Rate-optimal Tests for Jumps in Diffusion Processes

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ABSTRACT

Suppose one has given discrete observations of a continuous-time random process (like e.g. stock market data) and one wants to test for the presence of jumps. Then the power of the tests will depend on the frequency of observations. We show, that if the data are observed at intervals of length $1/n$, at best one can detect jumps of height $\sqrt{\log(n)/n}$. We construct a test which achieves this rate in the case of diffusion-type processes.

Keywords: High Frequency Data, Jump, Likelihood Test.

1. INTRODUCTION

Continuous diffusion models are the "workhorses" of models for financial time series. These diffusion models are simple, flexible and powerful in modeling. However, data are only observed at discrete times, so we do not have the "full" information on the trajectory of the process. Technological progress allows us to analyze high frequency data. Still, we may have modeling errors due to the discreteness of the observations, but this problem can be significantly mitigated.

High frequency data, however, generate their own challenges: We cannot be sure that the process modeling the data is continuous, there may be jumps. Furthermore, many data (when returns are measured in short intervals - say 1-5 minutes) contain some contamination commonly called the "market microstructure". Our aim is to propose an optimal test for the null hypothesis of continuous diffusion models against an alternative hypothesis of jump diffusion models considering market microstructure. In literature there are many tests for jump detection such as the ones by Barndorff-Nielsen and Shephard (2002) and Ait-Sahalia and Jacod (2009), but the power of these tests is rarely discussed. We derive a rate-optimal test from the general assumptions on data generating process.

2. LOCAL POWER BOUND

As null, we consider the usual diffusion model:

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where $W = (W_t : t \in [0, 1))$ is an Brownian motion, μ_t and σ_t are non-anticipating random processes fulfilling the usual requirements of Ito-calculus. Later on we will maintain even more assumption on

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μ and σ (We assume them to be smooth to a certain extent, so that the process X_t specified by the above equation has some nice properties.)

As an alternative, we want to consider jumps to diffusion model :

$$dX_t = \mu_t dt + \sigma_t dW_t + J_t d\kappa_t,$$

where J_t is a non-zero random variable whose absolute value specifies the jump size and κ_t is a counting process governing whether there is a jump or not.

The problem, however, is that we cannot observe the whole process, but only at discrete times $t = i/n$, where n is a natural number and

$$0 \leq i \leq n.$$

The first test for this testing problem - and still the "gold standard" for all tests was developed by Barndorff-Nielsen and Shephard (2002) only a few years ago. Since this problem is of enormous practical importance, a whole lot of research was (and is done) in this field. An alternative test was developed by Ait-Sahalia and Jacod (2009), and an informal "testing procedure" was given in Lee and Mykland (2008).

None of the results, however, discusses the power of the tests. Here we will show two things:

1. Clearly, when n, the number of observations increases, we should expect our test to have "better" power. In particular, we want to consider the power against local alternatives for the jumping processes. So we consider for each n the alternative

$$J_t^{(n)} = c_n$$

where we assume that c_n is a sequence converging to zero, and assume the process κ_t remains (uniformly) bounded. (So we assume there is only a maximum number of jumps). Let $\varepsilon > 0$ be arbitrary. Then we show that - even if we know that $\sigma_t = \sigma$ - it is impossible to construct tests that have nontrivial power against alternatives with

$$c_n = (1 - \varepsilon)\sigma \frac{\sqrt{\ln n}}{\sqrt{n}} \tag{2}$$

2. As a main result, we show that one can construct a test so that (even in the general case)

$$c_n = (\sqrt{2} + \varepsilon)\sigma_t \frac{\sqrt{\ln n}}{\sqrt{n}}$$

the power of the test converges to one. So in a certain way, our tests attain the "optimal rate". This is an advantage over the classical BNS or AJ tests: their local alternatives shrink with the order $n^{-1/4}$ (or - in the case of AJ - with the order of $n^{-1/2+1/p}$, where p is a positive number determining the test statistic).

Let us first deal with our first assertion. Let assume we even deal with the simplest case, namely $\mu_t = 0$ and $\sigma_t = 1$, so our underlying process X_t is a Wiener process. Then let us assume that - under the alternative - we only have one jump, and the time of the jump is distributed uniformly in the interval $[0, 1]$. We first will show that even under this rather ideal conditions we will be unable to construct tests with nontrivial power if the c_n are following (2).

THEOREM 2.1. *We want to test the null of X_t being a Wiener process W_t ,of known variance, against the alternative of*

$$X_t = W_t + c_n I(\tau \geq t),$$

where τ is an independent random variable following an uniform distribution. Suppose we observe the process X_t only at the time points $0, 1/n, 2/n, ...1$. Suppose c_n follows (2) (or is smaller then this bound). Then it is impossible to construct nontrivial tests.

Proof. We did assume the variance of the Wiener process W to be known. Without limitation of generality, we can assume this variance to be 1. Let P_n the probability measure of $(X_0, X_{1/n}, X_{2/n}, ..., X_1)$ under the null, and Q_n be the measure under the alternative. Let the z_i be defined as

$$z_i = \left(X_{i/n} - X_{(i-1)/n}\right)\sqrt{n}$$

Then the we can easily see that the z_i are i.i.d. standard normal, and that

$$\frac{dQ_n}{dP_n} = \frac{1}{n}\sum_{i=1}^{n} \exp((c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2)$$

Since each of the z_i is standard normal, the expectation of each $\exp((c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2)$ equals one. Moreover, $E\left(\exp((c_n\sqrt{n})z_i - \frac{1}{2}(c_n\sqrt{n})^2)\right)^2 = E(\exp(2(c_n\sqrt{n})z_i - (c_n\sqrt{n})^2)) = \exp((c_n\sqrt{n})^2)$. Hence the variance of $\frac{dQ_n}{dP_n}$ is smaller than $\exp((c_n\sqrt{n})^2)/n$, which converges to zero if c_n follows (2). Therefore

$$\frac{dQ_n}{dP_n} \to 1$$

in probability. Therefore for an arbitrary $\eta > 0$

$$P_n\left(\left[\left|\frac{dQ_n}{dP_n} - 1\right| > \eta\right]\right) \to 0.$$

Now let A_n be a sequence of events. Then we have

$$\begin{aligned}(1-\eta)P_n(A_n) - P_n\left(\left[\left|\frac{dQ_n}{dP_n} - 1\right| > \eta\right]\right) &< Q_n(A_n)\\ &< (1+\eta)P_n(A_n) + P_n\left(\left[\left|\frac{dQ_n}{dP_n} - 1\right| > \eta\right]\right).\end{aligned}$$

Since η was arbitrary, we can conclude that

$$P_n(A_n) - Q_n(A_n) \to 0.$$

Since A_n is an arbitrary sequence of events, we can conclude that the total variation between P_n and Q_n converges to zero, hence for all measurable functions φ_n with $0 \leq \varphi_n \leq 1$ we have

$$\int \varphi_n dP_n - \int \varphi_n dQ_n \to 0.$$

But this is exactly what we wanted to show: For every sequence of tests, the power under the null (P_n) is the same as under the alternative (Q_n). $\square$

Now we want to present a test statistic, for which we will show that we can reach this bound. The result above indicates that for fixed c_n our best statistic is an exponential sum of the z_i. We should, however, keep in mind that our z_i are increments over smaller and smaller time intervals. So - in order to consider relevant alternatives - we might be interested in alternatives where c becomes "large". In this case, the test statistic gives more and more influence to bigger values. So it might be a good idea to look at the "largest" value of the increments of X_t: Standard theory of diffusion processes guarantees that, when divided by σ_t, these increments are approximately normal. Since we do not know σ_t, we have to estimate it. Since σ_t is varying over time, a moving average of the squares of the increments seems to be natural. So we propose the test statistic: Let us define (for an arbitrary n) the $r_i = r_{i,n}$ by

$$r_i = r_{i,n} = \left(X_{i/n} - X_{(i-1)/n}\right)$$

Then choose an integer l (the "length of the window") and reject when

$$\tau_n = \sup_i \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-l}^2)/l}$$

becomes "too large". This is quite analogous to the test statistics of Lee and Mykland (2008): We standardize the return by an estimator for $\sigma^2(t, X_t)$. We use, however, the usual quadratic estimator instead of the bipower estimator. One might argue that jumps might distort our estimator. We think, however, that the much simpler form is justifiable, essentially for two reasons:

1. We assume that the jumps are separated events: *Before* the first jump, our estimator for $\sigma^2(t, X_t)$ will not be influenced by it.
2. We only use a window of size l for estimating $\sigma^2(t, X_t)$. So a jump will only influence a small number of estimated values. Our test would get only distorted if we had two jumps within a time-frame of length l/n, which we assume converges to zero.

The main reason, however, for using this specific estimator is convenience. Specifically, only lemma A.1 is essential for our proof. We think an analogous result will hold for a more general class of estimators.

3. CRITICAL VALUE AND POWER OF TEST

For the computation of the critical values, the following lemma is very helpful.

Let $z_i, i = 1, ..n$ be independent, identically distributed according to a standard normal distribution. Assume that for each n we have given an $l = l(n)$, and let us denote by $\mathcal{F}_i$ the $\sigma-$algebra generated by $z_i, z_{i-1}, ..$ Then let us define

$$w_i = \sum_{j=1}^{l} z_{i-j}^2,$$

$$\widehat{\sigma}_i^2 = w_i/l$$

and

$$\tau_i = z_i^2/\widehat{\sigma}_i^2.$$

Then we have the following lemma:

LEMMA 3.1. *Suppose*

$$l = o\,(n)\,,$$

but also

$$l \geq 2\log n.$$

Define for each $c > 0$ $K_n^* = K_n^*(c)$ *so that*

$$2E\left(\exp\left(-K_n^{*2}\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^{*2}\widehat{\sigma}_i^2}\right) = c/n \tag{3}$$

Then,

$$P\left(\max_{i=\ell+1,..n}\tau_i > K_n^*\right) \to 1-\exp\left(-c\right) \text{ as } n\to\infty.$$

Proof. First of all let us observe that $P\left(\max_{i=\ell+1,..n}\tau_i > K_n^*\right) = 1 - P\left(\max_{i=\ell+1,..n}\tau_i \leq K_n^*\right)$ *and*

$$P\left(\max_{i=\ell+1,..n}\tau_i \leq K_n^*\right) = E\left(\prod_{i=\ell+1,..n} I(\tau_i \leq K_n^*)\right).$$

It can immediately be seen that the τ_i *are* $\mathcal{F}_i$ *measurable. We will now repeatedly apply the optional sampling theorem for various stopping times. Let* $\varepsilon > 0$ *be arbitrary, and let* $M(\varepsilon)$ *be defined as in (14), (15), (16).*

Let us define the the stopping time ν *in the following way: Define* ν *to be the first index* $m \leq n-1$ *so that*

$$\sum_{j=\ell+1}^{m+1}\log E\left(I(\tau_i \leq K_n^*)/\mathcal{F}_{i-1}\right) < -c(1+\varepsilon)^3 \quad \text{or } \widehat{\sigma}_{m+1}^2 < M\left(\varepsilon\right)^2/K_n \text{ or } \sum_{j=\ell+1}^{m+1}\log E\left(I(\tau_i \leq K_n^*)/\mathcal{F}_{i-1}\right) > -c(1-\varepsilon)$$

and

$$n \text{ if no such } m \text{ exists.}$$

First of all let us observe that ν *is indeed a stopping time adapted to* $\mathcal{F}_i$*.: Since for* $i \leq m+1$ $E\left((\tau_i \leq K_n^*)/\mathcal{F}_{i-1}\right)$ *as well as* $\widehat{\sigma}_{m+1}^2$ *are* $\mathcal{F}_m$*-measurable, the event*

$$[\nu = n] \in \mathcal{F}_m.$$

We contend that

$$\lim_{n\to\infty} P([\nu = n]) = 1. \tag{4}$$

For showing (4). it is sufficient to first show that

$$P(\left[\inf\widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n\right]) \to 1 \tag{5}$$

and then - since $\log E\left(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}\right) \le 0$ -

$$P(\left[\sum_{j=\ell+1}^{n} \log E\left(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}\right) \ge -c(1+\varepsilon)^3\right] \cap \left[\inf \widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n\right]) \to 1. \tag{6}$$

(5) is an immediate consequence of lemma A.1. This lemma shows that

$$P\left[\inf \widehat{\sigma}_i^2 \le M\left(\varepsilon\right)^2/K_n\right] \le nP\left[\widehat{\sigma}_i^2 \le M\left(\varepsilon\right)^2/K_n\right] \to 0.$$

For the proof of (6), first observe that

$$E\left(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}\right) = 2\Phi(\sqrt{K_n^*\widehat{\sigma}_i^2}) - 1.$$

If $\widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n$ *, we can use inequality (16) and conclude that*

$$\begin{aligned} &\log\left(2\Phi(\sqrt{K_n^*\widehat{\sigma}_i^2}) - 1\right) \\ \ge \quad & -2(1+\varepsilon)^2 \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2} \end{aligned}$$

Hence

$$\begin{aligned} &\left[\sum_{j=\ell+1}^{n} \log E\left(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}\right) \ge -c(1+\varepsilon)^3\right] \cap \left[\inf \widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n\right] \\ \subseteq \quad & \left[-2(1+\varepsilon)^2 \sum_{j=\ell+1}^{n} \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2} \ge -c(1+\varepsilon)^3\right] \cap \left[\inf \widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n\right] \end{aligned}$$

Since we already know that $P\left(\left[\inf \widehat{\sigma}_i^2 > M\left(\varepsilon\right)^2/K_n\right]\right) \to 1$*, it is sufficient to show that*

$$P\left(\left[2\sum_{j=\ell+1}^{n} \exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2} \le c\left(1+\varepsilon\right)\right]\right) \to 1 \tag{7}$$

Let us now introduce the Y_j *by*

$$Y_j = 2\exp\left(-K_n^*\widehat{\sigma}_i^2/2\right)/\sqrt{2\pi K_n^*\widehat{\sigma}_i^2}$$

Then we can easily see that (7) is fulfilled if

$$\sum_{j=\ell+1}^{n} Y_j \to c \tag{8}$$

in probability. By our definition of K_n^**, ,* $EY_j = c/n$*. Moreover, we know that* $\widehat{\sigma}_i^2$ *is distributed according to a scaled* χ^2 *distribution with* l *degrees of freedom. Hence it is an elementary, but elementary exercise to show that* $EY_j^2 = O(1/n^2)$*, and that* Y_j *and* Y_k *are independent if*

$$|j-k| > \ell + 1.$$

As $\ell/n \to 0$, we can easily see that the variance of $\sum Y_j$ converges to zero.
We now have established (4). Now it is rather easy to establish our lemma: We have to show that

$$E\left(\prod_{i=\ell+1,..n} I(\tau_i \le K_n^*)\right) \to \exp(-c)$$

Using again (4), it is sufficient to show

$$E\left(\prod_{i\le\nu} I(\tau_i \le K_n^*)\right) \to \exp(-c)$$

Trivially,

$$E\left(\frac{I(\tau_i \le K_n^*)}{E(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}}/\mathcal{F}_{i-1}\right) = 1.$$

A straightforward argument, perfectly analogous to the optional sampling theorem yields

$$E\left(\frac{E\prod_{i\le\nu} I(\tau_i \le K_n^*)}{\prod_{i\le\nu} E(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1})}\right) = 1. \tag{9}$$

According to the definition of ν,

$$-(1+\varepsilon)^2 \sum_{j=\ell+1}^{\nu} Y_j \le \log \prod_{i\le\nu} E(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}) \le -(1-\varepsilon)^2 \sum_{j=\ell+1}^{\nu} Y_j \tag{10}$$

and

$$\log \prod_{i\le\nu} E(I(\tau_i \le K_n^*)/\mathcal{F}_{i-1}) \ge -c(1+\varepsilon)^3 \tag{11}$$

Moreover, (4) implies that $P\left(\left[\sum_{j=\ell+1}^{\nu} Y_j = \sum_{j=\ell+1}^{n} Y_j\right]\right) \to 1$. Therefore $\sum_{j=\ell+1}^{\nu} Y_j \to c$, too. Hence it can easily be seen that (11) and (10) allow us to deduct from (9) that

$$\begin{aligned}
\exp(-(1+\varepsilon)^2 c) &\le \liminf E \prod_{i\le\nu} I(\tau_i \le K_n^*) \\
&\le \limsup E \prod_{i\le\nu} I(\tau_i \le K_n^*) \le \exp(-(1-\varepsilon)^2 c).
\end{aligned}$$

Now one can easily see that (4) allows us to replace ν with n in the above inequalities, which proves our theorem. □

So we now have computed the distribution of our test statistic for a very specific process, namely when the parameters $\mu_t = 0$ and $\sigma_t = 1$. We now have to reduce the general case described by (1) to the specific case discussed above. For this purposed, we will have to make assumptions on μ_t and σ_t.

THEOREM 3.2. *Suppose μ_t and $\ln \sigma_t$ are diffusion-type processes with a.s. uniformly bounded diffusion coefficients. Then - provided - $l_n/\ln n$ converges to a constant different from 0 - the difference between the test statistic applied to X_t and W_t converges to zero in probability.*

Proof. Since the proof is rather technical, we give it in appendix B. □

The above lemma and the above theorem show that our construction - rejecting when the τ_i are larger than K_n^* - is indeed a test. Moreover, it is an easy, but tedious exercise to establish the order of magnitude of K_n^*. The distribution of $\widehat{\sigma}_i^2$ is a scaled χ^2, so the right hand side of (3) can be evaluated using the Gamma function.* Then it is an easy task to show that

$$K_n^* / \left(2 \ln n\right) = 1.$$

Then it is an elementary task to establish our assertion that the test is consistent against jumps of the order

$$\left(1+\varepsilon\right) \sigma_t \frac{\sqrt{2 \ln n}}{\sqrt{n}}.$$

4. POWER OF THE COMPETING TESTS

As mentioned in the introduction, various tests for this problems have been developed. Two of the most prolific ones are the tests of Barndorff-Nielsen and Shephard (2006) and of Ait-Sahalia and Jacod (2009). These tests are based on the following test statistics:

Definition 4.1. *BNS test statistic(Barndorff-Nielsen and Shephard (2006))*

$$\widehat{\tau}_{BNS}^{LIN} = \frac{\sqrt{n}\left(RV - \frac{\pi}{2} BPV\right)}{\sqrt{\int_0^1 \sigma_u^4 du}}, \widehat{\tau}_{BNS}^{ADJ} = \frac{\sqrt{n}\left(1 - \frac{\pi BPV}{2RV}\right)}{\sqrt{\max\left[1, \int_0^1 \sigma_u^4 du / \left\{\int_0^1 \sigma_u^2 du\right\}^2\right]}} \quad \textit{where}$$

$$RV = \sum_{j=1}^{1/\Delta} r_{t+j\Delta}^2 \textit{ and } BPV = \sum_{j=2}^{1/\Delta} \left|r_{t+j\Delta}\right| \left|r_{t+(j-1)\Delta}\right|$$

Another alternative was proposed by Ait-Sahalia and Jacod. This test is based on the p-th power variation, and compares the estimates for the variation for different time scales.

*Given the $\alpha-$level of significance, we can plug-in $c = -\log\left(1-\alpha\right)$. Then we can find the critical value K such that

$$\left(\frac{K}{l}+1\right)^{-\frac{(l-1)}{2}} \left(\frac{K}{l}\right)^{-1/2} \frac{\Gamma\left(l/2-1/2\right)}{\Gamma\left(1/2\right)\Gamma\left(l/2\right)} + \frac{\log\left(1-\alpha\right)}{n} = 0 \tag{12}$$

If the sample size is small and/or the average window size l is small, then the approximation of lemma 3.1 can be improved with a small sample corrected critical value K such that

$$\left(\frac{K}{l}+1\right)^{-\frac{(l-1)}{2}} \left(\frac{K}{l}\right)^{-1/2} \frac{\Gamma\left(l/2-1/2\right)}{\Gamma\left(1/2\right)\Gamma\left(l/2\right)} \left(\frac{l-1}{l}\right) + \frac{\log\left(1-\alpha\right)}{n-l} = 0 \tag{13}$$

DEFINITION 4.2. *AJ test statistic(Ait-Sahalia and Jacod (2009))*[†]

$$\begin{aligned}
\widehat{\tau}_{AJ}^{p,k} &= \left(k^{p/2-1} - \widehat{S}\left(p,k,\Delta\right)\right) / \sqrt{\widehat{V}_{p,k}} \text{ where } p > 3, k \geq 2, \\
\widehat{S}\left(p,k,\Delta\right) &= \widehat{B}\left(p,k\Delta\right) / \widehat{B}\left(p,\Delta\right), \ \widehat{B}\left(p,k\Delta\right)_t = \sum_{i=1}^{n/k} \left|r_{t+ik\Delta}\right|^p \text{ and} \\
& \widehat{V}_{p,k} \text{ is the variance of } \widehat{S}\left(p,k,\Delta\right) \text{ under the null.}
\end{aligned}$$

The behavior of these test statistics - under our kind of alternatives can easily be analyzed. We just add to one of the returns the jump. For the BNS test, this is easily be done. One can easily see, that if the jump is of $o(n^{-1/4})$, the difference between the test statistic under the null and the alternative converge against 0. Hence the test will be much less powerful against our kind of alternatives. The same is true for the AJ test: Here (after some calculations), one can see that the corresponding bound is $o\left(n^{-1/2+1/p}\right)$. So - in a bit of contrast to the inventors of the test - we think that larger order p deserve attention (Our simulation results, however, indicate that the limiting distribution for higher order p is not a good approximation of the sampling distribution). In any case, we think that this subject merits further research.

Despite the fact that the tests have "low" power against "our" alternatives, it should be noted that there are situations where these tests have large advantages over our test. Assume one has not only one jump, but many. So let us assume that we have L jumps of size J (rather evenly distributed, let us assume the time between jumps is bigger than 1. Then it is easily seen from the definition of the BNS statistic that the test is consistent (the power converges to 1) if

$$\sqrt{n} L J^2 \to \infty.$$

An analogous result holds for the AJ - test. This fact is easily explainable, if one takes into account that the test statistics are constructed from sums: So small jumps can accumulate, in contrast to our test. So one should consider these tests not as tests against simple jumps, but as tests against Levy-type alternatives. It might be a worthwhile task to investigate the power of the test against specific alternatives of this type.

[†]For $\widehat{V}_{p,k}$, they suggest two estimators:
$\widehat{V}_{p,k}^c = \frac{\Delta_n M(p,k) \widehat{A}(2p,\Delta)_t}{\widehat{A}(p,\Delta)_t^2}, \widetilde{V}_{p,k}^c = \frac{\Delta_n M(p,k) \widetilde{A}\left(\frac{p}{p+1}, 2p+2, \Delta\right)_t}{\widetilde{A}\left(\frac{p}{p+1}, p+1, \Delta\right)_t^2}$ where
$M\left(p,k\right) = \frac{1}{m_p^2}\left(k^{p-2}\left(1+k\right) m_{2p} + k^{p-2}\left(k-1\right) m_p^2 - 2k^{p/2-1} m_{k,p}\right),$
$m_p = E\left(\left|Z_1\right|^p\right) = \pi^{-1/2} 2^{p/2} \Gamma\left(\frac{p+1}{2}\right),$
$m_{k,p} = E\left[\left|Z_1\right|^p \left|Z_1 + \sqrt{k-1} Z_2\right|^p\right],$
$Z_i \sim^{iid} N\left(0,1\right),$
$\widehat{A}\left(p,\Delta_n\right)_t = \frac{\Delta_n^{1-p/2}}{m_p} \sum \left|\Delta_i^n X\right|^p 1\left\{\left|\Delta_i^n X\right| \leq \alpha \Delta_n^{\varpi}\right\}, \varpi \in \left(0,1/2\right),$
$\widetilde{A}\left(r,q,\Delta_n\right)_t = \frac{\Delta_n^{1-qr/2}}{m_r^q} \sum_{i=1} \Pi_{j=1}^q \left|\Delta_{i+j-1}^n X\right|^r$

5. SIMULATIONS

First, we consider a very simple ideal condition.

$$r_{i/n} = \int_{(i-1)/n}^{i/n} dp_t = \int_{(i-1)/n}^{i/n} \sigma dW_t + \int_{(i-1)/n}^{i/n} J d\kappa_t \qquad \text{(Model 1)}$$

Second, we borrowed the model from Barndorff-Nielsen and Shephard (2006)

$$\begin{aligned} dp_t &= \sigma(s) W(ds) + J d\kappa_t \qquad \text{(Model 2)} \\ \sigma^2(t) &= w_1 \sigma_1^2(t) + w_2 \sigma_2^2(t) \\ \sigma_k^2(t) &= -\int_0^t \lambda_k(s) \left\{ \sigma_k^2(s) - \xi(s) \right\} ds + \int_0^t \omega(s) \sigma_k(s) B_k(ds) \text{ where } k = 1, 2 \end{aligned}$$

We consider 4 test statistics : Lee-Ploberger (LP), Barndorff-Nielsen & Shephard(BNS), Ait-Sahalia & Jacod(AJ), and Lee-Mykland (LM). We assume $J \sim N\left(0, \sigma_c^2\right)$ and consider three cases of jump size : no jump$\left(\sigma_c^2 = 0\right)$, 20% jump$\left(\sigma_c^2 = 0.2\xi(s)\right)$, $\ln(n)/n$ jump$\left(\sigma_c^2 = \ln(n)/n * \xi(s)\right)$. The sample sizes considered are 72, 288, 1440, 2880, 8640 which are corresponding to 20 minutes, 5 minutes, 1 minutes, 30 seconds, and 10 seconds, respectively. The number of replication is 50,000. The parameters are calibrated by Barndorff-Nielsen and Shephard (2002) as follow : $\xi(s) = 0.509$, $\omega^2(s) = 0.461$, $w_1 = 0.218$, $w_2 = 1 - p_1$, $\lambda_1 = 0.0429$, $\lambda_2 = 3.74$. For MA(1) microstructure noise, we assume $m_t \sim N\left(0, \sigma_m^2\right)$ with $\sigma_m^2 = \frac{2.5}{n}\sigma_r^2$ which means we set the variance of MA(1) microstructure noise is 2.5 times larger than that of one second return. If r_t and m_t is independent, then the realized volatility with high frequency data will be 6 times larger than that with low frequency data with that assumption. We also set the minimum price variation $c = 0.02$ which is smaller than pre-decimalization tick sizes of NYSE ($1/8 and $1/16) but is larger than foreign exchange market minimum variation.

Let's consider the rejection probabilities under the null, no jump case. Table1 shows that our tests have better rejection probabilities than other tests in the continuous pure diffusion model. Most other tests are precise with 10 seconds data but they are imprecise with small samples. Our tests have precise size even with 20 minutes data. The difference between our rejection probabilities and nominal level of significance is less than 0.5 percentage point which can be explained by simulation variations. The next best test is the adjusted BNS test. It is better than other tests but to some extent it overjects the null with small samples.

Table2 also shows our tests are better in the continuous stochastic volatility model. With small samples, our tests show moderate size distortion around 1 percentage point but it is smaller than that of other tests. Note that our test and LM test statistics looks similar but their performance is different. LM test needs larger window size for normalization whose required order is the square root of sample size. So its performance with small averaging windows is distorted a lot even in the pure diffusion case, table1. However our test is still valid with small windows. Both tests assume some continuity of volatility within that averaging window. Since our averaging window is smaller, our test is more robust to the rapidly changing volatility model. So our tests show better performance comparing LM in the stochastic volatility model.

Let's consider the power of test. Our tests have better power controlling the size distortion. In some cases, other tests have larger rejection probabilities under the alternative hypothesis, but they also have larger rejection probabilities under the null hypothesis. If we control the size distortion, our test have better power in most cases. Especially our tests have non-trivial power with the jump whose order is $\frac{\ln(n)}{n}$ where n is the sample size while BNS and AJ tests have trivial power for that case. LM tests also have non-trivial power but their sizes are not as reliable as ours and their averaging window requirement is restricted. As we see in figure3 and table3, the size adjusted power curve of our test envelops those of other tests, which is the sign of optimality of our test. In following section, we apply our tests to various important financial data.

6. EMPIRICAL APPLICATIONS

We apply our test to stock index and foreign exchange rate in the FOREX database and individual stock data in the TAQ database. We consider USD index, USD/JPY, EUR/USD, GBP/USD, USD/CAN from 1999 to 2000. 1998-2000 DJDA, NASDAQ COMP, NASDAQ100, S&P500, S&P100, RUSSEL2000 are examined. Dow jones 30 stocks in 2005 NYSE TAQ are also considered. The empirical results are reported in table 6-9. We find the followings.

First, we can find the strong evidence of jump in the most markets. For the European and US foreign exchange market data, AJ test shows 15-45% trading days have the jump. The ratios of jumping days increases to 20-35% for BNS test, 30-65% for LP test, and 70-85% for LM test. We observe a similar pattern in other markets : The ratios of jumping days in the whole foreign exchange market are 20-60% for AJ test, 25-40% for BNS test, 45-85% for LP test, and 76-95% for LM test. Those in the stock indices are 2-8% for AJ test, 10-25% for BNS test, 15-40% for LP test, and 45-60% for LM test. Those in the Dow 30 stocks are 15-20% for AJ and BNS test, 20-30% for LP test, and 55-65% for LM test.

Second, we can order the tests by the ratio of jumping days. The AJ test shows the least number of jumping days. The next is the BNS tests and our LP test. The LM test has largest ratio of jumping days. Our previous simulated power curve shows the similar pattern. Note that the LM test has largest power because of size distortion under the null. So we can say the LP test has the most power in some sense.

Third, we can detect more jump with more data. The 5 minutes data shows more jumping days than the 15 minutes data in most cases. In the foreign exchange market data, we consider 13 hours data and 24 hours data. We can detect the more jump with 24 hours data. We can see the similar pattern in the previous simulated power curve example. The powers of tests increase with number of sample.

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APPENDIX A. NORMAL AND χ^2 DISTRIBUTIONS FOR SMALL AND LARGE VALUES

It is well known that for the standard distribution function $\Phi(x)$

$$\lim_{x\to\infty} (1-\Phi(x))\sqrt{2\pi}x\exp\left(x^2/2\right) = 1$$

or equivalently

$$\lim_{x\to\infty} (\log(2\Phi(x)-1))\sqrt{2\pi}x\exp\left(x^2/2\right)/2 = -1$$

we can define $M(\varepsilon)$ as the smallest value so that for all

$$x > M(\varepsilon) \tag{14}$$

$$(1-\varepsilon) \leq \left|\sqrt{2\pi}x\exp\left(x^2/2\right)(1-\Phi(x))\right| \leq (1+\varepsilon). \tag{15}$$

and

$$-2(1+\varepsilon)^2 \leq (\log(2\Phi(x)-1))\sqrt{2\pi}x\exp\left(x^2/2\right) \leq -2(1-\varepsilon)^2 \tag{16}$$

Lemma A.1. *So let us now choose an arbitrary $\varepsilon > 0$, and let l, n, w_i be the integers defined in the main section of the paper. If*

$$l \geq 2\ln n$$

and

$$K \to \infty,$$

then

$$p_n = P\left[w_i \leq lM(\varepsilon)^2/K\right] = o\left(n^{-1}\right).$$

.

Proof. Since w_i is distributed according to a χ^2distribution with l degrees of freedom, we have

$$p_n = \frac{1}{\Gamma(l/2)}\int_0^{lM(\varepsilon)^2/(K)} x^{l/2-1}\exp(-x/2)\,dx.$$

Since $\exp(-x/2) \leq 1$, we have with

$$C = \frac{M(\varepsilon)^2}{K}$$

$$p_n \leq \frac{1}{\Gamma(l/2)} \frac{1}{l/2} C^{l/2} l^{l/2},$$

and therefore

$$\log p_n \leq -\log \Gamma(l/2) - \log(l/2) + \frac{l}{2} \log C + l/2 \log l.$$

The well known formula of Stirling implies that for $l \to \infty$ (with $m = l/2 - 1$)

$$\log \Gamma(l/2) - \Big(m (\log(m) - m + \log(\sqrt{2\pi m}\Big) \to 0.$$

Therefore

$$\log p_n \leq ((l/2) \log l - m (\log(m)) + (m + \frac{l}{2} \log C) + O(\log l)$$

One can easily see that $((l/2) \log l - m (\log(m)) = (l/2) (\log(l/m)) + O(\log m)$. So the terms linear in l dominate the right hand side of the inequality, Moreover, as $M(\varepsilon)$ is fixed and $K \to \infty$, we may conclude that $C \to 0$. Therefore, $\frac{l}{2} \log C$ will become negative. Therefore, $\frac{l}{2} \log C$ will become negative and dominate other parts. Therefore it can immediately be seen that lim sup

$$\frac{-\log p_n}{l/2 (-\log C)} \geq 1,$$

which implies $p_n \leq \exp(-l/2) \frac{M(\varepsilon)^2}{K}$. □

APPENDIX B. THE PROOF OF THEOREM 3.2.

Our proof of Theorem 3.2 is based on the following lemma.

Lemma B.1. *Suppose we have given a standard Wiener process W, an adapted process f and a constant α so that*

$$\int_z^b f^2 dt \leq B.$$

Then, where $\int_a^b f dW$ *is the usual Ito-integral,*

$$P\left(\left[\left|\int_a^b f dW\right| \geq C\right]\right) \leq 2\exp(-\frac{C^2}{2B}).$$

Proof. Novikov's theorem guarantees that for all u

$$E\left(\exp(u\int_a^b fdW - \frac{u^2}{2}\int_a^b f^2dt\right) = 1$$

Hence

$$E(\exp(u\int_a^b fdW - \frac{u^2}{2}B) \leq 1$$

and therefore

$$\exp(uC - \frac{u^2}{2}B)P(\left[\int_a^b fdW > C\right]) \leq 1.$$

Setting

$$u = \frac{C}{B}$$

and repeating the same idea with $-\int_a^b fdW$ proves our proposition. ∎

We are now prove Theorem 3.2 applying Lemma B.1.

Proof. of Theorem 3.2. We have

$$d\mu_t = A_t dt + B_t dV_t^{(1)},$$

$$d(\log\sigma_t) = C_t dt + D_t dV_t^{(2)},$$

where A_t, B_t, C_t, D_t are continuous processes and $V_t^{(1)}$,$V_t^{(2)}$ are (standard) Wiener processes. First of all let us demonstrate that without limitation of generality we can assume that A_t, B_t, C_t, D_t and $\mu_t, \log\sigma_t$ as well are uniformly bounded.

Since the processes A_t, B_t, C_t, D_t and $\mu_t, \ln\sigma_t$ are continuous, for every $\varepsilon > 0$ there exists a $M = M(\varepsilon)$ so that

$$P\left([\sup|A_t|, \sup|B_t|, \sup|C_t|, \sup|D_t|, \sup|\mu_t|, \sup|\ln\sigma_t| < M(\varepsilon)]\right) > 1 - \varepsilon.$$

Let us now define the stopping time $\tau^{(\varepsilon)}$ be defined as the first time one of the absolute values of A_t, B_t, C_t, D_t and $\mu_t, \ln\sigma_t$ becomes larger than $M(\varepsilon)$, or 1 if the absolute values of the processes remain below $M(\varepsilon)$ all the time. Then

$$P\left(\left[\tau^{(\varepsilon)} = 1\right]\right) > 1 - \varepsilon. \tag{17}$$

Let $r_{i,n} = \left(X_{i/n} - X_{(i-1)/n}\right)$ and $s_{i,n} = \left(W_{i/n} - W_{(i-1)/n}\right)$. Then let

$$\rho_n = \sup_i \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-l}^2)/l},$$

$$\rho_n^{(\varepsilon)} = \sup_{i \le \tau^{(\varepsilon)}} \frac{r_i^2}{(r_{i-1}^2 + r_{i-2}^2 + ..r_{i-l}^2)/l},$$

$$\xi_n = \sup_i \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-l}^2)/l},$$

and

$$\xi_n^{(\varepsilon)} = \sup_{i \le \tau^{(\varepsilon)}} \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-l}^2)/l}.$$

Then - by definition, ρ_n and ξ_n are our test statistics applied to $X_{i/n}$ and $W_{i/n}$, respectively. Moreover, (17) guarantees that

$$P\left(\left[\rho_n = \rho_n^{(\varepsilon)}\right]\right) > 1 - \varepsilon$$

and

$$P\left(\left[\xi_n = \xi_n^{(\varepsilon)}\right]\right) > 1 - \varepsilon,$$

too. Hence it is sufficient to show that for all $\varepsilon > 0$ the difference between converges to zero. For showing this, let us first observe that

$$\min\left(\frac{\sigma_{(i-k)/n}^2}{\sigma_{i/n}^2}\right) \le \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-l}^2)/l} \Big/ \frac{\sigma_{i/n}^2 s_i^2}{(\sigma_{(i-1)/n}^2 s_{i-1}^2 + \sigma_{(i-2)/n}^2 s_{i-2}^2 + ..\sigma_{(i-l)}^2 s_{i-l}^2)/l} \le \max\left(\frac{\sigma_{(i-k)/n}^2}{\sigma_{i/n}^2}\right).$$

For analyzing the difference of the left and right side of the above inequality and one, it is sufficient to consider

$$\sup_{k \le l} \left| \ln\left(\frac{\sigma_{(i-k)/n}^2}{\sigma_{i/n}^2}\right) \right|.$$

Now observe that $\ln(\sigma_{i/n}^2) - \ln(\sigma_{(i-k)/n}^2) = \int_{(i-k)/n}^{i/n} C_t dt + D_t dV_t^{(2)}$. For $i < \tau^{(\varepsilon)}$ $\left|\int_{(i-k)/n}^{i/n} C_t dt\right| \le kM/n$.Moreover, we have due to Lemma B.1

$$P\left(\left[\left|\int_{(i-k)/n}^{i/n} D_t dV_t^{(2)}\right| > 2\sqrt{Ml}\sqrt{\frac{\ln n}{n}}\right]\right) \le \frac{1}{n^2}$$

and hence

$$P\left(\left[\sup_{i \le \tau^{(\varepsilon)}, k \le l} \left|\int_{(i-k)/n}^{i/n} D_t dV_t^{(2)}\right| > 2\sqrt{Ml}\sqrt{\frac{\ln n}{n}}\right]\right) \le \frac{l}{n} \to 0.$$

Hence we can conclude that

$$P\left(\left[\sup_{k \le l} \left|\ln\left(\frac{\sigma_{(i-k)/n}^2}{\sigma_{i/n}^2}\right)\right| > 4\sqrt{Ml}\sqrt{\frac{\ln n}{n}}\right]\right) \to 0.$$

Since

$$\sup \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-l}^2)/l} = O(\ln n),$$

we can conclude that the difference between

$$\sup \frac{s_i^2}{(s_{i-1}^2 + s_{i-2}^2 + ..s_{i-l}^2)/l}$$

and

$$\sup \frac{\sigma_{i/n}^2 s_i^2}{(\sigma_{(i-1)/n}^2 s_{i-1}^2 + \sigma_{(i-2)/n}^2 s_{i-2}^2 + ..\sigma_{(i-l)}^2 s_{i-l}^2)/l}$$

converges to zero.
It now remains to show that the differences

$$\left| r_{i,n} - \sigma_{(i-1)/n} s_{i,n} \right|$$

remain small. Now observe that

$$\begin{aligned}
\left| r_{i,n} - \sigma_{(i-1)/n} s_{i,n} \right| &= \left| \int_{(i-1)/n}^{i/n} \left(\mu_t dt + \sigma_t dW_t - \sigma_{(i-1)/n} dW_t \right) \right| \\
&= \left| \int_{(i-1)/n}^{i/n} \mu_t dt \right| + \left| \int_{(i-1)/n}^{i/n} \left(\sigma_t - \sigma_{(i-1)/n} \right) dW_t \right| \\
&\leq \max |\mu_t| \frac{1}{n} + \left| \int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right) dW_u \right| .
\end{aligned}$$

For the analysis of

$$\left| \int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right) dW_u \right|$$

we will apply Lemma B.1. Since σ_u is a diffusion process, where drift and diffusion coefficients were assumed to be bounded, we can conclude that for all $\alpha > 0$ there exists a M so that

$$P \left(\text{for all } i \text{ and } (i-1)/n \leq u \leq i/n \ \left| \sigma_u - \sigma_{(i-1)/n} \right| \leq M \left| u - (i-1)/n \right|^{1/2-\alpha} \right) \to 1.$$

Hence

$$P \left[\left(\int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right)^2 du \right) \leq 2Mn^{-2+\alpha} \right] \to 1.$$

To apply Lemma B.1, however, we need to guarantee an uniform bound on the integral $\int_{(i-1)/n}^{i/n} \left(\sigma_u - \sigma_{(i-1)/n} \right)^2 du$. This can easily be achieved by using a stopping time.

We stop the process at time S, where

$$i/n \geq S \geq (i-1)/n,$$

if for the first time

$$\int_{(i-1)/n}^{S} \left(\sigma_u - \sigma_{(i-1)/n} \right)^2 du = 2Mn^{-2+\alpha},$$

otherwise we set

$$S = 1.$$

Obviously the definition of M guarantees that

$$P\left(S = 1\right) \geq 1 - \varepsilon$$

Hence if we define

$$\sigma_u^* = \begin{cases} \sigma_u \text{ for } u \leq S \\ \sigma_S \text{ otherwise,} \end{cases}$$

we have

$$P\left([\sigma_u^* = \sigma_u \text{ for all } u]\right) \geq 1 - \varepsilon.$$

Hence it is sufficient to give estimates for $\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u$. For this task, however, we can apply Lemma B.1 and conclude that

$$P\left(\left|\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u\right| > \sqrt{8Mn^{-2+\alpha}\ln n}\right) \leq \frac{2}{n^2}$$

Since $\alpha > 0$ was arbitrary, we can conclude that for arbitrary $\beta > 0$

$$P\left[\sup\left|\int_{(i-1)/n}^{i/n} \left(\sigma_u^* - \sigma_{(i-1)/n}^*\right) dW_u\right| > n^{-1+\beta}\right] \to 0,$$

which demonstrates that these terms are negligible. □

APPENDIX C. DATA

We use the FOREX historical database which has intraday transactions data of stock indices and foreign exchange rates. We use the intraday data of US Dollar Index(DXA0), USD/JPY(Japanese yen, JPYA0), EUR/USD(EURA0), GBP/USD(British pound, GBPA0), and USD/CAN(Canadian dollar, CADA0). Since the European and US foreign exchange markets are larger than other markets, we first consider the data from 2:00 to 16:00 eastern time. We exclude the first and last 30 minutes data because we observe relatively infrequent trading. We also consider the whole 24 hour data including data from the Asian and pacific market.

For stock indices, we analyze Dow Jones Industrial Average (DJIA), Nasdaq Composite Index (COMPQ), NASDAQ-100 Index(NDX)[‡], S&P 500 Index(SPX), S&P 100 Index(OEX), and Russell 2000 Index (RUT). Contrary to other market value weighted indices, the Dow Jones index is a price weighted index and represents well-established blue-chip stocks. The Nasdaq Composite is the index of all of the common stocks and similar securities listed on the NASDAQ stock market, so it measures the performance of technology stocks. The NASDAQ-100 is the index of 100 of the largest non-financial companies listed on the NASDAQ. The S&P 500 is a large-cap stock market index of 500 of largest common stocks actively traded in the US stock market. The S&P 100 chooses 100 largest companies in the S&P 500 considering sector balance. The Russell 2000 Index is a small-cap stock market index of the bottom 2,000 stocks in the Russell 3000 Index which measures the performance of the small-cap segment of the US stock market. Since US stock market opens at 9:30 and closes at 16:00 eastern time, we consider that time span. But we exclude the first and last 30 minutes because of infrequent transactions.

We also use the New York Stock Exchange (NYSE) Trade and Quote (TAQ) database which covers intraday transactions data for securities listed on the major stock exchanges. Because of limited

[‡]Note that NASDAQ100(NDX) starts from 2.24. 1998 in FOREX database. So number of sample is smaller.

accessibility, we mainly consider data of 2005 year. Dow Jones 30 stocks are chosen as main subjects because they are generally leading blue-chip stocks represent their industry and constitute a popular stock market indicator, Dow Jones Industrial Average (DJIA). We use the transaction data from the New York Stock Exchange (NYSE), American Stock Exchange (AMEX), and Nasdaq National Market System (NMS). (We choose data whose Ex field is "N","A", or "T") Furthermore, only regular way sales are selected. We exclude special sales like Bunched sales (B), Automatic Executed sales(E), and Burst Basket Executed sales(F). TAQ database deals with intraday data which may have trading error or canceled transactions. By choosing trades whose CORR field is equal to either zero or one, we exclude erroneous data like cancelled trades and obvious error records. About 99.71 percentage data have the proper CORR field. When we see multiple trades with different prices at the same time, we choose a volume-weighted average of the trade price. Since we consider jumps, we did not filter data based on the size of price change.[§] During the opening and closing hours, we observe volatile movement of prices. We consider rather a clean time horizon excluding near opening and closing hours : from 10:30 to 15:30. We also exclude holidays and some trading days which had few transactions, say Labor day, Thanksgiving day, Black Friday, Christmas, and so on. On Dec. 1th, AT&T substitutes SBC. I used the return of SBC until Nov. 30th and used that of AT&T after Dec. 1st.

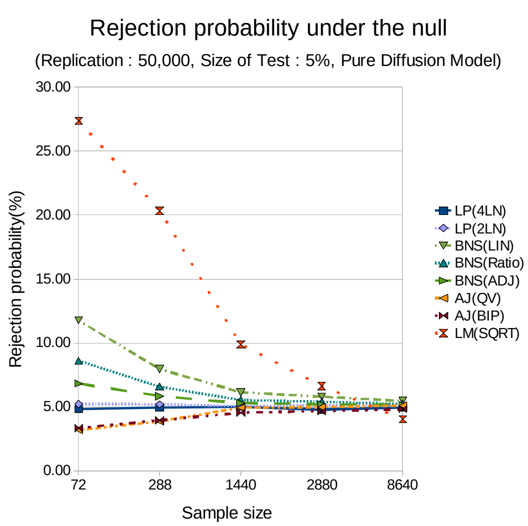
[§]Standard filtering rules exlcude trades which are less than 50% or greater 150% of the previous prices. (Boehmer,Saar, and Yu(JF2005))

APPENDIX D. TABLES AND FIGURES

Following tables summarize rejection probabilities under 5% size.

Model1-1 : Pure Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	4.83	5.24	11.76	8.61	6.83	3.19	3.35	27.37	6.03	27.37
288	4.95	5.21	7.97	6.58	5.85	3.87	3.95	20.34	11.03	40.13
1440	5.02	5.05	6.15	5.54	5.31	4.94	4.56	9.90	15.73	61.03
2880	4.81	5.16	5.81	5.40	5.21	4.99	4.67	6.63	19.23	71.74
8640	4.95	5.12	5.47	5.23	5.15	5.11	4.82	4.05	23.10	80.10
Model1-2 : Pure Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	35.55	27.89	32.84	28.88	26.36	6.00	9.65	50.71	35.33	50.71
288	58.83	52.54	47.96	46.13	45.32	19.13	35.31	66.41	62.32	74.81
1440	78.67	74.84	64.92	64.31	64.06	34.56	67.75	80.79	82.15	91.81
2880	84.23	81.38	70.71	70.33	70.20	38.68	76.21	85.34	87.44	95.65
8640	90.36	88.75	78.27	78.09	78.05	42.75	84.93	90.84	92.63	98.14
Model1-3 : Pure Diffusion W LN(N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	14.18	10.86	17.42	13.74	11.46	3.83	5.11	35.01	14.89	35.01
288	14.27	11.18	12.16	10.33	9.45	5.23	6.94	29.52	20.79	47.41
1440	14.69	11.12	8.36	7.64	7.31	6.30	7.73	20.64	25.87	66.13
2880	14.45	11.00	7.49	7.02	6.81	6.22	7.51	17.83	29.24	75.63
8640	14.86	11.43	6.39	6.13	6.05	6.17	7.10	15.90	33.18	82.89

Table1 : Simulated rejection probability of Model 1



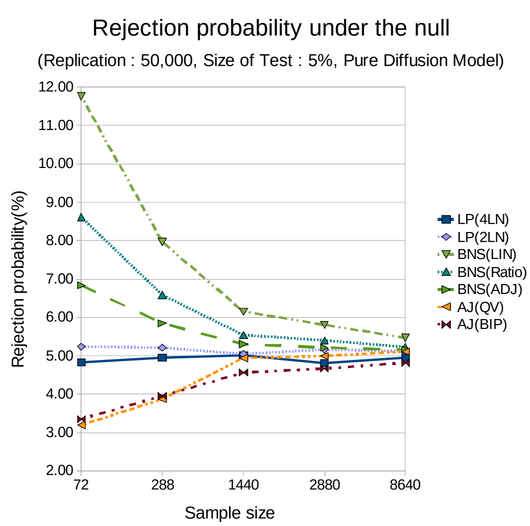


Figure1: Simulated rejection probability under the null
(model 1, 5% level of significance)

Model2-1 : CIR SV-Diffusion W/O JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	6.50	5.98	12.10	9.14	7.30	3.20	3.84	28.91	7.85	28.91
288	6.07	5.39	8.15	6.78	6.25	4.07	4.23	22.18	12.80	41.27
1440	5.40	5.11	6.38	5.73	5.58	4.83	4.57	10.78	16.77	61.29
2880	5.20	5.12	5.94	5.48	5.46	5.28	4.77	7.40	19.82	71.99
8640	5.02	4.89	5.40	5.17	5.15	5.43	4.84	4.49	23.64	80.36
Model2-2 : CIR SV-Diffusion W 20% JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	37.56	29.86	33.74	29.85	27.69	5.74	9.72	52.50	37.66	52.50
288	59.69	53.28	48.15	46.28	45.70	18.33	33.97	67.73	63.67	75.70
1440	79.03	75.34	64.94	64.28	64.18	34.22	66.94	81.09	82.50	91.73
2880	84.40	81.67	70.78	70.39	70.35	38.18	75.41	85.59	87.62	95.64
8640	90.46	88.86	78.10	77.93	77.92	42.32	84.41	91.02	92.84	98.19
Model2-3 : CIR SV-Diffusion W LN(N)/N JUMP										
	LP		BNS			AJ		LM		
n	4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
72	16.53	12.50	18.18	14.57	12.38	3.71	5.45	37.27	17.33	37.27
288	16.03	12.10	12.59	10.88	10.18	5.26	6.94	31.85	23.20	48.94
1440	16.38	12.26	8.65	7.88	7.73	6.19	7.54	22.65	28.01	67.12
2880	16.27	12.18	7.79	7.23	7.19	6.57	7.60	19.88	30.98	76.15
8640	16.27	12.42	6.38	6.11	6.10	6.42	7.07	17.59	34.57	83.38

Table2 : Simulated rejection probability of Model2

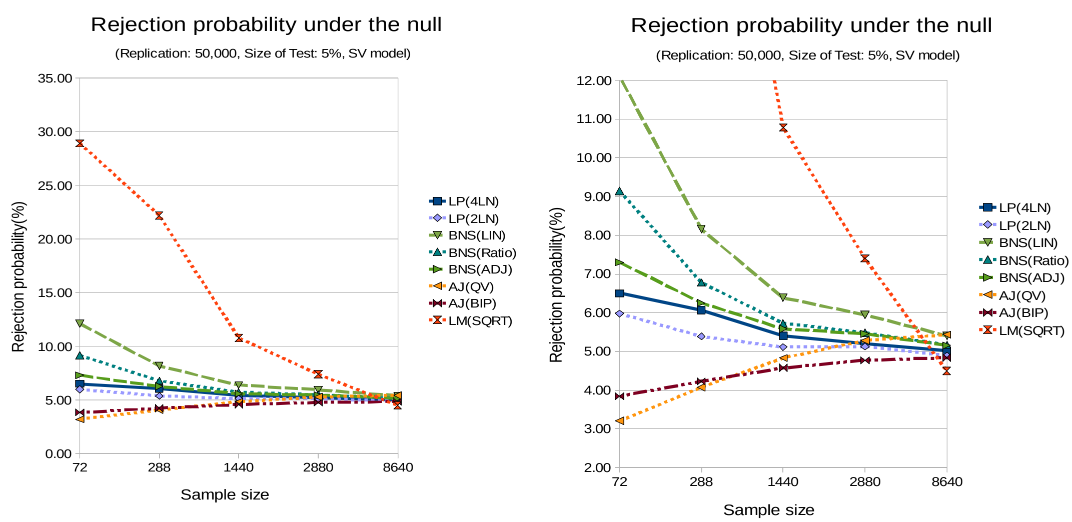
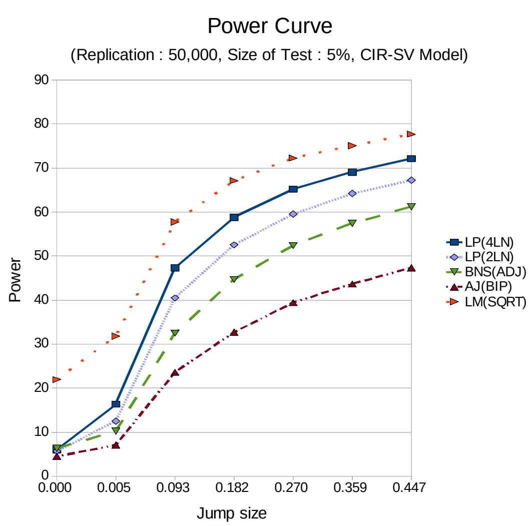


Figure2: Simulated rejection probability under the null
(model 2, 5% level of significance)

Jump/MeanVol	Meaning	LP		BNS			AJ		LM		
		4LN	2LN	LIN	Ratio	ADJ	QV	BIP	SQRT	4LN	2LN
0.000	No Jump	6.04	5.65	8.38	6.88	6.31	4.48	4.48	21.93	12.46	41.40
0.005	LN(N)/N	16.31	12.50	12.77	10.85	10.26	5.44	7.09	31.82	23.22	49.53
0.093		47.31	40.50	35.31	33.19	32.48	13.30	23.61	57.75	52.29	68.49
0.182		58.81	52.55	47.25	45.35	44.72	18.00	32.69	67.13	62.87	75.52
0.270		65.21	59.52	54.66	52.92	52.37	20.88	39.43	72.26	68.53	79.29
0.359		69.08	64.19	59.46	57.96	57.44	23.33	43.67	75.05	72.12	81.43
0.447	20%	72.11	67.22	63.18	61.70	61.24	25.17	47.34	77.70	74.90	83.27
Size Distortion		1.04	0.65	3.38	1.88	1.31	-0.52	-0.52	16.93	7.46	36.40
Size Adjusted Power											
Jump/MeanVol											
0.000	No Jump	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
0.005	LN(N)/N	15.27	11.85	9.38	8.97	8.95	5.96	7.61	14.88	15.76	13.13
0.093		46.27	39.85	31.92	31.31	31.17	13.82	24.13	40.81	44.83	32.09
0.182		57.77	51.90	43.87	43.47	43.41	18.53	33.21	50.20	55.41	39.11
0.270		64.17	58.87	51.28	51.04	51.06	21.41	39.96	55.33	61.07	42.88
0.359		68.04	63.54	56.08	56.09	56.13	23.86	44.20	58.11	64.66	45.03
0.447	20%	71.07	66.58	59.79	59.82	59.93	25.70	47.87	60.76	67.44	46.87

Table3 : Simulated rejection prob of Model 2 with 5 min frequency



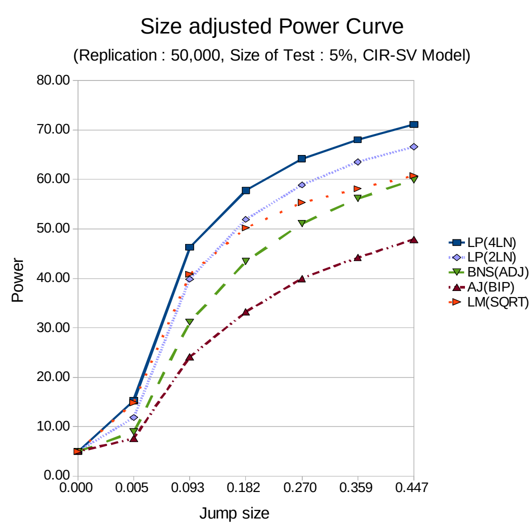


Figure2: Simulated power curve of Model 2 with 5 min frequency

15min	CODE	LP(4LN)	LP(2LN)	BNS	AJ	LM	n/day	days
USD	DXA0	46.22(4H)	36.61(2H)	28.83	16.56	70.76	51.3	489
JPY	JPYA0	44.02(4H)	34.06(2H)	22.11	20.32	69.52	51.7	502
EUR	EURA0	48.21(4H)	39.64(2H)	30.08	15.94	73.51	51.7	502
GBP	GBPA0	39.84(4H)	31.47(2H)	23.11	16.33	66.73	51.7	502
CAN	CADA0	48.70(4H)	36.73(2H)	27.15	19.16	74.45	51.6	501
5min	CODE	LP(4LN)	LP(2LN)	BNS	AJ	LM	n/day	days
USD	DXA0	65.03(54M)	50.31(44M)	33.33	22.09	83.03	151.6	489
JPY	JPYA0	64.34(55M)	54.38(45M)	32.67	37.65	85.66	154.1	502
EUR	EURA0	62.95(55M)	52.19(45M)	35.06	27.29	85.06	154.4	502
GBP	GBPA0	55.18(55M)	46.61(45M)	30.68	31.08	80.28	153.7	502
CAN	CADA0	60.68(54M)	55.89(44M)	26.75	45.51	85.23	150.9	501

Table4 : Empirical Rejection ratio of FX rates (13HR)

15min	CODE	LP(4LN)	LP(2LN)	BNS	AJ	LM	n/day	days
USD	DXA0	63.80(4H31M)	47.65(2H20M)	31.29	19.22	80.98	82.9	489
JPY	JPYA0	58.17(4H41M)	47.61(2H26M)	27.49	24.50	78.29	89.8	502
EUR	EURA0	66.33(4H41M)	49.00(2H26M)	31.27	20.32	84.46	89.9	502
GBP	GBPA0	63.55(4H40M)	50.80(2H26M)	28.88	21.31	85.06	89.1	502
CAN	CADA0	65.07(4H37M)	52.69(2H23M)	27.74	28.34	89.62	87.2	501
5min	CODE	LP(4LN)	LP(2LN)	BNS	AJ	LM	n/day	days
USD	DXA0	72.8(1H53M)	61.35(58M)	41.31	25.77	90.59	238.9	489
JPY	JPYA0	83.67(1H54M)	71.71(59M)	39.64	50.00	95.02	265.6	502
EUR	EURA0	79.88(1H54M)	66.53(59M)	41.43	35.26	92.43	265.5	502
GBP	GBPA0	75.3(1H54M)	66.93(59M)	36.25	36.45	91.24	259.5	502
CAN	CADA0	76.45(1H52M)	66.47(58M)	27.35	57.49	93.41	240.7	501

Table5 : Empirical Rejection ratio of FX rates (24HR)

15min	CODE	LP(4LN)	LP(2LN)	BNS	AJ	LM	n/day	days
DJIA	INDU	17.38(3H14M)	12.03(1H45M)	13.37	7.49	42.51	21.8	748
NASDAQ COMP	COMPQ	23.93(3H15M)	13.1(1H45M)	12.17	4.01	44.39	21.9	748
NASDAQ100	NDX	21.09(3H15M)	12.85(1H45M)	10.89	5.45	38.13	21.9	716
S&P500	SPX	17.87(3H15M)	10.93(3H15M)	14.53	5.87	39.73	21.9	750
S&P100	OEX	18.00(3H15M)	10.93(1H45M)	12.80	8.00	41.73	21.9	750
RUSSEL2000	RUT	29.12(3H15M)	20.61(3H15M)	10.24	3.32	52.26	21.9	752
5min	CODE	LP(SQRT)	LP(2LN)	BNS	AJ	LM	n/day	days
DJIA	INDU	24.87(1H25M)	17.25(45M)	21.12	8.16	47.46	65.3	748
NASDAQ COMP	COMPQ	31.02(1H25M)	18.98(45M)	19.52	3.61	53.34	64.6	748
NASDAQ100	NDX	24.16(1H25M)	14.11(45M)	15.92	4.89	46.09	65.5	716
S&P500	SPX	27.73(1H25M)	20.4(45M)	26.27	6.53	50.67	65.5	750
S&P100	OEX	26.4(1H25M)	17.47(45M)	20.53	8.00	47.07	65.5	750
RUSSEL2000	RUT	41.89(1H25M)	30.32(45M)	9.97	1.33	61.04	65.4	752

Table6 : Empirical Rejection ratio of stock indices

freq	LP(4LN)	LP(2LN)	BNS	AJ	LM
15min	0.2223(3H15M)	0.1823 (1H45M)	0.1471	0.1404	0.5559
5min	0.3197(1H25M)	0.2418(45M)	0.1701	0.1694	0.6315

Table7 : Empirical Rejection ratio of stocks of Dow30 (Average)