

An Academic Response to Basel 3.5

Risk Aggregation and Model Uncertainty

Paul Embrechts

RiskLab, Department of Mathematics, ETH Zurich

Senior SFI Professor

www.math.ethz.ch/~embrechts/

Joint work with A. Beleraj, G. Puccetti, L. Rüschendorf and R. Wang

Regulation

Three regulatory documents

- R1: BCBS-Consultative Document, May 2012,
Fundamental review of the trading book (\Leftarrow Basel 3.5)
- R2: United States Senate, March 15, 2013,
JPMorgan Chase Whale trades: a case history of
derivatives risks and abuses
- R3: UK House of Lords/House of Commons, June 12, 2013,
Changing banking for good, Volumes I and II
- (In total, about 1000 pages!)

Regulation

Some statements:

From R1: Page 20. *Choice of risk metric:*

"... However, a number of **weaknesses** have been identified with VaR, including its **inability to capture "tail risk"**. The Committee therefore believes it is necessary to consider **alternative risk metrics** that may overcome these weaknesses."

From R2: Pages 13 and 172. *VaR models changes:*

"\$7 billion, or more than 50% of the total \$13 billion RWA reduction, could be achieved by modifying risk related models." "**The change in VaR methodology effectively masked the significant changes in the portfolio.**"

Regulation

From R3: Volume II, page 119. *Output of a "stress test" exercise, from HBOS:*

"We actually got an external advisor [to assess how frequently a particular event might happen] and they came out with one in 100,000 years and we said "no", and I think **we submitted one in 10,000 years**. But that was a year and a half before it happened. It doesn't mean to say it was wrong: **it was just unfortunate that the 10,000th year was so near.**"

Basel 3.5 Question

In this talk we focus on the following question raised by the Basel Committee:

From R1, Page 41, Question 8:

“What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?”

- A challenge for financial mathematicians and financial statisticians!

Basel 3.5 Question

We focus on the mathematical and statistical aspects, avoiding discussion on practicalities and operational issues.

From R1, Page 3:

“The Committee recognises that moving to ES could entail certain **operational challenges**; nonetheless it believes that these are **outweighed by the benefits** of replacing VaR with a measure that **better captures tail risk**.”

A more recent document

R4: BCBS-Consultative Document, October 2013,
Fundamental review of the trading book: A revised market
risk framework.

The Basel Committee went already a step beyond its
consultative document May 2012:

From R4: Page 3, *Approach to risk management*:
"the Committee has its intention to pursue two
key **confirmed** reforms outlined in the first
consultative paper [May 2012]: Stressed
calibration . . . **Move from Value-at-Risk (VaR) to
Expected Shortfall (ES).**"

VaR and ES

Definition

$\text{VaR}_\alpha(X)$, for $\alpha \in (0, 1)$,

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Definition

$\text{ES}_\alpha(X)$, for $\alpha \in (0, 1)$, if $\mathbb{E}[X] < \infty$,

$$\text{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_\delta(X) d\delta \stackrel{(F \text{ cont.})}{=} \mathbb{E}[X | X > \text{VaR}_\alpha(X)].$$

VaR versus ES, extreme value theory

- For all $\alpha \in (0, 1) \Rightarrow \text{ES}_\alpha(X) \geq \text{VaR}_\alpha(X)$.
- For light tailed distributions (such as $X \sim N(\mu, \sigma^2)$),

$$\lim_{\alpha \rightarrow 1} \frac{\text{ES}_\alpha(X)}{\text{VaR}_\alpha(X)} = 1.$$

- For heavy tailed distributions:

$P(X > x) = x^{-1/\xi}L(x)$, $0 < \xi < 1$, L slowly varying,

$$\lim_{\alpha \rightarrow 1} \frac{\text{ES}_\alpha(X)}{\text{VaR}_\alpha(X)} = \frac{1}{1 - \xi}.$$

VaR versus ES, 0.99 vs 0.975

From R4: Page 22, *Moving to expected shortfall*:

“... using an ES model, the Committee believes that moving to a confidence level of 97.5% (relative to the 99th percentile confidence level for the current VaR measure) is appropriate.”

VaR_{0.99} vs ES_{0.975}

- Example: $X \sim \text{Normal}(0,1)$.

$$\text{ES}_{0.975}(X) = 2.3378,$$

$$\text{VaR}_{0.99}(X) = 2.3263.$$

They are quite close for all normal models!

VaR versus ES, 0.99 vs 0.975

From EVT: approximately,

- for **heavy-tailed** risks, $ES_{0.975}$ yields a more conservative value than $VaR_{0.99}$;
- for **light-tailed** distributions, $ES_{0.975}$ yields an equivalent regulation principle as $VaR_{0.99}$;
- for risks that do not have a very heavy tail, it holds $ES_{0.975}(X) \approx VaR_{0.99}(X)$.

▶ details

VaR Aggregation

Consider:

- One-period risk positions X_1, \dots, X_d with **known** distribution functions (dfs) $F_i, i = 1, \dots, d$;
- Portfolio position $X_d^+ = X_1 + \dots + X_d$;
- $\text{VaR}_\alpha(X_i), i = 1, \dots, d$, the marginal VaR's at the common confidence level $\alpha \in (0, 1)$.

Task:

Calculate $\text{VaR}_\alpha(X_d^+)$

Problem:

- We need a *joint* model for the random vector $\mathbf{X} = (X_1, \dots, X_d)'$

VaR Aggregation

- **X elliptical**

$$\text{VaR}_\alpha(X_d^+) \leq \sum_{i=1}^d \text{VaR}_\alpha(X_i)$$

Examples: multivariate Gaussian, multivariate Student t.

- **X comonotone** i.e. there exist increasing functions $\psi_i, i = 1, \dots, d$ and a random variable Z so that

$$X_i = \psi_i(Z)$$

then

$$\text{VaR}_\alpha(X_d^+) = \sum_{i=1}^d \text{VaR}_\alpha(X_i)$$

i.e. VaR_α (like ES_α) is **comonotone additive**.

- **Diversification benefit**: one often uses

$$(1 - \delta) \sum_{i=1}^d \text{VaR}_\alpha(X_i), \quad 0 < \delta < 1.$$

VaR Bounds

The Fréchet (unconstrained) problem

$$\underline{\text{VaR}}_{\alpha}(X_d^+) = \inf_F \{ \text{VaR}_{\alpha}(X_1^F + \cdots + X_d^F) : X_i \stackrel{d}{\sim} F_i, i = 1, \dots, d \}$$

$$\overline{\text{VaR}}_{\alpha}(X_d^+) = \sup_F \{ \text{VaR}_{\alpha}(X_1^F + \cdots + X_d^F) : X_i \stackrel{d}{\sim} F_i, i = 1, \dots, d \}$$

VaR Bounds

Equivalently, for \mathcal{C}_d the space of all d -copulas

$$\underline{\text{VaR}}_\alpha(X_d^+) = \inf_{C \in \mathcal{C}_d} \{\text{VaR}_\alpha(X_1^C + \cdots + X_d^C) : X_i \stackrel{d}{\sim} F_i, i = 1, \dots, d\}$$

$$\overline{\text{VaR}}_\alpha(X_d^+) = \sup_{C \in \mathcal{C}_d} \{\text{VaR}_\alpha(X_1^C + \cdots + X_d^C) : X_i \stackrel{d}{\sim} F_i, i = 1, \dots, d\}$$

Recall from Sklar's Theorem: $F = C(F_1, \dots, F_d)$.

VaR Bounds

$d = 2$

The **sharp** bounds $\overline{\text{VaR}}_\alpha(X_2^+)$ and $\underline{\text{VaR}}_\alpha(X_2^+)$ are known for *any* type of marginal distributions F_1, F_2 . Analytic formulas are given in Makarov (1981) and Rüschendorf (1982).

▶ details

VaR Bounds

$d \geq 3$, Homogeneous case

- $\overline{\text{VaR}}_\alpha(X_d^+)$: **A dual bound technique** introduced in Embrechts and Puccetti (2006).
- Analytical results obtained for both $\overline{\text{VaR}}_\alpha(X_d^+)$ and $\underline{\text{VaR}}_\alpha(X_d^+)$ under a **tail-monotone** condition on F (mostly satisfied in practice) by Wang, Peng and Yang (2013), based on the concept of **complete mixability**.
- **Sharpness** of the dual bound of $\overline{\text{VaR}}_\alpha(X_d^+)$ under same conditions obtained by Puccetti and Rüschendorf (2013).

▶ details

VaR Bounds

$d \geq 3$, Heterogeneous case

- **Rearrangement Algorithm** of Embrechts, Puccetti, Rüschendorf (2013) yields a powerful computational tool for the calculation of $\overline{\text{VaR}}_\alpha(X_d^+)$ and $\underline{\text{VaR}}_\alpha(X_d^+)$, and possibly $d \geq 1000$.
- Analytical approximation and connection with convex order are given by Bernard, Jiang and Wang (2014).

Dependence Uncertainty

Worst-dependence scenarios:

$$\overline{\text{VaR}}_\alpha(X_d^+) = \sup_F \{ \text{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \stackrel{d}{\sim} F_i, 1 \leq i \leq d \}.$$

$$\begin{aligned} \overline{\text{ES}}_\alpha(X_d^+) &= \sup_F \{ \text{ES}_\alpha(X_1^F + \cdots + X_d^F) : X_i \stackrel{d}{\sim} F_i, 1 \leq i \leq d \} \\ &= \sum_{i=1}^d \text{ES}_\alpha(X_i). \end{aligned}$$

Dependence Uncertainty

Two important measures

Measure 1 Superadditivity ratio

$$\bar{\Delta}_{\alpha,d}(X_d^+) = \frac{\overline{\text{VaR}}_{\alpha}(X_d^+)}{\sum_{i=1}^d \text{VaR}_{\alpha}(X_i)}.$$

Measure 2 Ratio between worst-ES and worst-VaR

$$\mathcal{B}_{\alpha,d}(X_d^+) = \frac{\overline{\text{ES}}_{\alpha}(X_d^+)}{\overline{\text{VaR}}_{\alpha}(X_d^+)} = \frac{\sum_{i=1}^d \text{ES}_{\alpha}(X_i)}{\overline{\text{VaR}}_{\alpha}(X_d^+)}.$$

Dependence Uncertainty

Superadditivity ratio: some examples

- Short tailed risks

- LogNormal(2,1)-distributed risks $\Rightarrow \bar{\Delta}_{0.999,d}(X_d^+) \approx 1.4$.

- Gamma(3,1)-distributed risks $\Rightarrow \bar{\Delta}_{0.999,d}(X_d^+) \approx 1.1$.

- Heavy tailed risks

- Pareto(2)-distributed risks $\Rightarrow \bar{\Delta}_{0.999,d}(X_d^+) \approx 2$.

In QRM applications often Pareto(θ) with $\theta \in [0.5, 5]$:

- [0.5, 1] catastrophe insurance,

- [3, 5] market return data,

- $\theta \geq 0.5$ for operational risk.

VaR versus ES: Dependence Uncertainty

Asymptotic equivalence for large dimensions of the risk portfolio, under some general conditions:

$$\lim_{d \rightarrow \infty} \frac{\overline{\text{ES}}_{\alpha}(X_d^+)}{\overline{\text{VaR}}_{\alpha}(X_d^+)} = 1$$

▶ details

- In the case of F_i being identical:

$$\overline{\Delta}_{\alpha,d}(X_d^+) \approx \frac{\text{ES}_{\alpha}(X_1)}{\text{VaR}_{\alpha}(X_1)}.$$

Application: Operational Risk

Definition

Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems, or external events.

Remark: This definition includes legal risk but excludes reputational and strategic risk.

Application: Operational Risk

The LDA Operational risk capital calculation under Basel II

The ingredients:

- Risk measure VaR_α
- Holding period: 1 year
- Confidence level: 99.9%, $\alpha = 0.999$
- The data 7×8 matrix; 8 Business lines, 7 Loss types
- Often: aggregate column-wise $\Rightarrow \text{VaR}_\alpha^{(1)}, \dots, \text{VaR}_\alpha^{(8)}$

$$\text{Aggregate: } \sum_{i=1}^8 \text{VaR}_\alpha^{(i)} = \text{VaR}_\alpha^+.$$

Example: Pareto(2) risks

Sharp bounds on the VaR and ES for the sum of d Pareto(2) distributed rvs for $\alpha = 0.999$; VaR_α^+ corresponds to the comonotonic case.

	$d = 8$	$d = 56$
$\underline{\text{VaR}}_\alpha$	31	53
$\underline{\text{ES}}_\alpha$	178	472
VaR_α^+	245	1715
$\overline{\text{VaR}}_\alpha$	465	3454
$\overline{\text{ES}}_\alpha$	498	3486
$\overline{\Delta}_\alpha(X_d^+)$	1.898	2.014
$\mathcal{B}_\alpha(X_d^+)$	1.071	1.009

An inhomogeneous Portfolio

Sharp bounds on the VaR and ES for an inhomogeneous portfolio divided into 3 homogeneous subgroups i.e $d = 3k$ having marginals distributed as $F_1 = \text{Pareto}(2)$, $F_2 = \text{Exp}(1)$, $F_3 = \text{LogN}(0,1)$, $\alpha = 0.999$.

	$k = 1$	$k = 3$	$k = 10$	$k = 20$
$\underline{\text{VaR}}_\alpha$	31	31	36	71
$\underline{\text{ES}}_\alpha$	64	107	190	264
VaR_α^+	60	179	595	1190
$\overline{\text{VaR}}_\alpha$	77	277	979	1982
$\overline{\text{ES}}_\alpha$	100	301	1003	2006
$\overline{\Delta}_\alpha(X_d^+)$	1.2833	1.5475	1.6454	1.6655
$\mathcal{B}_\alpha(X_d^+)$	1.299	1.087	1.025	1.012

Backtesting

Recall from R1, Page 41, Question 8
"*... robust backtesting ...*"

Backtesting:

- (i) estimate a risk measure from past observations;
- (ii) test whether (i) is appropriate using future observations;
- (iii) purpose: test and update risk measure forecasts.

Backtesting

Example - VaR backtesting:

- (1) suppose the estimated/modeled VaR_α is V at $t = 0$;
- (2) consider $A_t = I_{\{X_t > V\}}$ based new iid observations $X_t, t > 0$;
- (3) standard hypothesis testing methods for $H_0: A_t$ are iid Bernoulli($1 - \alpha$) random variables.

For ES such simple and intuitive backtesting techniques do not exist!

Backtesting

Elicitability

- A new notion for comparing risk measure forecasts: **elicitability**; Gneiting (2011).
- Roughly speaking, a risk measure (statistical functional) $\rho : \mathcal{P} \rightarrow \mathbb{R}$ is elicitable if ρ is the unique solution to the following equation:

$$\rho(L) = \operatorname{argmin}_{x \in \mathbb{R}} \mathbb{E}[s(x, L)],$$

where

- $s : \mathbb{R}^2 \rightarrow [0, \infty)$ is a **strictly consistent scoring function**;
- for example, the mean is elicitable with $s = (x - L)^2$.

Backtesting

Elicitability and backtesting

- suppose the estimated/modeled ρ is ρ_0 at $t = 0$;
- based on new iid observations $X_t, t > 0$, consider the statistics $s(\rho_0, X_t)$; for instance, test statistic can typically be chosen as $T_n(\rho_0) = \frac{1}{n} \sum_{t=1}^n s(\rho_0, X_t)$;
- $T_n(\rho_0)$: a statistic which indicates the goodness of forecasts.
- updating ρ : look at a minimizer for $T_n(\rho)$;
- the above procedure is **model-independent**.

Elicitable statistics are **straightforward** to backtest.

Backtesting

VaR vs ES: elicibility

Theorem (Gneiting (2011)).

Under general conditions,

- VaR is elicitable;
- ES is not elicitable.

Backtesting

Remarks:

- under specific EVT-based conditions, backtesting of ES is possible; see McNeil et al. (2005), p.163;
- the relevance of elicibility for risk management purposes is heavily contested:
 - Emmer, Kratz and Tasche (2014): alternative method for [backtesting ES](#); favors ES.
 - Davis (2014): backtesting based on [prequential principle](#); favors quantile-based statistics (VaR-type).

Robustness

Robustness - some quotes

A precise definition matters!

- Cont et al. (2010): "Our results illustrate in particular, that using recently proposed risk measures such as **CVaR/Expected Shortfall** leads to a **less robust** risk measurement procedure than Value-at-Risk."
- Kou et al. (2013): "**Coherent** risk measures are **not robust**", proposed Median Shortfall (VaR-like).
- Emmer et al. (2014): "The fact that VaR does not cover tail risks 'beyond' VaR is a more serious deficiency although **ironically** it makes **VaR** a risk measure that is **more robust** than the other risk measures we have considered."

Robustness

Example: different internal models

- Same data set, two different parametric models (e.g. normal vs student-t).
- Estimation of parameters, and compare the VaR and ES for two models.
- VaR is more robust in this setting, since **it does not take the tail behavior into account** (normal and student-t do not make a big difference).
- ES is less robust (heavy reliance on the model's tail behavior).
- Capital requirements: heavily depends on the internal models.

Robustness

Opposite opinions

- Cambou and Filipovic (2014): "ES is robust, and VaR is non-robust based on the notion of ϕ -divergence".
- Krätschmer et al. (2014): "We argue here that Hampel's classical notion of qualitative robustness is **not suitable** for risk measurement ..." (Introduce an index of qualitative robustness).
- BCBS (2013, R4): "This confidence level [97.5th ES] will provide a broadly similar level of risk capture as the existing 99th percentile VaR threshold, while providing a number of benefits, including generally **more stable** model output and often **less sensitivity** to extreme outlier observations."

Much more work is needed!

VaR versus ES: Summary

Value-at-Risk

- 1 Always exists
- 2 Only frequency
- 3 Non-coherent risk measure
(non-subadditive)
 - Heavy tailed
 - Very skew
 - Special dependencies
- 4 Backtesting
straightforward
- 5 Estimation (EVT)
- 6 Model uncertainty
- 7 Robust with respect to
weak topology

Expected Shortfall

- 1 Needs first moment
- 2 Frequency and severity
- 3 Coherent risk measure
(diversification benefit)
- 4 Backtesting an issue
(non-elicitability)
- 5 Estimation (EVT)
- 6 Model uncertainty
- 7 Robust with respect to
Wasserstein distance

The Holy Triangle of Risk Measures

There are many desired properties of a good risk measure.
Some properties are without debate:

- cash-invariance: $\rho(X + c) = \rho(X) + c, c \in \mathbb{R}$;
- monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$;
- identity: $\rho(1) = 1$;
- law-invariance: $\rho(X) = \rho(Y)$ if $X =_d Y$.

(A **standard** risk measure; those properties are not restrictive)

The Holy Triangle of Risk Measures

In my opinion, in addition to being standard, the three key elements of being a good risk measure are

- (C) Coherence (subadditivity): $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
[diversification benefit/aggregate regulation/capturing the tail]
- (A) Comonotone additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if X and Y are comonotone. [economical interpretation/distortion representation/non-diversification identity]
- (E) Elicitability [robust estimation/backtesting straightforward].

The War of the Two Kingdoms

- Some financial mathematicians
 - appreciate coherence (subadditivity);
 - favor ES in general.
- Some financial statisticians
 - appreciate backtesting and statistical advantages;
 - favor VaR in general.

A natural question is to find a standard risk measure which is both coherent (subadditive) and elicitable.

Expectiles

Expectiles

- For $0 < \tau < 1$ and $X \in L^2$,

$$e_\tau(X) = \operatorname{argmin}_{x \in \mathbb{R}} \mathbb{E}[\tau \max(X - x, 0)^2 + (1 - \tau) \max(x - X, 0)^2].$$

- $e_\tau(X)$ is the unique solution x of the equation for $X \in L^1$:

$$\tau \mathbb{E}[(X - x)^+] = (1 - \tau) \mathbb{E}[(x - X)^+].$$

- $e_{1/2}(X) = \mathbb{E}[X]$.

Expectiles

The risk measure e_τ has the following properties:

- 1 homogeneous and standard,
- 2 **subadditive** for $1/2 \leq \tau < 1$, superadditive for $0 < \tau \leq 1/2$,
- 3 **elicitable**,
- 4 **coherent** for $1/2 \leq \tau < 1$,
- 5 **not comonotone additive** in general.

Bellini et al. (2014), Ziegel (2014), Delbaen (2014).

The War of the Three Kingdoms

In summary:

- VaR has (A) and (E): often criticized for not being subadditive: **diversification/aggregation problems and inability to capture the tail!**
- ES has (C) and (A): criticized for **estimation, backtesting and robustness problems!**
- Expectile has (C) and (E): criticized for **lack of economical meaning, distributional computation and over-diversification benefits!**

The War of the Three Kingdoms

The following holds (Bellini and Bignozzi (2014)):

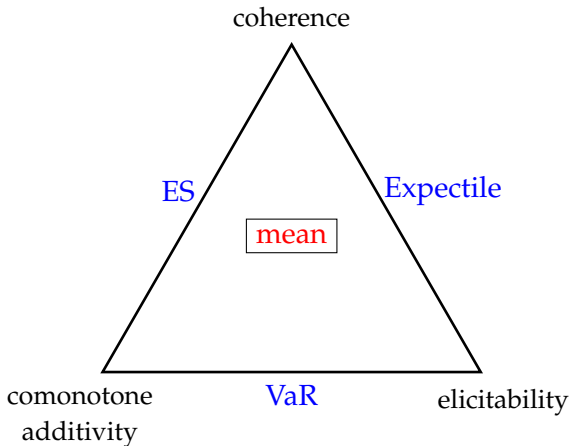
- if ρ is coherent, and elicitable with a convex scoring function, then ρ is an expectile;
- any spectral risk measure (coherent and comonotone additive) must not be elicitable, expect for the mean.

In summary:

The only standard risk measure that has (C), (A) and (E) is [the mean](#), which is not a tail risk measure, and does not have a risk loading.

- Remark: the very old-school risk measure/pricing principle $\rho(X) = (1 + \theta)\mathbb{E}[X]$, $\theta > 0$ has (C), (A) and (E), although it is not standard.

The Holy Triangle of Risk Measures



Extreme-scenario Measure

Wang, Bignozzi and Tsanakas (2014)

Coherence is indeed a natural property desired by a **good risk measure**. Even when a non-coherent risk measure is used for a portfolio, its extreme behavior under **dependence uncertainty** leads to coherence.

Example: the extreme behavior of VaR is the corresponding ES.

◀ details






Conclusion

- C1 Q8 and Basel 3.5: a short question with many ramifications. No clear answer so far.
- C2 On ES or VaR? **ES!** . . . however . . .
- C3 Concerning MU and VaR bounds:
 - Find sharp couplings
 - Are they realistic in practice?
 - Impose extra dependence assumptions
 - Add statistical uncertainty
- C4 Many more examples needed
- C5 Expectiles as an alternative?






References I

-  BCBS (2012). Consultative Document, May 2012. Fundamental review of the trading book. Bank for International Settlements, Basel.
-  BCBS (2013). Consultative Document, October 2013. Fundamental review of the trading book: A revised market risk framework. Bank for International Settlements, Basel.
-  Bernard, C., X. Jiang, and R. Wang (2014). Risk aggregation with dependence uncertainty. *Insurance: Mathematics and Economics*, **54**(1), 93–108.
-  Bellini, F. and V. Bignozzi (2014). Elicitable Risk Measures. Available at SSRN: <http://ssrn.com/abstract=2334746>





References II

-  Bellini, F., Klar, B., Müller, A. and Rosazza Gianin, E. (2014). Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, **54**(1), 41-48.
-  Cambou, M. and D. Filipovic (2014). Scenario aggregation for solvency regulation. *Preprint, EPFL Lausanne*.
-  Cont, R., R. Deguest, and G. Scandolo (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative Finance*, 10(6), 593–606.
-  Davis, M.H.A. (2014). Consistency of risk measure estimates. *Preprint, Imperial College London*.
-  Delbaen, F. (2014). A remark on the structure of expectiles. *Preprint, ETH Zurich*.





References III

-  Embrechts, P., G. Puccetti, and L. Rüschendorf (2013). Model uncertainty and VaR aggregation. *Journal of Banking and Finance*, 37(8), 2750–2764.
-  Embrechts, P., B. Wang, and R. Wang (2014). Diversification penalty of Value-at-Risk. *Preprint, ETH Zurich*.
-  Emmer, S., M. Kratz, and D. Tasche (2014). What is the best risk measure in practice? A comparison of standard measures. *Preprint, Bank of England*.
-  Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106, 746–762.
-  Kou, S., X. Peng, and C.C. Heyde (2013). External risk measures and Basel accords. *Mathematics of Operations Research*, 38(3), 393–417.




References IV

-  Krätschmer, V., A. Schied, and H. Zähle (2014). Comparative and qualitative robustness for law-invariant risk measures. *Preprint, University of Mannheim*.
-  Makarov, G.D. (1981). Estimates for the distribution function of the sum of two random variables with given marginal distributions. *Theory of Probability and its Applications*, 26, 803–806.
-  McNeil, A., R. Frey, and P. Embrechts (2005). *Quantitative Risk Management*. Princeton University Press.
-  Puccetti, G., and L. Rüschendorf (2013). Sharp bounds for sums of dependent risks. *Journal of Applied Probability*, 50(1), 42-53.

References V

-  Rüschemdorf, L. (1982). Random variables with maximum sums. *Advances in Applied Probability*, 14(3), 623–632.
-  UK (2013). UK House of Lords/House of Commons. Report, Document June 12, 2013. Changing banking for good. Volumes I and II.
-  USS (2013). United States Senate. Report, Document March 15, 2013. JPMorgan Chase Whale Trades: A Case History of Derivatives Risks and Abuses.
-  Wang, B. and R. Wang (2011). The complete mixability and convex minimization problems with monotone marginal densities. *J. Multivariate Anal.* 102(10), 1344–1360.

References VI

-  Wang, R., V. Bignozzi, and A. Tsakanas (2014). How superadditive can a risk measure be? *Preprint, University of Waterloo*.
-  Wang, R., L. Peng, and J. Yang (2013). Bounds for the sum of dependent risks and worst value-at-risk with monotone marginal densities. *Finance Stoch.* 17(2), 395–417.
-  Ziegel, J. (2014). Coherence and elicibility. *Mathematical Finance*, to appear.

THANK YOU!

VaR versus ES, 0.99 vs 0.975

- In general: for $\xi \in [0, 1)$ ($\xi = 0$ indicates a light tail),

$$\frac{\text{ES}_{0.975}(X)}{\text{VaR}_{0.975}(X)} \approx \frac{1}{1 - \xi},$$

and

$$\frac{\text{VaR}_{0.99}(X)}{\text{VaR}_{0.975}(X)} \approx 2.5^\xi.$$

Putting the above together,

$$\frac{\text{VaR}_{0.99}(X)}{\text{ES}_{0.975}(X)} \approx 2.5^\xi(1 - \xi).$$

VaR versus ES, 0.99 vs 0.975

- $\xi \in [0, 1)$,

$$\frac{\text{VaR}_{0.99}(X)}{\text{ES}_{0.975}(X)} \approx 2.5^\xi(1 - \xi) \leq e^\xi(1 - \xi) \leq 1.$$

Approximately, $\text{ES}_{0.975}$ yields a more conservative regulation principle than $\text{VaR}_{0.99}$.

- For a particular X , it is not always $\text{ES}_{0.975}(X) \geq \text{VaR}_{0.99}(X)$.

VaR versus ES, 0.99 vs 0.975

- Light-tailed distributions: as $\xi \rightarrow 0$,

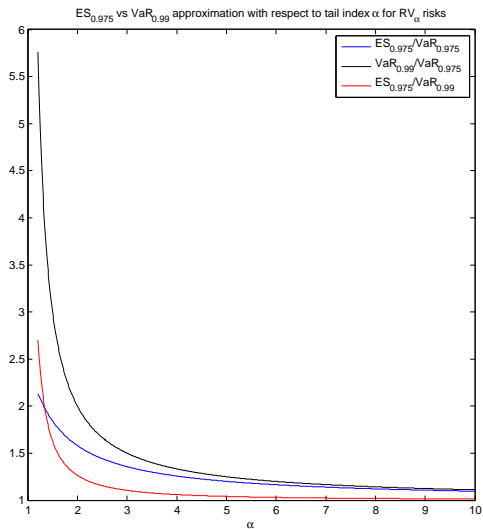
$$\frac{\text{VaR}_{0.99}(X)}{\text{ES}_{0.975}(X)} \approx 2.5^\xi(1 - \xi) \rightarrow 1.$$

For **light-tailed** distributions, $\text{ES}_{0.975}$ yields an (**approximately**) equivalent regulation principle as $\text{VaR}_{0.99}$.

- It seems that the value

$$c = 2.5 = (1 - 0.975)/(1 - 0.99)$$

is chosen such that c is close to $e \approx 2.72$, so that the approximation $c^\xi(1 - \xi) \approx 1$ holds most accurate for small ξ ; note that $e^{-\xi} \approx 1 - \xi$ for small ξ .

VaR versus ES, 0.99 vs 0.975 ($\alpha = 1/\xi$)

VaR Bounds

Makarov and Rüschendorf

For $d = 2$, sharp tail bound for any $s \in \mathbb{R}$ is:

$$\sup\{P(X_1 + X_2 \geq s) : X_i \sim F_i\} = \inf_{x \in \mathbb{R}} \{\bar{F}_1(x-) + \bar{F}_2(s - x)\},$$

where $\bar{F}_i(x) = 1 - F_i(x) = P(X_i > x)$ and $\bar{F}_1(x-) = P(X_1 \geq x)$.

◀ back

VaR Bounds

Sharp VaR bounds (Wang, Peng and Yang (2013))

Suppose that the density function of F is decreasing on $[b, \infty)$ for some $b \in \mathbb{R}$. Then, for $\alpha \in [F(b), 1)$, and $X \stackrel{d}{\sim} F$,

$$\overline{\text{VaR}}_{\alpha}(X_d^+) = d\mathbb{E}[X | X \in [F^{-1}(\alpha + (d-1)c_{d,\alpha}), F^{-1}(1 - c_{d,\alpha})]],$$

$c_{d,\alpha}$ is the smallest number in $[0, \frac{1}{d}(1 - \alpha)]$ s.t.

$$\int_{a+(d-1)c}^{1-c} F^{-1}(t) dt \geq \frac{1-\alpha-dc}{d} (F^{-1}(\alpha + (d-1)c) + F^{-1}(1 - c)).$$

Red part clearly has an ES-type form ($c_{d,\alpha} = 0$ leads to ES).

VaR Bounds

Sharp VaR bounds II

Suppose that the density function of F is decreasing on its support. Then for $\alpha \in (0, 1)$ and $X \stackrel{d}{\sim} F$,

$$\underline{\text{VaR}}_{\alpha}(X_d^+) = \max\{(d-1)F^{-1}(\alpha) + F^{-1}(0), d\mathbb{E}[X|X \leq F^{-1}(\alpha)]\}.$$

Red part has a Left-Tail-ES-type form.

Complete Mixability

Definition (Complete mixability, Wang and Wang (2011))

A distribution function F on \mathbb{R} is called d -completely mixable (d -CM) if there exist d random variables X_1, \dots, X_d identically distributed as F such that

$$P(X_1 + \dots + X_d = dk) = 1, \quad (1)$$

for some $k \in \mathbb{R}$.

- Some examples of CM distributions: Normal, Student t , Cauchy, Uniform, Binomial.
- Most relevant result: F has **monotone densities on a finite interval** with a mean condition (depends on d) is d -CM.
 - Examples: truncated Pareto, Gamma, Log-normal.

Asymptotic Equivalence

Theorem (Embrechts, Wang and Wang (2014))

Suppose the continuous distributions F_i , $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $\alpha \in (0, 1)$,

- (i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ is uniformly bounded for some $k > 1$;
- (ii) $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{ES}_\alpha(X_i) > 0$.

Then

$$\lim_{d \rightarrow \infty} \frac{\overline{\text{ES}}_\alpha(X_d^+)}{\overline{\text{VaR}}_\alpha(X_d^+)} = 1.$$

◀ back

Extreme-scenario Measure

- For any risk measure ρ , denote its worst-case value under dependence uncertainty as $\bar{\rho}(X_d^+)$.
- For $X \sim F$, let

$$\Gamma_\rho(X) = \limsup_{d \rightarrow \infty} \frac{1}{d} \bar{\rho}(X_d^+),$$

where $X_d^+ = X_1 + \dots + X_d$ and $X_i \sim F, i = 1, \dots, d$.

- Γ_ρ is called an **extreme-scenario measure** induced by ρ .
- Γ_ρ represents the limiting worst-case value of ρ for a homogeneous portfolio.
- Example: $\Gamma_{\text{VaR}_\alpha} = \text{ES}_\alpha$.

Extreme-scenario Measure

Theorem (Wang, Bignozzi and Tsanakas (2014))

For commonly used classes of risk measures ρ , Γ_ρ is a coherent risk measure. Moreover, it is

- (a) the smallest subadditive risk measure that dominates ρ ;
- (b) a spectral risk measure if ρ is a distortion risk measure;
- (c) an expectile if ρ is a shortfall risk measure;
- (d) the mean if ρ is a superadditive distortion risk measure.

◀ back