The first two effects tend to reduce the market share of OS relative to P, and the third effect tends to increase it.

To solve the first stage of the game, we calculate $f(n_{os}) = \pi_{os}(n_{os}) - \pi_p(n_{os} - 1)$, where $\pi(n_{os}) = p_i s_i - c x_i$. The equilibrium conditions are the same than in the standard logit case.

Figure 4 shows the graph of $f(n_{os})$ for different parameter values. There are three interesting observations to be made. First, as σ increases for given γ and δ (OS varieties become more similar), the equilibrium number of firms in OS decreases (Figure 4a). Second, if σ is high enough, there are equilibria with only P firms (Figure 4b). Third, for some parameter values the model exhibits multiple equilibria (in Figure 4c there is an equilibrium with $n_{os} = 2$ and another with $n_{os} = 10$). In this case, there may be a coordination problem if the OS project fails to attract a large number of contributors.

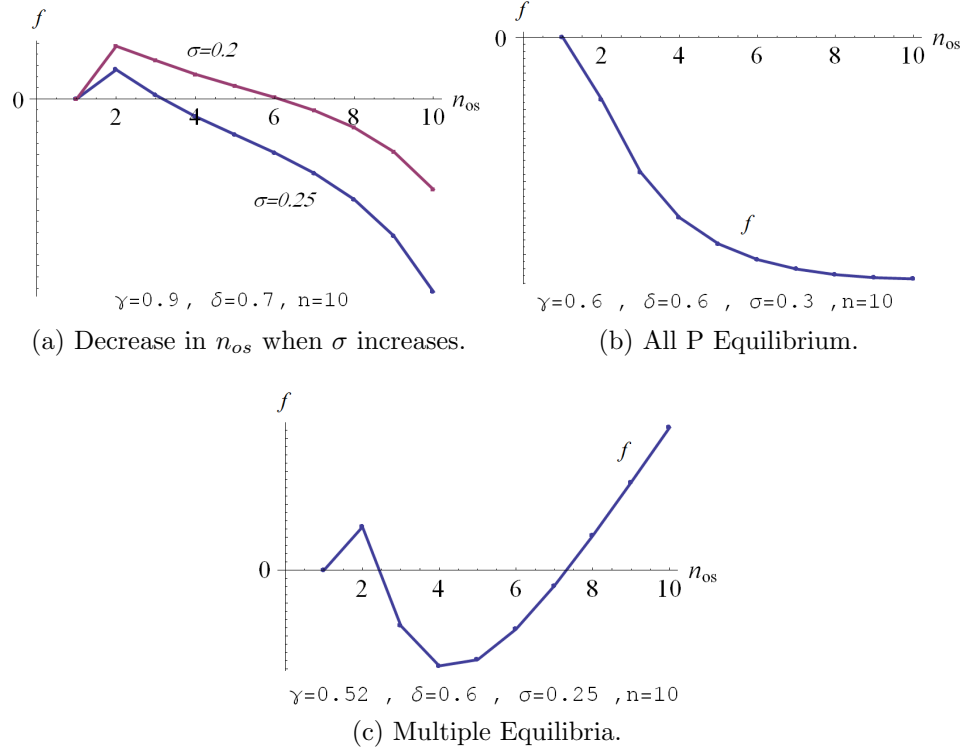


Figure 4: Equilibrium of the nested logit model.

Figure 5 shows the equilibrium regions for different values of γ and σ , given $\delta = 0.6$, $\mu = 1$ and $n = 10$ (in case of multiple equilibria we take the equilibrium with highest n_{os}). OS will subsist if the differentiation between OS varieties is high enough (σ is low enough). Also, there are

values of γ such that as σ increases the equilibrium goes from all OS, to coexistence and then to all P. For coexistence, we need a combination of low σ and high γ . Finally, our simulations show that whenever OS and P coexist, the quality and market share of P firms is larger than that of OS firms, which means that the main result of the paper still holds.

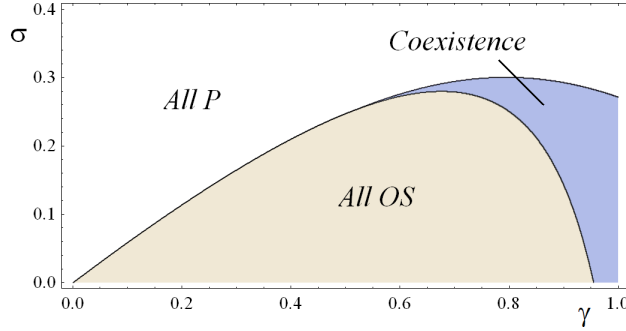


Figure 5: Equilibrium regions for the nested logit.

7. DIRECT INVESTMENT IN THE COMPLEMENTARY GOOD

In the baseline model, the quality of the complementary good is determined by individual investments in the primary good (x_i). This means that firms cannot increase the quality of the complementary good without increasing the quality of the primary good at the same time.

In this section, we extend the basic model to allow for direct investment in the complementary good. The indirect utility of consumer j from consuming package i is still:

$$v_{ij} = \alpha a_i + \beta b_i + y - p_i + \varepsilon_{ij},$$

where $a_i = \ln(x_i)$ for P firms and $a_{os} = \ln(\sum_{i \in os} x_i)$ for OS firms, but now, $b_i = \omega \ln(x_i) + (1 - \omega) \ln(z_i)$ for all firms, where z_i is the direct investment in the complementary good, and $\omega \in [0, 1]$ measures the importance of the learning effect vis-a-vis the direct investment. The total cost of the investment is $c(x_i + z_i)$.

Demands are still given by (2), and firms maximize $\pi = p_i s_i - c(x_i + z_i)$. Equilibrium price is:

$$p_i = \frac{\mu}{1 - s_i}$$

for all firms, and equilibrium investments in the primary good for OS and P firms are:

$$\begin{aligned} x_{os} &= \frac{\alpha + \beta\omega}{c} s_{os} \left(1 - \frac{\alpha}{\alpha + \beta\omega} \frac{n_{os} - 1}{n_{os}(1 - s_{os})} \right), \\ x_p &= \frac{\alpha + \beta\omega}{c} s_p. \end{aligned}$$

Finally, equilibrium investment in the complementary good is:

$$z_i = \frac{\beta(1 - \omega)}{c} s_i$$

for all firms.

An increase in the importance of direct investment with respect to the learning effect will decrease the optimal investment in the primary good and increase the optimal investment in the complementary good, for given s_i . Substituting prices and investments in the ratio of market shares, taking logs and rearranging terms we get:

$$\begin{aligned} (16) \quad & (1 - \delta) \ln \left(\frac{s_{os}}{s_p} \right) + \frac{1}{1 - s_{os}} - \frac{1}{1 - s_p} \\ & = \delta (\omega + (1 - \omega)\gamma) \ln \left(1 - \frac{\gamma}{\omega + (1 - \omega)\gamma} \frac{n_{os} - 1}{n_{os}(1 - s_{os})} \right) + \delta \gamma \ln(n_{os}), \end{aligned}$$

where γ and δ are defined as in Section 2.2. Notice that we can get the baseline model by making $\omega = 1$ in equation (16). As ω decreases, the market share of OS firms decreases for given n_{os} . Thus, the change in the composition of investments as ω decreases has a higher impact on OS firms because of the lower appropriability of the investment in the primary good.

To solve the first stage of the game, we calculate $f(n_{os}) = \pi_{os}(n_{os}) - \pi_p(n_{os} - 1)$, where $\pi(n_{os}) = p_i s_i - c x_i$. The equilibrium conditions are the same than in the baseline model.

Figure 6 shows the equilibrium for $\gamma = 0.9$, $\delta = 0.7$, $n = 10$, and different values of ω . We can see that as ω decreases, the equilibrium number of firms in OS decreases. In the limit, when $\omega = 0$, the equilibrium has both kinds of firms (coexistence). Simulations show that this result holds for any value of δ and γ .

The analysis of this section shows the effects of allowing firms to invest solely in the complementary good. Direct investment in the complementary good is fully appropriable. As the importance of this kind of investment relative to the learning effect increases, the quality differential between OS and P firms decreases, reducing the incentives to participate in OS. However, the main result of the paper still holds: there are parameter values for which both kinds of firms coexist in

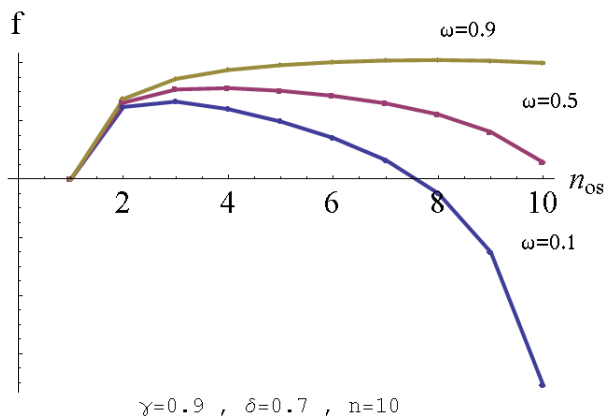


Figure 6: Equilibria with direct investment in the complementary good.

equilibrium, and these equilibria are characterized by an asymmetric market structure, with a few large P firms and many small OS firms.

8. COMPATIBILITY BETWEEN OS AND P

In previous sections, we assumed that complementary goods could only be used with the primary good for which they were developed. In this section, we analyze what happens when complementary goods can be combined with any primary good, that is, when OS and P goods are compatible.

A direct implication is that primary and complementary goods will have separate prices. Let r_k be the price of primary good k , and p_i be the price of complementary good i . P firms can set a positive price for both goods. OS firms, on the other hand, are selling an homogeneous primary good, and price competition implies that $r_{os} = 0$. Therefore, OS firms can set a positive price only for the complementary good.

As a result of compatibility, the model becomes highly complex. Therefore, to keep the model tractable, we will assume there is only one P firm, competing against n_{os} OS firms. In other words, we will fix the number of OS and P firms and we will focus on studying equilibrium investment and pricing decisions on the second stage of the previous models.

The model is based on the nested logit. There are two nests, one corresponding to users of the OS primary good, and the other corresponding to users of the P primary good. Within each nest, consumers can choose between the complementary goods of all firms.

The quality of the OS primary good is $a_{os} = \ln(\sum_{i \in os} x_i)$ and the quality of the P primary good is $a_p = \ln(x_i)$, where x_i are individual investments in the primary good.

The quality of the complementary good is determined by a learning effect as in the basic model, but this learning effect is discounted when the complementary good is used with the primary good for which it was not developed. Specifically, the quality of complementary good i developed for primary good k is $b_{ik} = \ln(x_i)$ when used with k , and $b_{ik'} = \ln(\theta x_i)$ when used with $k' \neq k$.

$\theta \in [0, 1]$ measures the degree of compatibility between the primary and complementary goods of different firms. When $\theta = 0$, goods of different firms are incompatible, and we obtain the model of previous sections. $0 < \theta < 1$ implies partial compatibility, and $\theta = 1$ implies full compatibility.

Consumer j 's indirect utility from consuming good i with primary good k is:

$$v_{kij} = \alpha a_k + \beta b_{ik} + y - r_k - p_i + \sigma \eta_{kj} + (1 - \sigma) \varepsilon_{ij}.$$

The distributions of the taste shocks η_{kj} and ε_{ij} are given in Assumption 2.

Let S_{os} and S_p be the market shares of the OS and P primary goods. The market share of complementary good i can be decomposed in the following way:

$$s_i = S_{os} s_{i|os} + S_p s_{i|p},$$

where $s_{i|os}$ is the market share of firm i inside the OS nest, and $s_{i|p}$ is the market share of firm i inside the P nest.

The market share of complementary good i inside nest k is:

$$s_{i|k} = \frac{\exp\left(\frac{\beta b_{ik} - p_i}{(1 - \sigma)\mu}\right)}{\sum \exp\left(\frac{\beta b_{ik} - p_i}{(1 - \sigma)\mu}\right)}.$$

As in Section 6, the market share inside a nest depends only on the relative quality and price of the different complementary goods. This is true for OS and P firms. The only difference is that now all firms sell complementary goods in both nests.

The market share of the OS primary good is:

$$S_{os} = \frac{\exp(V_{os}/\mu)}{\exp(V_{os}/\mu) + \exp(V_p/\mu)},$$

and $S_p = 1 - S_{os}$, where V_{os} and V_p are the average values of the complementary goods within each nest:

$$\begin{aligned} V_{os} &= \alpha a_{os} + \mu(1 - \sigma) \ln \left(\sum \exp \left(\frac{\beta b_{ios} - p_i}{(1 - \sigma)\mu} \right) \right), \\ V_p &= \alpha a_p - r + \mu(1 - \sigma) \ln \left(\sum \exp \left(\frac{\beta b_{ip} - p_i}{(1 - \sigma)\mu} \right) \right). \end{aligned}$$

The profit of OS firms is $\pi_{os} = s_i p_i - c x_i$, whereas the profit of the P firm is $\pi_p = s_p p_p + S_p r_p - c x_p$. Notice that when $\theta = 0$, $S_p = s_p$ and the P firm chooses a single price equal to $p_p + r_p$, as in previous sections. As in Section 6, we assume $\sigma < 1 - \delta$, which guarantees the quasiconcavity of the maximization problem of OS and P firms.

The optimal price and investment for OS firms are:

$$\begin{aligned} p_{os} &= \frac{\mu(1 - \sigma)}{1 - s_{os} - \sigma S_{os} S_p \frac{(s_{i|os} - s_{i|p})^2}{s_{os}}} \\ x_{os} &= \frac{\alpha}{c} S_{os} S_p (s_{i|os} - s_{i|p}) \frac{p_{os}}{n_{os} \mu} + \frac{\beta}{c} s_{os}. \end{aligned}$$

The optimal price for the primary and complementary good of the P firm, and its optimal investment are:

$$\begin{aligned} p_p &= \frac{\mu(1 - \sigma) s_{i|os}}{S_{os} s_{i|os} (1 - s_{i|os}) + S_p s_{i|p} (1 - s_{i|p})} \\ r_p &= \frac{\mu}{(1 - S_p)} - (s_{i|p} - s_{i|os}) p_p \\ x_p &= \frac{\alpha}{c} S_p + \frac{\beta}{c} s_p; \end{aligned}$$

Replacing optimal prices and investments into the market share equations, and noticing that $S_p = 1 - S_{os}$, $s_{p|os} = 1 - n_{os} s_{os|os}$, and $s_{p|p} = 1 - n_{os} s_{os|p}$, we can get a system of three equations with three unknowns characterizing the equilibrium market shares. These equilibrium expressions are difficult to analyze because of their analytical complexity. Therefore, it is useful to begin by analyzing the full compatibility case ($\theta = 1$), which provides a tractable set of equilibrium conditions.

8.1. Full compatibility. When $\theta = 1$, $s_i = s_{i|os} = s_{i|p}$ and equilibrium prices and investments become:

$$\begin{aligned} p_{os} &= \frac{\mu(1-\sigma)}{1-s_{os}}, \\ p_p &= \frac{\mu(1-\sigma)}{1-s_p}, \\ r_p &= \frac{\mu}{1-S_p}, \\ x_{os} &= \frac{\beta}{c} s_{os}, \\ x_p &= \frac{\alpha}{c} S_p + \frac{\beta}{c} s_p. \end{aligned}$$

The ratios of market shares are:

$$\begin{aligned} \frac{s_{os}}{s_p} &= \exp\left(\frac{\beta}{\mu(1-\sigma)} \ln\left(\frac{x_{os}}{x_p}\right) - \frac{p_{os} - p_p}{\mu(1-\sigma)}\right), \\ \frac{S_{os}}{S_p} &= \exp\left(\frac{\alpha}{\mu} \ln\left(\frac{n_{os} x_{os}}{x_p}\right) + \frac{r}{\mu}\right). \end{aligned}$$

Operating, we get:

$$(17) \quad \ln\left(\frac{s_{os}}{s_p}\right) = -\frac{\beta}{\mu(1-\sigma)} \ln\left(\frac{\alpha}{\beta} \frac{S_p}{s_{os}} + \frac{s_p}{s_{os}}\right) - \frac{1}{1-s_{os}} + \frac{1}{1-s_p},$$

$$(18) \quad \ln\left(\frac{S_{os}}{S_p}\right) = \frac{\alpha}{\mu} \ln(n_{os}) - \frac{\alpha}{\mu} \ln\left(\frac{\alpha}{\beta} \frac{S_p}{s_{os}} + \frac{s_p}{s_{os}}\right) + \frac{1}{1-S_p}.$$

The equilibrium is characterized by the previous equations and the condition that the sum of market shares is equal to 1:

$$(19) \quad S_{os} + S_p = 1,$$

$$(20) \quad n_{os} s_{os} + s_p = 1.$$

Proposition 6. *A symmetric equilibrium for the full compatibility case exists and is unique. Equilibrium market shares solve (17) to (20). In equilibrium, $s_p \geq s_{os}$ and $S_{os} \geq S_p$, and the profit of the P firm is always higher than the profit of an OS firm.*

It is important to remark that the result that $S_{os} > S_p$ for all parameter values depends on the fact that there is only one P firm, and may not longer hold if we introduce more P firms in the market.

Lemma 6 provides simple results which can be compared with those of the nested logit model of Section 6, where it is assumed that $\theta = 0$. When $\theta = 0$, OS firms could have a higher market share and profits than P firms, depending on γ , δ and σ . When $\theta = 1$, on the other

hand, the complementary goods market share and profit of the P firms will always be higher than the market share of OS firms.

These results suggest that OS will perform better when the complementary good is more specific to the primary good. Good examples of these kinds of complementary goods are support and training services, customizations, platform-specific software, and mobile devices (like MP3 players, PDAs or cell phones), among others.

8.2. Partial compatibility. We now turn to the analysis of the effects of changes in θ on the equilibrium. As θ increases, the degree of specificity of the complementary goods decreases. This means that OS firms tend to have similar market shares in the OS and P primary goods markets, and therefore have less incentives to invest in the primary good. The P firm, on the other hand, keeps its incentives to invest in the primary good because it sells this good for a positive price. Therefore, as θ increases, the investment of the P firm increases relative to the investment of the OS firms.

The above argument implies that as θ increases, we should see an increase in the market shares of the primary and complementary goods of the P firm, and also an increase in the profits of the P firm relative to the profit of the OS firms.

Figure 7 shows the effects of changes in θ on the natural logarithm of the ratios s_{os}/s_p , S_{os}/S_p and π_{os}/π_p , for $n_{os} = 10$, $\gamma =$, $\delta =$, $\mu = 1$, $\sigma =$ and $c = 1$. We can see that these three ratios decrease with θ , which confirms our previous assertions. Our simulations for different values of the parameters indicate that $S_{os}/S_p > 1$ for all θ .

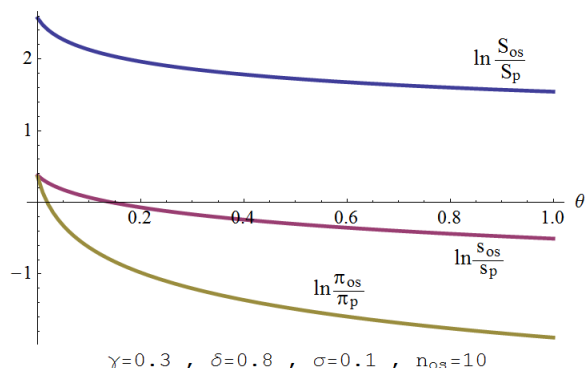


Figure 7: Effects of changes in θ on equilibrium.

In Figure 7, OS firms have higher market shares in the complementary goods and higher profits than the P firm for small θ , but have lower market shares and profits for high θ . More generally, we know that s_{os}/s_p and π_{os}/π_p may be larger or smaller than 1 when $\theta = 0$, which depends on γ , δ and σ . However, what is more important is that these two ratios are decreasing in θ and that they will always be smaller than 1 when $\theta = 1$, by Lemma 6.

Finally, it is important to remark that introducing compatibility strengthens our results of an asymmetric market structure, where P firms have a higher market share than OS firms (in the complementary goods market).

9. CONCLUSION

This paper investigates the motivations of commercial firms to participate in OS, and the implications of direct competition between OS and P firms on R&D investments and equilibrium market shares. We present a model in which firms decide whether to become OS or P and their investment in R&D and price. Both kinds of firms sell packages composed by a primary good (like software) and a complementary private good (like support and training services or hardware). The difference between both kinds of firms is that OS share their investments in R&D, while P develop their products on their own.

Our main contribution is to determine conditions under which OS and P coexist in equilibrium. These equilibria are characterized by an asymmetric market structure: P firms invest more in R&D and obtain a larger market share than OS firms. OS firms, on the other hand, benefit from lower development costs. We also show that the results are robust to the introduction of initial asymmetries in firm size, lower differentiation among OS varieties, direct investment in the complementary good, and compatibility.

Our model points to several important characteristics of OS. In particular, the success of OS will depend on (i) the strength of the complementarity between primary and complementary goods, and (ii) the possibility to differentiate the firm's OS variant from other OS and P products, (iii) the size of the learning effect when investing in the primary good, and (iv) the degree of compatibility between the primary and complementary goods of different firms.

The welfare analysis shows the equilibrium is suboptimal for two reasons: there is too little collaboration (caused by P firms) and too little investment in R&D (caused by OS firms). We show that a subsidy to OS development can improve welfare not only because it increases

the investment in R&D, but also because it encourages commercial firms to participate in OS, enhancing collaboration as a result. This explains the active involvement of governments in promoting OS.

There is a large need for more empirical research on OS. This paper provides several testable implications, among which are the following:

1. OS firms should tend to have smaller investment and market share than P firms.
2. The market share of OS should be higher when:
 - i. the degree of complementarity between the primary and complementary goods is higher,
 - ii. the degree of horizontal product differentiation is higher,
 - iii. initial differences in market shares and investments are smaller, and
 - iv. complementary goods are primary good-specific (the degree of compatibility between goods of different firms is small).

Our objective was to present a tractable model to analyze the coexistence of OS and P firms. We believe our paper is an important first step in the analysis of the behavior of profit maximizing firms in OS, which can be extended in several directions. First, the model could be modified to endogenize bundling and compatibility decisions. Second, consumer preferences could be modified to introduce network effects. Third, an important technological difference between OS and P firms is that OS benefit more from user innovation than P firms. In OS, users can access the source code, which allows them to customize the software program to their needs, and also to correct bugs at a faster rate.

OS is a promising and exciting area of research, which deserves further study. We believe this paper contributes to our understanding of this phenomenon, particularly in those aspects related to the design of an optimal business strategy.

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APPENDIX A: PROOFS OF THEOREMS IN TEXT

Proposition 1. *A second-stage equilibrium exists and is unique. Given n_{os} , the equilibrium market shares solve (6) and (7).*

Proof. The first order conditions with respect to p_i and x_i are:

$$(21) \quad \frac{\partial \pi_i}{\partial p_i} = \frac{\partial s_i}{\partial p_i} p_i + s_i \leq 0 \quad \text{with equality if } p_i > 0,$$

$$(22) \quad \frac{\partial \pi_i}{\partial x_i} = \frac{\partial s_i}{\partial x_i} p_i - c \leq 0 \quad \text{with equality if } x_i > 0.$$

For the moment, assume that $p_i > 0$ and $x_i > 0$ in equilibrium, so the first order conditions hold with equality. Later, we will show there are no corner equilibria. Working with equation (21) we get the optimal price:

$$(23) \quad p_i = \mu / (1 - s_i).$$

Equation (23) holds for both kinds of firms (OS and P). To find the optimal investment in R&D we need to calculate $\partial s_i / \partial x_i$, which in the case of OS firms is:

$$\frac{\partial s_i}{\partial x_i} = \frac{s_i(1 - s_i)}{\mu} \left(\alpha \frac{\partial a}{\partial x_i} + \beta \frac{\partial b}{\partial x_i} \right) - \sum_{j \in OS_{-i}} \alpha \frac{s_i s_j}{\mu} \frac{\partial a}{\partial x_i},$$

and in the case of P firms is:

$$\frac{\partial s_i}{\partial x_i} = \frac{s_i(1 - s_i)}{\mu} \left(\alpha \frac{\partial a}{\partial x_i} + \beta \frac{\partial b}{\partial x_i} \right).$$

Here we can see that the difference between both kinds of firms is the public good nature of a_i for OS firms. The improvement in the quality of the primary good due to firm i 's investment benefits the rest of OS firms and therefore the increase in market share is less than what it would be if the firm was P.

Imposing symmetry and introducing these expressions in (22) we get:

$$(24) \quad x_{os} = \frac{1}{c} s_{os} \left(\alpha + \beta - \alpha \frac{n_{os} - 1}{n_{os}} \frac{1}{1 - s_{os}} \right),$$

$$(25) \quad x_p = \frac{1}{c} s_p (\alpha + \beta),$$

and the ratio of optimal investments in equilibrium:

$$(26) \quad \frac{x_{os}}{x_p} = \frac{s_{os}}{s_p} \left(1 - \frac{\alpha}{\alpha + \beta} \frac{n_{os} - 1}{n_{os}} \frac{1}{1 - s_{os}} \right).$$

From (2) we get that the ratio of market shares between OS and P firms is:

$$(27) \quad \frac{s_{os}}{s_p} = \exp\left(\frac{\alpha a_{os} + \beta b_{os} - \alpha a_p - \beta b_p + p_p - p_{os}}{\mu}\right),$$

$$(28) \quad \ln\left(\frac{s_{os}}{s_p}\right) = \frac{1}{\mu} \Delta + \frac{1}{1-s_p} - \frac{1}{1-s_{os}},$$

where $\Delta = \alpha a_{os} + \beta b_{os} - \alpha a_p - \beta b_p$ represents quality differences. From the definitions of a and b :

$$(29) \quad \Delta = (\alpha + \beta) \ln\left(\frac{x_{os}}{x_p}\right) + \alpha \ln(n_{os}).$$

From equations (26), (28) and (29), we get equation (6), which is an implicit equation determining the relation of market shares between OS and P firms in equilibrium. This equation, together with the equation establishing that the sum of the market shares is equal to 1, completely characterizes the equilibrium.

Now we will show there are no corner solutions ($x_i > 0$ and $p_i > 0$ in the symmetric equilibrium). If $p_i = 0$, then profits are zero and the firm would find profitable to increase p_i . To analyze $x_i = 0$ we have to specify what happens with s_i when $x_i = 0$. Assume that if $x_i = 0$ and $x_j > 0$ for at least one $j \neq i$, then $s_i = 0$. When $x_i = 0$ for all i , on the other hand, $s_i = \frac{\exp(-p_i/\mu)}{\sum \exp(-p_i/\mu)}$. There are 3 cases: $x_p = 0$ and $x_{os} > 0$, $x_p > 0$ and $x_{os} = 0$, and $x_p = 0$ and $x_{os} = 0$. If $x_p = 0$ and $x_{os} > 0$, then $s_p = 0$ and a P firm makes zero profits. But a P firm can deviate to $p_i = \frac{\mu}{1-s_i}$ and $x_i = \frac{s_i(\alpha+\beta)}{c}$ with $s_i > 0$. Such a deviation is profitable if $s_i > 1 - \frac{1}{\delta}$ which always holds. If $x_p > 0$ and $x_{os} = 0$, then $s_{os} = 0$ and an OS firm makes zero profits. But an OS firm can deviate to $p_i = \frac{\mu}{1-s_i}$ and $x_i = \frac{s_i(\alpha+\beta)}{c}$ obtaining positive profits. If $x_p = 0$ and $x_{os} = 0$, then $s_p = s_{os} = \frac{1}{n}$. An OS or a P can deviate to $x_i = \epsilon > 0$ obtaining a discontinuous jump in revenue ($s_i = 1$) and a small increase in costs.

Finally, to show existence and uniqueness, we need to prove two things: (1) there is only one fixed point of the system of equations in Proposition 1 (there is only one symmetric equilibrium), and (2) the profit function is concave at the equilibrium (the second order conditions for optimality hold).

Let's first show there is only one fixed point in term of equilibrium market shares. Define the function $g(s_{os})$ by plugging equation (7) in equation (6).

$$(30) \quad g(s_{os}) = (1 - \delta) \ln\left(\frac{(n - n_{os})s_{os}}{1 - n_{os}s_{os}}\right) - \delta \ln\left((1 - \gamma) + \gamma \frac{1 - n_{os}s_{os}}{(1 - s_{os})n_{os}}\right) + \\ - \delta \gamma \ln(n_{os}) - \frac{n - n_{os}}{1 + n_{os}(1 - s_{os}) - n} + \frac{1}{1 - s_{os}}.$$

By construction, s_{os} solves equations (6) and (7) if and only if $g(s_{os}) = 0$. Existence follows from a standard application of the mean value theorem. First, $\lim_{s_{os} \rightarrow 0} g(s_{os}) = -\infty$ and $\lim_{s_{os} \rightarrow \frac{1}{n_{os}}} g(s_{os}) = \infty$. Then, continuity of g implies there exists at least one s_{os} such that $g(s_{os}) = 0$. Next, we will show there exists only one such s_{os} . For this, it is sufficient to show that g is strictly increasing, for

which we will calculate its derivative:

$$\begin{aligned} \frac{\partial g}{\partial s_{os}} &= \frac{1 - \delta}{s_{os}(1 - n_{os}s_{os})} + \frac{\delta \gamma (n_{os} - 1)/(1 - s_{os})}{(1 - \gamma)(n_{os} - 1) + 1 - s_{os}n_{os}} + \\ &+ \frac{(n - n_{os})n_{os}}{(1 + n_{os}(1 - s_{os}) - n)^2} + \frac{1}{(1 - s_{os})^2}. \end{aligned}$$

All terms are positive because $s_{os}n_{os} \leq 1$. It follows that there exists a unique (s_{os}, s_p) solving the system of equations.

To prove that the profit function is concave at the equilibrium candidate, we will evaluate the determinant of the Hessian of the profit function at the equilibrium price and market share, and show that it is positive definite. The determinants of the Hessian of both kinds of firms are:

$$\begin{aligned} |H_p| &= \frac{(\alpha + \beta)s_p^2}{\mu x_p^2} \left(1 - \frac{(\alpha + \beta)(1 - s_p)^2}{\mu} \right), \\ |H_{os}| &\geq \frac{s_{os}^2}{\mu x_{os}^2} \left(\left(\frac{1 - n_{os}s_{os}}{(1 - s_{os})n_{os}^2} \alpha + \beta \right) - \frac{(1 - s_{os})^2}{\mu} \left(\frac{1 - n_{os}s_{os}}{(1 - s_{os})n_{os}} \alpha + \beta \right)^2 \right). \end{aligned}$$

A sufficient condition for both determinants to be positive is $\mu \geq \alpha + \beta$, which has been assumed throughout the paper. Thus, the concavity of the profit function at the equilibrium is guaranteed for both kinds of firms. \blacksquare

Lemma 1. $s_p > s_{os}$ if $\gamma > \hat{\gamma}(n_{os}, n)$, and $s_p < s_{os}$ in the opposite case, where $\hat{\gamma}(n_{os}, n)$ is increasing in n_{os} and n and solves:

$$\gamma \frac{n_{os}^\gamma}{n_{os}^\gamma - 1} \frac{n_{os} - 1}{n_{os}} = \frac{n - 1}{n}.$$

Proof. To prove the first part of the lemma we only have to check the sign of $g\left(\frac{1}{n}\right)$, where g is defined in (30). If $g\left(\frac{1}{n}\right) < 0$, then $s_{os} > 1/n$ and therefore $s_{os} > s_p$. Then

$$g\left(\frac{1}{n}\right) = -\delta \left(\ln \left(1 - \gamma \frac{n}{n-1} \frac{n_{os}-1}{n_{os}} \right) + \gamma \ln(n_{os}) \right),$$

and $g\left(\frac{1}{n}\right) < 0$ if and only if:

$$1 - \gamma \frac{n}{n-1} \frac{n_{os}-1}{n_{os}} > n_{os}^{-\gamma}.$$

Rearranging this expression we get the desired result.

To show that $\hat{\gamma}(n_{os}, n)$ is increasing in n and n_{os} , let $h(\gamma, n_{os}) = \gamma \frac{n_{os}^\gamma}{n_{os}^\gamma - 1} \frac{n_{os} - 1}{n_{os}}$. Computing the derivatives:

$$(31) \quad \frac{\partial h}{\partial \gamma} = \frac{n_{os}^\gamma}{(n_{os}^\gamma - 1)^2} \frac{n_{os} - 1}{n_{os}} (n_{os}^\gamma - 1 - \gamma \ln(n_{os}))$$

$$(32) \quad \frac{\partial h}{\partial n_{os}} = \frac{\gamma}{n_{os}^{2-\gamma} (n_{os}^\gamma - 1)^2} (n_{os}^\gamma - 1 - \gamma(n_{os} - 1))$$

First, we will show that $\frac{\partial h}{\partial \gamma} \geq 0$, which is enough to determine that $\hat{\gamma}$ is increasing in n . $\frac{\partial h}{\partial \gamma} \geq 0$ if and only if $n_{os}^\gamma - 1 \geq \ln(n_{os}^\gamma)$. Let $x = n_{os}^\gamma$, $f_1(x) = x - 1$ and $f_2(x) = \ln(x)$. x ranges from 1 to n_{os} . When $x = 1$, $f_1 = f_2$, but then f_1 grows faster than f_2 for any x . This means that $n_{os}^\gamma - 1 \geq \ln(n_{os}^\gamma)$ and $\frac{\partial h}{\partial \gamma} \geq 0$.

Next, we will show that $\frac{\partial h}{\partial n_{os}} \leq 0$, which implies that $\hat{\gamma}$ is increasing in n_{os} following a simple application of the implicit function theorem. $\frac{\partial h}{\partial n_{os}} \leq 0$ if and only if $\gamma(n_{os} - 1) \geq n_{os}^\gamma - 1$. Let $g_1(\gamma) = \gamma(n_{os} - 1)$ and $g_2(\gamma) = n_{os}^\gamma - 1$. It is easy to check that $g_1(0) = g_2(0)$, $g_1(1) = g_2(1)$, and that both functions are increasing but g_2 is strictly convex and g_1 is linear. Therefore, $g_1(\gamma) \geq g_2(\gamma)$ and $\frac{\partial h}{\partial n_{os}} \leq 0$. ■

Lemma 2. s_{os} is increasing in δ if $\gamma < \hat{\gamma}(n_{os}, n)$, and decreasing in δ in the opposite case.

Proof. Suppose $\gamma < \hat{\gamma}$. Then, by lemma 1, $h(n_{os}, \gamma) < (n - 1)/n$ and $s_{os}^* > 1/n$ in equilibrium. Let partial derivatives of g be denoted by subscripts, where g is defined in (30). By the implicit function theorem, $\partial s_{os}/\partial \delta = -g_\delta/g_{s_{os}}$. In the proof of proposition 1 it has been shown that $g_{s_{os}} > 0$. Next, we will determine the sign of g_δ . It can be shown that g_δ is decreasing in s_{os} , then if $g_\delta(1/n) \leq 0$, and given that $s_{os}^* > 1/n$, we can deduce that $g_\delta(s_{os}^*) \leq 0$. Let us compute $g_\delta(1/n)$:

$$g_\delta(1/n) = -\ln(n_{os}^\gamma) - \ln\left(\frac{(n-1)n_{os} - n(n_{os}-1)\gamma}{(n-1)n_{os}}\right).$$

This expression is negative if and only if,

$$\frac{(n-1)n_{os}}{n_{os}^\gamma((n-1)n_{os} - n(n_{os}-1)\gamma)} < 1.$$

Rearranging terms, this expression is equivalent to $h(n_{os}, \gamma) < (n - 1)/n$ which holds by assumption. Thus $g_\delta(s_{os}^*) \leq 0$ and $\frac{\partial s_{os}}{\partial \delta} \geq 0$. The proof for $\gamma > \hat{\gamma}$ is analogous, but reversing the inequalities. ■

Lemma 3. There exists $\gamma_d \in (0, \hat{\gamma})$ such that s_{os} is increasing in γ for $\gamma < \gamma_d$, and decreasing in γ for $\gamma > \gamma_d$.

Proof. By the implicit function theorem, $\partial s_{os}/\partial \gamma = -g_\gamma/g_{s_{os}}$, where g is defined in (30). We know $g_{s_{os}} > 0$. With respect to g_γ :

$$g_\gamma = \ln(n_{os}) - \frac{n_{os} - 1}{\gamma + (1 - s_{os} - \gamma)n_{os}}$$

Therefore, $\partial s_{os}/\partial \gamma = 0$ when $g_\gamma = 0$. Solving for the value \hat{s}_{os} that makes $g_\gamma = 0$ we get:

$$\hat{s}_{os} = \frac{\ln(n_{os})(n_{os}(1 - \gamma) + \gamma) + 1 - n_{os}}{n_{os} \ln(n_{os})}$$

Introducing this in $g = 0$ we get an equation determining the value γ_d that makes the derivative equal to zero.

To prove that to the right of γ_d the graph of $s_{os}(\gamma)$ is decreasing, assume this is not the case, so $g_\gamma > 0$. Then, for $\gamma > \gamma_d$ it has to be the case that $s_{os} > \hat{s}_{os}$, but this implies that $g_\gamma < 0$, which is a contradiction. This means that $\partial s_{os}/\partial \gamma < 0$ for $\gamma > \gamma_d$. A similar reasoning implies that $\partial s_{os}/\partial \gamma > 0$ for $\gamma < \gamma_d$. ■

Proposition 2. A Subgame Perfect Equilibrium for the two-stage game exists and is unique.

Proof. For $n_{os} = 1$ to be an equilibrium we only need $f(2) \leq 0$. Likewise, for $n_{os} = n$ to be an equilibrium we only need $f(n) \geq 0$. In order to have an equilibrium with both kinds of firms ($1 < n_{os} < n$), we need that $f(n_{os}) \geq 0$ and $f(n_{os}+1) \leq 0$ at the equilibrium n_{os} . If $f(2) \geq 0$ and $f(n) \leq 0$ then it is guaranteed that there is at

least one such equilibrium. Therefore, existence of an equilibrium with $1 \leq n_{os} \leq n$ is guaranteed. Simulations show that the equilibrium is unique for any value of the parameters. ■

Proposition 3. *Given $n > 3$ and δ , there exist $0 < \bar{\gamma} < \hat{\gamma} < 1$, such that in equilibrium:*

- i. *If $\gamma > \hat{\gamma}$, both kinds of firms co-exist and P have higher quality and market share than OS.*
- ii. *If $\bar{\gamma} < \gamma \leq \hat{\gamma}$, all firms decide to be OS, but a P firm would have higher quality and market share.*
- iii. *If $\gamma \leq \bar{\gamma}$, all firms decide to be OS, and a P firm would have lower quality and market share.*

Proof. Simulations show that $f(2) > 0$ for any γ and δ . This means that the equilibrium always has at least 2 firms participating in OS. Lemma 5 will prove very important in characterizing the Subgame Perfect Equilibrium of the game. In order for an OS firm to find it profitable to become P ($f(n_{os}) < 0$), it has to be the case that the increase in market share from becoming P is large enough to compensate for the increase in cost. If $\gamma < \hat{\gamma}(n_{os}-1, n)$, then OS firms have a larger market share so it is not profitable for them to deviate ($f(n_{os}) > 0$). Corollaries 1 and 2 are two important implications of this lemma.

Lemma 5 (Sufficient condition for positive f). *If $\gamma < \hat{\gamma}(n_{os}-1, n)$ then $f(n_{os}) > 0$.*

Proof. Rearranging $f(n_{os})$ and dividing by μ we get:

$$\frac{f(n_{os})}{\mu} = \frac{s_{os}}{1-s_{os}}(1-\delta(1-s_{os})) - \frac{\tilde{s}_p}{1-\tilde{s}_p}(1-\delta(1-\tilde{s}_p)) + \delta\gamma \frac{s_{os}}{1-s_{os}} \frac{n_{os}-1}{n_{os}}$$

where $s_{os} = s_{os}(n_{os})$ and $\tilde{s}_p = s_p(n_{os}-1)$. The value of μ does not influence the sign of f . The first two terms have the same functional form and are increasing in s . The last term is always positive. Therefore, if $s_{os}(n_{os}) \geq s_p(n_{os}-1)$, then $f(n_{os}) > 0$. A sufficient condition is that $s_{os}(n_{os}-1) \geq 1/n$ and $s_{os}(n_{os}) \geq 1/n$, which is equivalent to $\gamma < \hat{\gamma}(n_{os}-1, n)$ and $\gamma < \hat{\gamma}(n_{os}, n)$. However, $\hat{\gamma}(n_{os}, n)$ is decreasing in n_{os} , so $\gamma < \hat{\gamma}(n_{os}-1, n)$ implies $f(n_{os}) > 0$. ■

Corollary 1 (Necessary condition for an interior equilibrium). *At an interior equilibrium n_{os} it is necessary that $\gamma \geq \hat{\gamma}(n_{os}, n)$.*

Proof. For an interior equilibrium at n_{os} we need that $f(n_{os}) \geq 0$ and $f(n_{os}+1) \leq 0$, but Lemma 5 implies that for $f(n_{os}+1) \leq 0$ we need $\gamma \geq \hat{\gamma}(n_{os}-1, n)$. ■

Corollary 2 (Sufficient condition for an equilibrium with $n_{os} = n$). *If $\gamma \leq \hat{\gamma}(n-1, n)$ then all firms decide to be OS in equilibrium.*

Proof. If $\gamma \leq \hat{\gamma}(n-1, n)$ then $f(n) \geq 0$, so if $n_{os} = n$ then no firm would gain by becoming a P firm. ■

Corollary 1 states that in any interior equilibrium it has to be the case that the P firms have a larger market share than OS firms, and therefore a higher quality product. Corollary 2, on the other hand, shows that if the degree of public good of the investment is low enough, OS firms have a larger market share for any n_{os} , and therefore all firms decide to collaborate in the OS project. Lemma 6 complements Corollary 2, by providing the necessary and sufficient condition for an equilibrium with $n_{os} = n$.

Lemma 6 (Necessary and Sufficient condition for equilibrium with $n_{os} = n$). *Given $n > 3$ and δ , there exists $\bar{\gamma} \in (\hat{\gamma}, 1)$ such that $f(n) \geq 0$ if and only if $\gamma \leq \bar{\gamma}$.*

Proof. μ only affects the scale of $f(n_{os})$, so assume $\mu = 1$ for the rest of this proof. We know that $f(n) > 0$ for $\gamma < \hat{\gamma}(n-1, n)$. We need to determine the sign of $f(n)$ for the rest of values of γ . When $n_{os} = n$, $s_{os} = 1/n$. Therefore,

$$(33) \quad f(n) = \frac{1}{n-1} - \frac{\delta(1-\gamma)}{n} - \frac{\tilde{s}_p}{1-\tilde{s}_p} (1 - \delta(1 - \tilde{s}_p)),$$

where $\tilde{s}_p = s_p(n-1)$. We need to find the value of \tilde{s}_p that makes $f(n) = 0$. There are two roots of this equation. The only positive root is:

$$\tilde{s}_p = \frac{-n^2(1-\delta) - (1-\gamma)\delta - n\gamma\delta + \sqrt{n^4 - 2n^2z + z^2}}{2\delta n(n-1)}$$

where $z = \delta(n-1)(n-1+\gamma)$. The corresponding value for $s_{os}(n-1)$ is:

$$\tilde{s}_{os} = \frac{n^2 + s - \sqrt{n^4 - 2n^2z + z^2}}{2\delta n(n-1)^2}$$

Plugging this value in the equilibrium condition (30) and solving for γ we get the value $\bar{\gamma}$ where $f(n) = 0$. Lemma 5 implies that $\bar{\gamma} \geq \hat{\gamma}(n-1, n)$. Lemma 3 implies that $\partial\tilde{s}_p/\partial\gamma > 0$ in the relevant area. This means that $\bar{\gamma}$ is the unique value of γ such that $f(n) = 0$.

To finish the proof we need to show that $f(n) > 0$ for $\gamma < \bar{\gamma}$ and $f(n) < 0$ for $\gamma > \bar{\gamma}$. Given the continuity and monotonicity of s_p , it suffices to show there is some value to the right or to the left of $\bar{\gamma}$ such that these inequalities hold.

Consider first the case of $\gamma < \bar{\gamma}$. We know that at $\gamma = \hat{\gamma}(n-1, n)$, $f(n) > 0$. This proves that $f(n) > 0$ for $\gamma < \bar{\gamma}$. For $\gamma > \bar{\gamma}$, consider $\gamma = 1$. When $\gamma = 1$, the investment of OS firms is very low, and P firms have the largest advantage. In this case, $f(n) < 0$, which proves that this inequality holds for any $\gamma > \bar{\gamma}$. ■

Proposition 3 follows from a straightforward application of Corollaries 1 and 2 and Lemma 6. ■

Lemma 4. $s_{os} > s_p$ in a second-stage equilibrium if

$$\kappa > 1 - \left(1 - \gamma \frac{n}{n-1} \frac{n_{os}-1}{n_{os}}\right) n_{os}^\gamma,$$

and $s_p > s_{os}$ in the opposite case.

Proof. The proof follows the same steps than the proof of Lemma 1, but rearranging the resulting condition in a different way. ■

Proposition 4. *The optimal subsidy is $\kappa^* = \gamma$, which attains the first best levels of investment. In equilibrium, all firms decide to be OS.*

Proof. Assume that the subsidy is high enough, so that all firms decide to be OS. Individual investments are $x_{os} = (\alpha + \beta)(1 - \gamma)/(n c_{os})$. Assuming that the subsidy is financed with lump-sum or proportional taxes is equivalent for this model. The social welfare function is the same as before (taxes and subsidy cancel in the social welfare function). The government wants to find the subsidy which attains the optimal investment $x^* = (\alpha + \beta)/cn$. Thus, the optimal subsidy becomes $\kappa^* = \gamma$.

To finish the proof, we have to show that given this subsidy, all firms effectively choose to be OS. Remember that if $\kappa > 1 - (1 - \gamma n(n-2)(n-1)^{-2})(n-1)^\gamma$,

then $s_{os} > s_p$ for $n_{os} = n - 1$, and therefore all firms want to be OS. Given that $n(n-2)(n-1)^{-2} \in [0, 1]$, $\kappa > 1 - (1-\gamma)(n-1)^\gamma$ is a sufficient condition, which holds if $\kappa = \gamma$. \blacksquare

Proposition 5. *A second-stage equilibrium for the nested model exists and is unique. Given n_{os} , the equilibrium market shares solve (15) and (7).*

Proof. The first order conditions are (21) (22). As in the standard logit model, there are no corner solutions so the first order conditions hold with equality. Equilibrium prices and investment for P firms are identical to the logit model so we will focus on the OS firms. In the case of OS firms the partial derivative of the market share with respect to the price is

$$\frac{\partial s_i}{\partial p_i} = -\frac{1}{(1-\sigma)\mu} s_i (1 - \sigma s_{i|os} - (1-\sigma) s_i).$$

Then from the equation (21) and imposing symmetry we get the optimal price in equation (13). To find x_{os} we need to calculate $\partial s_i / \partial x_i$ for OS firms:

$$\frac{\partial s_i}{\partial x_i} = \frac{s_i (1 - \sum_{j \in OS} s_j)}{\mu} \alpha \frac{\partial a}{\partial x_i} + \frac{s_i (1 - \sigma s_{i|os} - (1-\sigma) s_i)}{(1-\sigma)\mu} \beta \frac{\partial b}{\partial x_i}.$$

From equation (22) and imposing symmetry we get equation (14), and the ratio of optimal investments in equilibrium:

$$(34) \quad \frac{x_{os}}{x_p} = \frac{s_{os}}{s_p} \left(1 - \frac{\gamma}{1-\sigma} \frac{(n_{os}-1)/n_{os}}{\left(1 - s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}\right)} \right).$$

The ratio of market shares between OS and P firms is:

$$(35) \quad \frac{s_{os}}{s_p} = n_{os}^{-\sigma} \exp \left(\frac{\alpha a_{os} + \beta b_{os} - \alpha a_p - \beta b_p + p_p - p_{os}}{\mu} \right),$$

$$(36) \quad \ln \left(\frac{s_{os}}{s_p} \right) = -\sigma \ln n_{os} + \frac{1}{\mu} \Delta + \frac{1}{1-s_p} - \frac{1}{1-s_{os} + \frac{\sigma}{1-\sigma} \frac{n_{os}-1}{n_{os}}},$$

where $\Delta = \alpha a_{os} + \beta b_{os} - \alpha a_p - \beta b_p$ represents quality differences. From the definitions of a and b :

$$(37) \quad \Delta = (\alpha + \beta) \ln \left(\frac{x_{os}}{x_p} \right) + \alpha \ln(n_{os}).$$

From equations (34), (36) and (37), we get equation (15), which is an implicit equation determining the relation of market shares between OS and P firms in equilibrium. This equation, together with the equation establishing that the sum of the market shares is equal to 1, completely characterizes the equilibrium.

Finally, to show existence and uniqueness, we need to prove two things: (1) there is only one fixed point of the system of equations in Proposition 5 (there is only one symmetric equilibrium), and (2) the profit function is concave at the equilibrium (the second order conditions for optimality hold).

To show (1), define $g(s_{os})$ by plugging (7) in equation (15). Then, the result follows from an application of the mean value theorem as in the standard logit model.

To prove that the profit function is concave at the equilibrium candidate, we will evaluate the determinant of the Hessian of the profit function at the equilibrium

price and market share, and show that it is positive definite. The determinant of the Hessian for OS firms is:

$$|H_{os}| \geq \frac{s_{os}^2}{(1-\sigma)\mu x_{os}^2} (1-\sigma s_{i|os}) \left((1-\sigma)\mu \left(\frac{1-\sigma-(1-\sigma)n_{os}s_{os}}{(1-\sigma s_{i|os}-(1-\sigma)s_{os})n_{os}^2} \alpha + \beta \right) + \right. \\ \left. - \left(\frac{1-\sigma-(1-\sigma)n_{os}s_{os}}{(1-\sigma s_{i|os}-(1-\sigma)s_{os})n_{os}} \alpha + \beta \right)^2 \right).$$

The determinant of the Hessian for P firms is equivalent to that of the standard logit model. A sufficient condition for both determinants to be positive is $\mu(1-\sigma) \geq \alpha + \beta$, which has been assumed for this section of the paper. Thus, the concavity of the profit function at the equilibrium is guaranteed for both kinds of firms. ■

Proposition 6. *A symmetric equilibrium for the full compatibility case exists and is unique. Equilibrium market shares solve (17) to (20). In equilibrium, $s_p \geq s_{os}$ and $S_{os} \geq S_p$, and the profit of the P firm is always higher than the profit of an OS firm.*

Proof. To show existence, we begin by noticing that for any value of $S_{os} \in [0, 1]$, there is a value of $s_{os} \in (0, 1/n_{os})$ that solves equation 17. This is because the left hand side is continuous for $s_{os} \in (0, 1/n_{os})$, and goes from $-\infty$ when $s_{os} \rightarrow 0$ to ∞ when $s_{os} \rightarrow 1/n$. By a similar argument, for any value of $s_{os} \in (0, 1/n_{os}]$ there is a value of $S_{os} \in [0, 1]$ that solves equation 18. This implies that an equilibrium exists.

To show uniqueness, let us denote equation (17) by $f(s_{os}, S_{os}) = 0$ and equation (18) by $g(s_{os}, S_{os}) = 0$. By the implicit function theorem we can write $S_{os} = f_1(s_{os})$ using the first equation, and $S_{os} = g_1(s_{os})$ using the second equation. It is straightforward to show that both functions are continuously increasing, but the slope of f_1 is always less than 1, while the slope of g_1 is always larger than 1. This means that the two curves cross only once, so the symmetric equilibrium is unique.

Now we show that $s_p > s_{os}$. In equilibrium, the following condition must hold:

$$(38) \quad \ln\left(\frac{s_{os}}{s_p}\right) + \frac{1}{1-s_{os}} - \frac{1}{1-s_p} = -\frac{\beta}{\mu(1-\sigma)} \ln\left(\frac{\alpha}{\beta} \frac{S_p}{s_{os}} + \frac{s_p}{s_{os}}\right).$$

When $s_p = s_{os} = 1/n$, the condition becomes

$$0 = -\frac{\beta}{\mu(1-\sigma)} \ln\left(1 + \frac{\alpha}{\beta} S_p n\right).$$

Define m as the right hand side of the previous equality. $s_p = s_{os}$ is not an equilibrium because $m < 0$ when $s_p = s_{os}$. For equation (38) to return to the equilibrium, s_p has to increase and s_{os} has to decrease relative to $s_p = s_{os}$. Therefore, the equilibrium has $s_p > s_{os}$.

Next, we show that $S_{os} > S_p$. We will prove this by contradiction. By a similar argument than before, for $S_{os} \leq S_p$ the following condition must hold:

$$-\ln(n_{os}) + \ln\left(\frac{1-n_{os}s_{os}}{s_{os}} + \frac{\alpha}{2\beta s_{os}}\right) - \frac{2\mu}{\alpha} \geq 0.$$

This implies that we should have $s_{os} \leq \frac{\alpha+2\beta}{2\beta(1+e^{2\mu/\alpha})n_{os}}$ in equilibrium. If we introduce $s_{os} = \frac{\alpha+2\beta}{2\beta(1+e^{2\mu/\alpha})n_{os}}$ in the equilibrium condition (17), and rearrange terms,

we get that for this value of s_{os} to be an equilibrium the following should hold:

$$(39) \quad 0 = -\frac{2\beta}{\alpha + 2\beta} \left(1 + e^{\frac{2\mu}{\alpha}}\right) + \log\left(\frac{\alpha + 2\beta}{2\beta e^{\frac{2\mu}{\alpha}} - \alpha}\right) + \frac{2\beta}{\alpha(1 - \sigma)} \\ + \frac{n_{os}}{n_{os} - \frac{(\alpha + 2\beta)(1 - \tanh(\frac{\mu}{\alpha}))}{4\beta}} + \left(\frac{\beta}{\mu(1 - \sigma)} - 1\right) \log(n_{os})$$

Let \hat{m} be the right hand side of the previous expression. It can be shown that $\hat{m} < 0$ for any value of the parameters. Therefore, $s_{os} > \frac{\alpha + 2\beta}{2\beta(1 + e^{2\mu/\alpha})n_{os}}$ in equilibrium, and therefore, $S_{os} \leq S_p$ cannot hold.

Finally, to see that the P firm will always have a higher profit than an OS firm, notice that the P firm can always choose the same investment and price than an OS firm. The P firm will have the same revenues due to the complementary good, and the same cost of R&D, but it will also have some revenue on the primary good side. ■