Climate Change in Solowia

J. P. Amigues[1]

[1]TSE LERNA, INRA, University of Toulouse

Toulouse Conference in honor of Michel Moreaux
November 18th 2011

Summary

Solowia

Amigues

Motivations

Framework

Growth after
the ceiling

Growth during
the ceiling

Growth before
the ceiling

- After the Club de Rome report, a strong theoretical reaction among economists (RES symposium, 1974).
- After the first IPCC reports, move to simulation models of climate change and growth (e.g. the Stern Review).
- Two problems :
  - How to interpret the results from IAMs ?
  - Which strong economic points have to be raised in the public debate ?
- A +2 ˚ objective is a constraint affecting negatively growth and welfare. But who ? When ? And how much ?

A Tale of Solowia

- Solowia derives consumption from durable capital and a polluting non renewable resource (coal).
- Solowia enjoys exogenous technical progress
- Carbon pollution accumulates in the atmosphere but may be regenerated.
- No direct effect of pollution upon welfare.
- The Royal Academy of Sciences managed to convince the King of Solowia to keep the atmospheric carbon concentration below some critical threshold.
- The King's economists bother about growth and welfare consequences of this constraint.

A Stiglitz like model of a polluting resource.

- The King's Planning Board face the planning problem :

$$\max_{x(t),c(t)} \quad \int_0^\infty u(c(t))e^{-\rho t}dt$$

$$\dot{K}(t) = e^{\delta t}f(K(t), x(t)) - c(t)$$

$$\dot{X}(t) = -x(t)$$

$$s.t. \quad \dot{Z}(t) = \zeta x(t) - \alpha Z(t) \qquad .$$

$$x(t) \geq 0 \ , \ c(t) \geq 0 \ ,$$

$$Z^0 \leq Z(t) \leq \bar{Z}$$

A simple model of a polluting resource

- The resource and capital are essential inputs
- The utility function is increasing, concave and satisfies the first Inada condition.
- King's economists envision a three phases scenario
  - A first pre-ceiling phase $[0, \underline{t})$.
  - A ceiling phase $[\underline{t}, \bar{t})$ during which $x(t) = \bar{x} \equiv \alpha \bar{Z}/\zeta$.
  - A post ceiling phase $[\bar{t}, \infty)$ during which $Z(t) < \bar{Z}$ and coal is ultimately exhausted

Efficiency.

- During the post ceiling phase, the standard Hotelling efficiency rule applies :

$$\frac{(e^{\delta t}f_x)\dot{}}{e^{\delta t}f_x} \;\; = \;\; e^{\delta t}f_K$$

- During the ceiling phase, Solowia growth path follows a $\bar{x}$ constrained Ramsey-Solow process

Efficiency

- During the pre-ceiling phase, efficiency requires that :

$$e^{\delta t} f_K = \frac{(e^{\delta t} f_x)\dot{}}{e^{\delta t} f_x} + \frac{\frac{d}{dt}\{\frac{(e^{\delta t} f_x)\dot{}}{e^{\delta t} f_x} - e^{\delta t} f_K\}}{\frac{(e^{\delta t} f_x)\dot{}}{e^{\delta t} f_x} - e^{\delta t} f_K} - \alpha$$

- of the form :

$$\frac{\dot{n}}{n} = \alpha - n \quad \text{where} : n \equiv \frac{(e^{\delta t} f_x)\dot{}}{e^{\delta t} f_x} - e^{\delta t} f_K$$

Optimality

- Short term conditions :

$$
\begin{aligned}
e^{-\rho t} u'(c) &= \pi \\
e^{\delta t} \pi f_x + \zeta \mu &= \lambda \quad (\mu < 0) \\
\nu \geq 0 \quad &, \quad \nu(\bar{Z} - Z) = 0 \, , \, \bar{Z} - Z \geq 0 \, .
\end{aligned}
$$

- Dynamic conditions :

$$
\begin{aligned}
-\frac{\dot{\pi}}{\pi} &= e^{\delta t} f_K \\
\dot{\mu} &= \alpha \mu + \nu
\end{aligned}
$$

- gives the Ramsey-Keynes condition :

$$
-\frac{u''(c)}{u'(c)} \dot{c} + \rho \quad = \quad -\frac{\dot{\pi}}{\pi} = e^{\delta t} f_K \quad \forall t \geq 0
$$

More assumptions

- Visiting Solowia, Cobb and Douglas managed to convince the King that their famous form was adequate to describe the Solowia production possibilities frontier :

$$y \;=\; e^{\delta t} K^{\beta} x^{\gamma} \quad \beta + \gamma < 1$$

- The King's favorite risk analyst advocated the use of a CRRA function to describe Solowians preferences

$$u(c) \;=\; \frac{1}{1-\eta} c^{1-\eta} \quad \eta > 0 \,,\; \eta \neq 1$$

- The King's econometricians estimates conclude that :

$$\beta < 1 < \eta$$

Change of variables

- The King's Planning Board adopted the Stiglitz approach and set :

$$a(t) \equiv \frac{c(t)}{K(t)} \quad , \quad b(t) \equiv \frac{y(t)}{K(t)}$$

- They get from the Ramsey-Keynes condition :

$$\frac{\dot{a}(t)}{a(t)} = a(t) - \frac{\eta - \beta}{\eta} b(t) - \frac{\rho}{\eta}$$

$$\frac{\dot{K}(t)}{K(t)} = b(t) - a(t)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta} \left[\beta b(t) - \rho \right]$$

The dynamics apply over all possible phases.

Implications of efficiency

- While $b(t)$ dynamics differ between phases :
  - During the pre-ceiling phase $[0, \underline{t})$:

$$\frac{\dot{b}(t)}{b(t)} = \frac{1 - \beta - \gamma}{1 - \gamma}a(t) - (1 - \beta)b(t) + \frac{\delta - \gamma n(t)}{1 - \gamma}$$

  - During the ceiling phase $[\underline{t}, \overline{t})$:

$$\frac{\dot{b}(t)}{b(t)} = (1 - \beta)a(t) - (1 - \beta)b(t) + \delta$$

  - During the post-ceiling phase $[\overline{t}, \infty)$

$$\frac{\dot{b}(t)}{b(t)} = \frac{1 - \beta - \gamma}{1 - \gamma}a(t) - (1 - \beta)b(t) + \frac{\delta}{1 - \gamma}$$

Solving procedure

- The Solowia economists followed a backward procedure :
  - Describe the growth path during the post-ceiling phase in the (a, b) space.
    Start : $x(\bar{t}) = \bar{x}, K(\bar{t}) = \bar{K}$.
  - Describe the growth path during the ceiling phase in the (a, b) space.
    Start : $x(t) = \bar{x}, X(\underline{t}) = \underline{X}, K(\underline{t}) = \underline{K}$.
  - Describe the growth path during the pre-ceiling phase in the (a, b, n) space.
    Start : $x(\underline{t}) = \bar{x}, Z(\underline{t}) = \bar{Z}, X(0) = X^0, K(0) = K^0$,
    $Z(0) = Z^0$.

- The Stiglitz survival condition holds :

$$\rho \leq \delta/\gamma$$

- The Solowia economy converges towards stationary values a^* of $a(t)$ and b^* of $b(t)$.

- and asymptotic growth rates of its main macroeconomic variables :

$$g^{K*} = g^{y*} = g^{c*} \quad = \quad \frac{1}{\eta}\left[\beta b^* - \rho\right] = \frac{\delta - \gamma\rho}{1 - \beta - \gamma + \gamma\eta}$$

- It is easily checked that :

$$\frac{\dot{x}(t)}{x(t)} \quad = \quad \frac{1}{1-\gamma}\left(\delta - \beta a(t)\right) < 0$$

Phase diagram of the post-ceiling phase
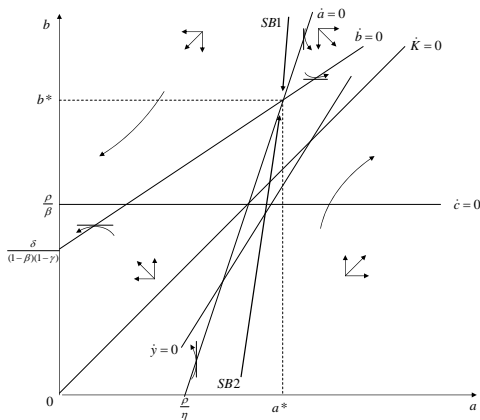
FIG.: **Optimal growth after the ceiling if $\gamma\rho < \delta$**

Sensitivity analysis

Proposition

Let $\bar{c} \equiv c(\bar{t})$, $g^h \equiv \dot{h}/h$ for any variable h, $\bar{X} = X(\bar{t})$, the required resource stock to follow the optimal trajectory from $\bar{t}$, then :

1. $\partial \bar{X}/\partial \bar{K} > 0$, $\partial \bar{X}/\partial \bar{x} > 0$.

2. $\partial g^c(t)/\partial \bar{K} < 0$, $\partial g^c(t)/\partial \bar{x} > 0$, $t \geq \bar{t}$.

3. $\partial \bar{c}/\partial \bar{K} > 0$, $\partial \bar{c}/\partial \bar{x} > 0$.

4. $\partial g^x(t)/\partial \bar{K} > 0$, $\partial g^x(t)/\partial \bar{x} < 0$, $t \geq \bar{t}$.

5. $\partial g^K(t)/\partial \bar{K} < 0$, $\partial g^K(t)/\partial \bar{x} > 0$, $t \geq \bar{t}$.

- The Solowia economics obeys a Ramsey Solow growth model constrained by the ceiling $\bar{x}$.

- The growth process would converge towards stationary levels $\hat{a}$ of $a(t)$ and $\hat{b}$ of $b(t)$.

- It is easily checked that :

$$a^* < \hat{a} \quad \text{and} \quad b^* < \hat{b}$$

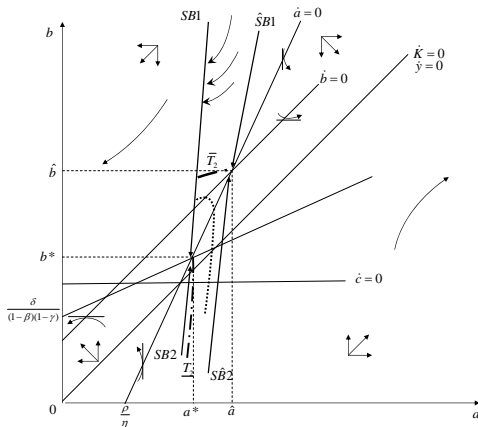- Concentrate upon transition trajectories connecting to the high saddle branch after the ceiling.

FIG.: **Optimal growth during the ceiling**

Closed form solution from $(\underline{X}, \underline{K}, \underline{t})$

■ Implications of the resource stock constraint :

$$\underline{X} = \bar{x}(\bar{t} - \underline{t}) + \bar{X}(\bar{K}, \bar{x})$$

■ It defines a decreasing relationship between $\bar{t}$ and $\underline{a}$, $\bar{t}_X(\underline{a})$

FIG.: **Admissible levels of $\bar{X}$ for a given $(\underline{K}, \underline{X})$.**

Closed form solution

- The $(a(t), b(t))$ Type 1 dynamics define an increasing relationship between $\bar{t}$ and $\underline{a}$, $\bar{t}_a(\underline{a})$

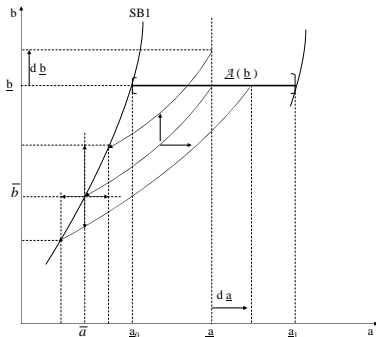FIG.: **Type 1 trajectories network.**

A fixed point argument

- For Type 1 trajectories, $(\underline{a}, \overline{t})$ are defined by the curves $t_X(\underline{a})$, $t_a(\underline{a})$:
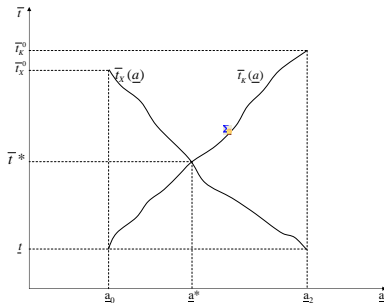

FIG.: **Determination of $\underline{a}$ and $\overline{t}$.**

Sensitivity analysis

Proposition

Let $\underline{c} \equiv c(\underline{t})$, then

1. *A higher initial resource stock level $\underline{X}$ induces :*
   - *A longer ceiling phase, $\partial \bar{t}/\partial \underline{X} > 0$;*
   - *A higher initial consumption level, $\partial \underline{c}/\partial \underline{X} > 0$;*
   - *A lower consumption growth rate, $\partial g^c(t)/\partial \underline{X} < 0$;*

2. *A higher initial capital stock level $\underline{K}$ induces :*
   - *A shorter ceiling phase, $\partial \bar{t}/\partial \underline{K} < 0$;*
   - *A lower initial consumption rate, $\partial \underline{c}/\partial \underline{K} < 0$,*
   - *A higher consumption growth rate, $\partial g^c(t)/\partial \underline{K} > 0$*

3. *A less stringent ceiling constraint, that is a higher level of $\bar{x}$ induces :*
   - *A shorter ceiling phase, $\partial \bar{t}/\partial \bar{x} < 0$;*
   - *An ambiguous effect over $\underline{c}$;*
   - *An ambiguous effect over $g^c(t)$ of the opposite sign of the effect over $\underline{c}$.*

The Economists Address to the King

- A stricter ceiling constraint does not necessarily mean lower consumption or lower growth during the ceiling phase
- However it always imply staying longer at the ceiling.
- A stricter ceiling has ambiguous effects upon capital accumulation during the ceiling phase and thus over growth and welfare after the ceiling.

A rather comfortable policy message... But what happens **before** the ceiling ?

- Growth has to be described in the 3-dimensional space (a, b, n).
- Fortunately, $\dot{n} = \alpha n - n^2$ is independent from a and b and a solution of the Ricatti equation is :

$$n(t) = \frac{\alpha n^0}{(\alpha - n^0)e^{-\alpha t} + n^0} \quad n(0) = n^0$$

- It is easily checked that $n(t)$ increases when the economy approaches the ceiling.
- The locus $\dot{b} = 0$ moves downwards with n increasing.
- Trajectories connecting to the ceiling trajectories are such that : $\dot{a} < 0$, $\dot{b} < 0$.
- This implies that $\dot{c} > 0$, $\dot{K} > 0$ and $\dot{y} > 0$ before the ceiling.

Closed form solution

- It has been shown that $(\underline{K}, \underline{X}, \underline{t})$ define a unique pair $(\underline{a}, \bar{t})$ and $\underline{b} = e^{\delta t} \underline{K}^{(\beta-1)} \bar{x}^{\gamma}$.
- Thus $(\underline{a}, \underline{b})$ are uniquely determined by $(\underline{K}, \underline{X}, \underline{t})$.
- Before the ceiling, $g^a(a(t), b(t))$, $g^b(a(t), b(t), n(t; n^0))$ define with $a(\underline{t}) = \underline{a}$, $b(\underline{t}) = \underline{b}$ a unique trajectory :

$$
\begin{aligned}
a_1(t; \underline{a}, \underline{b}, \underline{t}, n^0) &\equiv a_1(t, \underline{K}, \underline{X}, \underline{t}, n^0) \\
b_1(t; \underline{a}, \underline{b}, \underline{t}, n^0) &\equiv b_1(t; \underline{K}, \underline{X}, \underline{t}, n^0)
\end{aligned}
$$

- The extraction rate obeys the following dynamics before the ceiling :

$$
\frac{\dot{x}(t)}{x(t)} = \frac{\delta - \beta a(t) - n(t)}{1 - \gamma} .
$$

- This defines $g^x(t; \underline{K}, \underline{X}, \underline{t}, n^0)$.
- Last : $b_1(0) = (K^0)^{(\beta-1)} x(0)^{\gamma}$ gives $x^0(\underline{K}, \underline{X}, \underline{t}, n^0, K^0)$.

Closed form solution

- $(\underline{K}, \underline{X}, \underline{t}, n^0)$ are solution of :
  - Continuity condition over the extraction path at $\underline{t}$:

    $$\bar{x} \;\;=\;\; x^0(\underline{K}, \underline{X}, \underline{t}, n^0, K^0) e^{\int_0^{\underline{t}} g^x(t, \underline{K}, \underline{X}, \underline{t}, n^0) dt}$$

  - Capital accumulation condition before the ceiling :

    $$\underline{K} \;\;=\;\; K^0 e^{\int_0^{\underline{t}} g^K(t, \underline{K}, \underline{X}, \underline{t}, n^0) dt}$$

  - Resource stock condition :

    $$X^0 \;\;=\;\; x^0(\underline{K}, \underline{X}, \underline{t}, n^0, K^0) \int_0^{\underline{t}} e^{\int_0^t g^x(\tau, \underline{K}, \underline{X}, \underline{t}, n^0) d\tau} dt + \underline{X}$$

  - Pollution stock condition :

    $$\bar{Z} e^{\alpha \underline{t}} \;\;=\;\; Z^0 + \zeta x^0(\underline{K}, \underline{X}, \underline{t}, n^0, K^0) \int_0^{\underline{t}} e^{\int_0^t g^x(\tau, \underline{K}, \underline{X}, \underline{t}, n^0) d\tau} e^{-\alpha t} dt$$

Sensitivity analysis

- With respect to an unchanged $b(t)$ trajectory (or equivalently an unchanged consumption growth rate dynamics), it is is easily verified that a stricter ceiling means :
  - A sooner arrival at the ceiling
  - A slow down of coal extraction
  - Less investment in capital accumulation
  - A higher level of available coal reserves when arriving at the ceiling.
  - An increased difference between the rates of return of capital and the resource
  - An increased consumption level when arriving at the ceiling thus a higher $\underline{a}$

- Since $\underline{b}(\underline{a})$ is a decreasing function, a higher $\underline{a}$ requires to readjust b downwards.

A last economists address to the King

- After readjusting, a stricter ceiling will result in :
  - An ambiguous effect over the the arrival time at the ceiling
  - A slow down of coal extraction at least when approaching the ceiling
  - A higher level of available coal reserves when arriving at the ceiling
  - More investment in capital accumulation
  - A decreased consumption level and a lower growth rate
  - An increased difference between the rates of returns of the capital and the resource
- A higher capital stock and a higher level of the coal reserves put Solowia in better position to achieve growth when arriving at the ceiling...