The Economic Value of Realized Volatility*

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Abstract

Many studies have documented that daily realized volatility estimates based on intraday returns provide volatility forecasts that are superior to forecasts constructed from daily returns only. A few recent studies also find that density forecasts based on realized volatility are superior to those based on daily data. We investigate whether these forecasting improvements translate into economic value added. In order to address this question we develop a new class of discrete-time option valuation models that use daily returns as well as realized volatility, and that nest the daily Heston and Nandi (2000) GARCH model as a special case. We derive closed-form option valuation formulas and we assess the option valuation properties using S&P500 return and option data. We find that realized volatility reduces the pricing errors of the benchmark model significantly across moneyness, maturity and volatility levels.

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1 Introduction

ARCH models (Engle (1982)) and their extensions (Bollerslev (1986), Nelson (1991), Glosten, Jagannathan, and Runkle (1993)) have proven very successful in describing the time series behavior of conditional variances of financial asset returns. Statistical tools like the likelihood principle strongly favor ARCH and GARCH over models with constant variance, and the models have therefore found widespread use in finance to model stock returns, interest rates, exchange rates, and option prices.

One important criticism of GARCH models concerns their apparent shortcomings in forecasting volatility, as measured by the R^2 of a Mincer-Zarnowitz regression that uses squared daily return as a proxy for variance on the left-hand side and the GARCH forecast of volatility on the right-hand side. Andersen and Bollerslev (1998) make two important contributions in this regard. First, they prove theoretically and by simulation that when the GARCH model is the true underlying data generating process, the R^2s of Mincer-Zarnowitz regression can be expected to be low, and are in fact of similar magnitude than the empirically observed R^2s. Andersen and Bollerslev (1998) note that this apparent lack of predictive ability is due to the use of the squared daily returns as the dependent variable in the regression, because the squared daily return is a very noisy measure of the variance. Andersen and Bollerslev's second contribution is to demonstrate that realized volatility, measured as the sum of squared intra-daily returns, is a superior measure of volatility, and leads to much higher R^2s in the Mincer-Zarnowitz regression.

Following the realization that accurate measures of volatility can be obtained from high frequency data, a growing literature has developed that studies the properties of realized volatility. Andersen, Bollerslev, Diebold, and Labys (2003) propose time series models for realized volatility in order to more accurately predict volatility. Joint models for returns and realized volatility have been proposed, either ignoring the contribution of jumps (Forsberg and Bollerslev (2002)) or by incorporating them in the model (Bollerslev, Kretschmer, Pigorsch, and Tauchen (2009)).[1]

Likewise, a few authors jointly model returns and realized volatility for the purpose of option pricing. Following the density modeling approach in Forsberg and Bollerslev (2002), Stentoft (2008) assumes that the conditional distribution of realized volatility is Inverse Gaussian with time-varying mean, while returns are assumed to be conditionally normal with variance equal to current realized volatility. Corsi, Fusari, and La Vecchia (2009) follow a similar approach by jointly modeling returns and the two-scale realized volatility (Zhang, Mykland and Aït-Sahalia (2005)). These models are not affine, and therefore pricing European options is done using

[1] Other studies investigate the properties of realized volatility when the intra-day returns converge toward zero (e.g., Barndorff-Nielsen and Shephard (2002)), when there are market frictions (e.g., Zhang, Mykland, and Aït-Sahalia (2005)), and when one faces different types of jumps (see e.g., Aït-Sahalia and Jacod (2009) and the references therein).

simulation, making statistical inference challenging.

We develop a new class of affine discrete-time models that allow for closed-form option valuation formulas using the conditional moment-generating function. We model daily returns as well as realized volatility. The volatility dynamic for the resulting models contains a GARCH component that consists of daily lagged squared returns, but also a realized volatility component.[2] We refer to this model as the generalized realized volatility (GRV) model. The GRV model nests the daily Heston and Nandi (2000) GARCH model as a special case, and also nests a variance dynamic with realized volatility only as a special case, which we refer to as the RV model.

While realized volatility is a better proxy for spot volatility than lagged squared returns, it still may be too noisy when used in its raw form in option valuation. We therefore develop another model where expected realized volatility is used in the variance dynamic in conjunction with squared returns. We refer to this model as the GERV model, and the corresponding special case that only models expected realized volatility is referred to as the ERV model. We thus study five models in total: GRV, RV, GERV, ERV, and the benchmark Heston-Nandi GARCH model.

We implement and test our models using two different data sets. First, we estimate the models on S&P500 returns and realized volatility data using maximum likelihood. Second, we assess the option valuation properties of our models using S&P500 option data. We find that incorporating realized volatility leads to a better fit on returns and realized volatility, and that it reduces the option pricing errors of the benchmark model significantly across moneyness, maturity and volatility levels. Moreover, modeling expected realized volatility is superior to the use of raw realized volatility. We demonstrate that the improved performance of our newly proposed models is due to their ability to more adequately model higher moments, in particular the volatility of variance.

The paper proceeds as follows. Section 2 introduces the models. Section 3 presents model estimates obtained using a long sample of returns and realized volatilities. Section 4 discusses the models' risk neutralization. Section 5 discusses option valuation using a large sample of option data. Section 6 concludes. Some of the more technical material is collected in the appendix.

2 Modeling Return Dynamics Using Realized Volatility

This section builds two classes of affine dynamic models that employ the information embedded in daily realized volatility, while nesting the affine discrete time option pricing model of Heston and Nandi (2000). The first class of models focuses on realized volatility directly, and we refer

[2] Our model thus contains two components, and is somewhat related to the literature that argues that more than one volatility component is needed. See for instance Bates (2000), Duffie, Pan, and Singleton (2000), Christoffersen, Jacobs, Ornthanalai, and Wang (2008), Christoffersen, Dorion, Jacobs, and Wang (2009), and Christoffersen, Heston, and Jacobs (2009).

to it as the GRV model. The second class models the expectation of realized volatility instead, and we refer to it as the GERV model.

2.1 The Benchmark Heston and Nandi (2000) GARCH Model

Heston and Nandi (2000) assumes the following process for daily log returns

$$R_{t+1} \equiv \ln\left(S_{t+1}/S_t\right) = r + \lambda h_t - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_{t+1},$$

where r denotes the risk-free rate, λ denotes the price of risk, and h_t is the conditional variance for day $t+1$ which is known at the end of day t.[3] The i.i.d. standard normal error term is represented by ε_{t+1}. The autoregressive GARCH-type variance process takes the following form

$$h_{t+1} = \omega + \beta h_t + \alpha\left(\varepsilon_{t+1} - \gamma\sqrt{h_t}\right)^2,$$

where γ captures the asymmetric volatility response, often referred to as the leverage effect. We will refer to this model as HN below.

2.2 Augmenting GARCH with Realized Volatility: The GRV Model

The seminal paper by Andersen, Bollerslev, Diebold and Labys (2003) contains the important intuition that realized volatility helps in forecasting future volatility, because it provides a better assessment of current spot volatility. GARCH models instead need to infer today's volatility from a moving average of past daily squared returns. This intuition motivates us to build an option valuation model where realized volatility is used to construct today's spot volatility. This should in turn lead to better estimates of the volatility term structure, and thus to more accurate option prices.

In the GRV model, returns are specified as

$$R_{t+1} = r + \lambda\bar{h}_t - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon_{1,t+1}. \tag{1}$$

Daily volatility is defined as a weighted average of two components, one driven by daily return

[3] Note that our timing convention is a little different from Heston and Nandi (2000). Furthermore, they do not include the $-\frac{1}{2}h_t$ term thus $\lambda = -\frac{1}{2}$ corresponds to risk-neutrality in their setup whereas $\lambda = 0$ corresponds to risk-neutrality in ours. This will be discussed further below.

innovations and one by daily realized volatility

$$\begin{aligned} \bar{h}_t &= nh_t + (1-n)RV_t \quad (2) \\ h_{t+1} &= \omega_1 + \beta_1 h_t + \alpha_1 \left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2, \quad (3) \end{aligned}$$

where RV_t is observed at the end of day t. The HN model is nested in the GRV model by imposing $n = 1$. A purely realized-volatility based model obtains if $n = 0$. We refer to this model as RV.

In order to use the model for option valuation we need to specify the conditional distribution of the random variable RV_{t+1}. Staying with the Heston-Nandi type affine dynamics, we assume that

$$RV_{t+1} = \omega_2 + \beta_2 RV_t + \alpha_2 \left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2, \quad (4)$$

where we assume that the joint distribution of the innovations to return and realized volatility, $\varepsilon_{1,t+1}$ and $\varepsilon_{2,t+1}$, is bivariate standard normal with correlation ρ. Note that the daily return and the daily realized volatility are allowed to have separate leverage effects through γ_1 and $\rho\gamma_2$. Note also that each volatility component follows an affine dynamic, as in the Heston-Nandi GARCH model in Section 2.1.

Below we estimate these models using quasi-maximum likelihood. We therefore need the following moment expressions

$$\begin{aligned} E_t\left(R_{t+1}\right) &= r + \lambda\bar{h}_t - \frac{1}{2}\bar{h}_t \\ E_t\left(RV_{t+1}\right) &= \omega_2 + \alpha_2 + \alpha_2\gamma_2^2 n h_t + (\beta_2 + \alpha_2\gamma_2^2(1-n))RV_t \quad (5) \\ Var_t\left(RV_{t+1}\right) &= \alpha_2^2(3 + 4\gamma_2^2\bar{h}_t) \\ Cov_t\left(RV_{t+1}, R_{t+1}\right) &= -2\rho\gamma_2\alpha_2\bar{h}_t. \end{aligned}$$

The following moments drive the model's option pricing performance, and will be studied in more detail below.

First, the expected variance is given by the GARCH and RV factors

$$E_t\left(\bar{h}_{t+1}\right) = nE_t\left(h_{t+1}\right) + (1-n)E_t\left(RV_{t+1}\right),$$

where $E_t\left(RV_{t+1}\right)$ is provided above and where

$$E_t\left(h_{t+1}\right) = \omega_1 + \alpha_1 + \left(\beta_1 + \alpha_1\gamma_1^2 n\right)h_t + \alpha_1\gamma_1^2(1-n)RV_t. \quad (6)$$

Second, the conditional variance of variance takes the form

$$Var_t\left(\bar{h}_{t+1}\right) = n^2\alpha_1^2(3+4\gamma_1^2\bar{h}_t) + (1-n)^2\alpha_2^2(3+4\gamma_2^2\bar{h}_t) + 2\alpha_1\alpha_2\left(1+2\rho^2+4\gamma_1\gamma_2\bar{h}_t\rho\right).$$

Third, the conditional covariance between return and variance takes the form

$$Cov_t\left(R_{t+1}, \bar{h}_{t+1}\right) = -2\left(n\alpha_1\gamma_1 + (1-n)\rho\gamma_2\alpha_2\right)\bar{h}_t.$$

Notice that using RV as a factor not only provides a potentially more accurate picture of the current spot volatility and the volatility term structure, it also provides more flexible functional forms for variance of variance and the leverage effect which are key in capturing the dynamics of the higher moments of the return distribution.

2.3 Augmenting GARCH with Expected Realized Volatility: The GERV Model

The approach in (2.2) is the most straightforward way to incorporate realized volatility into a return dynamic that is suitable for option valuation. However, while realized volatility is a better proxy for spot volatility than squared returns, it still may be too noisy when used for option valuation directly within the weighted average conditional variance above. We thus develop another model where the expected realized volatility, $m_t = E_t\left[RV_{t+1}\right]$, is used instead. We refer to this model as GERV.

Returns are again defined as

$$R_{t+1} = r + \lambda\bar{h}_t - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon_{1,t+1}. \tag{7}$$

The conditional variance is now defined by

$$\begin{aligned} \bar{h}_t &= nh_t + (1-n)m_t \\ h_{t+1} &= \omega_1 + \beta_1 h_t + \alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2. \end{aligned} \tag{8}$$

Note again that the HN model in Section 2.1 appears as a special case when $n = 1$. A model purely based on expected realized volatility emerges if $n = 0$. We denote this special case by ERV.

We assume that expected realized volatility evolves as

$$m_t = \omega_2 + \theta m_{t-1} + \beta_2 RV_t \tag{9}$$

and that observed realized volatility is linked with its expectation via

$$RV_{t+1} = m_t + \alpha_2 \left[\left(\varepsilon_{2,t+1} - \gamma_2 \sqrt{\bar{h}_t} \right)^2 - \left(1 + \gamma_2^2 \bar{h}_t\right) \right]. \tag{10}$$

Note that the term inside the square bracket has a zero mean by construction. Note also that this model offers a richer dynamic for realized volatility than does the RV model above.

We again assume that $\varepsilon_{1,t+1}$ and $\varepsilon_{2,t+1}$ follow a bivariate standard normal distribution with correlation ρ. The model moments needed to estimate the model by quasi maximum likelihood are

$$\begin{aligned} Var_t\left(RV_{t+1}\right) &= \alpha_2^2(3 + 4\gamma_2^2 \bar{h}_t) \\ Cov_t\left(RV_{t+1}, R_{t+1}\right) &= -2\rho\gamma_2\alpha_2\bar{h}_t. \end{aligned}$$

The key moments for option valuation are as follows. First, the expected variance is given by

$$E_t\left(\bar{h}_{t+1}\right) = nE_t\left(h_{t+1}\right) + (1-n)E_t\left(m_{t+1}\right),$$

where

$$\begin{aligned} E_t\left(h_{t+1}\right) &= \omega_1 + \alpha_1 + \left(\beta_1 + \alpha_1\gamma_1^2 n\right) h_t + \alpha_1\gamma_1^2(1-n)m_t \\ E_t\left(m_{t+1}\right) &= \omega_2 + \left(\theta + \beta_2\right) m_t. \end{aligned}$$

Second, the conditional variance of variance implied by the model is

$$Var_t\left(\bar{h}_{t+1}\right) = n^2\alpha_1^2(3 + 4\gamma_1^2\bar{h}_t) + (1-n)^2 \alpha_2^2\beta_2^2(3 + 4\gamma_2^2\bar{h}_t) + 2\alpha_1\alpha_2\beta_2\left(1 + 2\rho^2 + 4\gamma_1\gamma_2\bar{h}_t\rho\right).$$

Third, the conditional covariance between return and variance is

$$Cov_t\left(R_{t+1}, \bar{h}_{t+1}\right) = -2\left(n\alpha_1\gamma_1 + (1-n)\rho\gamma_2\alpha_2\beta_2\right)\bar{h}_t.$$

The time series paths of these moments will be plotted once the models have been estimated.

3 Daily Return and Realized Volatility Empirics

We now estimate the two models introduced in Section 2, GRV and GERV, as well as the special cases RV and ERV, using daily return and realized volatility data. We also estimate the benchmark Heston-Nandi model. Recall that these five models are related as follows.

Model:	HN	RV	GRV	ERV	GERV
n:	1	0	free	0	free

We use daily close-to-close returns and realized variance data for the S&P 500 cash index for the period February 2, 1983 to August 31, 2006, which yields a total of 5,946 observations. The daily returns on the S&P500 index are plotted in the top panel of Figure 1. The 22% drop in October 1987 dominates the picture but the plot also illustrates the low-frequency variations in volatility. The low-volatility period in the mid 1990s is evident as is the period of high volatility in the late 1990s and early 2000s. The last part of the sample (2004-2006) is characterized by another episode of relatively low volatility.

The middle and bottom panels of Figure 1 plot the daily realized volatility (square root of RV) using 5-minute and 60-minute intraday returns respectively. It is interesting to note that the dominating October 1987 crash shows up as a 14% volatility day when using 60-minute RV but only as a 7% volatility day when using 5-minute returns to compute RV. Notice also the that 60-minute RV, which is computed from many fewer intraday observations than the 5-minute RV, tends to exhibit more high-frequency variation which may be driven by sampling error. We will focus on the 5-minute RVs below and include results on 60-minute RVs as a robustness check.

3.1 Maximum Likelihood Estimation

We estimate the five models using quasi maximum likelihood. The quasi-log-likelihood of returns at time $t+1$ conditional on information known at time t is

$$L(R_{t+1}|I_t) = -\frac{1}{2}\ln(2\pi Var_t[R_{t+1}]) - \frac{(R_{t+1} - E_t[R_{t+1}])^2}{2Var_t[R_{t+1}]}. \tag{11}$$

The quasi-log-likelihood of realized variance at time $t+1$ conditional on information known at time t is

$$L(RV_{t+1}|I_t) = -\frac{1}{2}\ln(2\pi Var_t[RV_{t+1}]) - \frac{(RV_{t+1} - E_t[RV_{t+1}])^2}{2Var_t[RV_{t+1}]}.$$

and the joint quasi-log-likelihood of returns and realized variance is

$$\begin{aligned} L(R_{t+1}, RV_{t+1}|I_t) &= -\ln(2\pi) - \frac{1}{2}\ln\left(Var_t[R_{t+1}]\,Var_t[RV_{t+1}] - Cov_t(RV_{t+1}, R_{t+1})^2\right) \\ &- \frac{\begin{pmatrix} Var_t[RV_{t+1}](R_{t+1} - E_t[R_{t+1}])^2 + Var_t[R_{t+1}](RV_{t+1} - E_t[RV_{t+1}])^2 \\ -2(RV_{t+1} - E_t[RV_{t+1}])(R_{t+1} - E_t[R_{t+1}])\,Cov_t(RV_{t+1}, R_{t+1}) \end{pmatrix}}{2\left(Var_t[R_{t+1}]\,Var_t[RV_{t+1}] - Cov_t(RV_{t+1}, R_{t+1})^2\right)}. \end{aligned}$$

The conditional moments required for these likelihood functions are provided in Section 2 above.

Estimation results are reported in Tables 1 and 2. For each RV series, we follow Hansen and Lunde (2005) by adding the squared overnight return to the high-frequency intraday returns. We then scale the realized variances to match the unconditional variance of returns. When using realized volatility data, the choice of aggregation interval is important. Table 1 reports results obtained using 5-minute intervals, and Table 2 using 60-minute intervals. The appendix provides the conditions for variance stationarity as well as equations for variance and equity premium targeting which is used to pin down estimates of λ, ω_1, and ω_2 as functions of the other parameters.

First consider the results obtained using 5-minute intervals in Table 1. Note that model-implied average volatility is the same in all five models, because of variance targeting. Similarly, λ is the same across models due to equity premium targeting. In the GRV model the point estimate of the parameter n is 0.45, and it is significantly different from zero. In the GERV model the point estimate of the parameter n is 0.20, and it is significant at conventional significance levels. The persistence of volatility and the volatility components is indicated at the bottom of the table. In the GRV model the persistence associated with the GARCH component is 0.70, whereas the persistence associated with the realized volatility component is 0.50. In the GERV model, the structure is somewhat different: the realized volatility component is slowly mean reverting, with a persistence of 0.99, while the GARCH component rapidly mean reverts with a persistence of 0.25. In the special cases of the RV and the ERV model, persistence is estimated at 0.88 and 0.98 respectively, which compares with the GARCH persistence of 0.96.

The estimates of γ_1 and γ_2 are positive in all cases, which is induced by the negative skewness of the return distribution. The estimate of ρ is approximately 0.1 in all models.

The log-likelihood values allow us to test the special cases of the RV model and the ERV model against the more general GRV model and GERV model respectively. Using a standard likelihood ratio statistic, the restrictions imposed by these models are resoundingly rejected, indicating that the GARCH and RV dynamic both contribute to the modeling of daily index returns.

Another interesting comparison is between the two models incorporating realized volatility, RV and GRV, and the two competing ones that model the conditional mean of realized volatility instead, ERV and GERV. This comparison is also interesting because the number of parameters in these models only differs by one. Based on the likelihood values, we conclude that modeling based on the expected value of realized volatility is preferable, and that the resulting reduction in noise is beneficial for the purpose of modeling returns and realized volatility.

It is less straightforward to statistically compare the GARCH model and the four newly proposed models. The GARCH likelihood does not contain a realized volatility component, and therefore cannot be meaningfully compared to the overall likelihood of the GRV and GERV models. The table reports the realized volatility component and return component separately for the four newly proposed models, but a comparison of the return component of the likelihood with the GARCH likelihood is also problematic, because in the four newly proposed models (11) is not considered separately. For the four newly proposed models, we therefore also report the likelihood from returns only. This is done by computing the conditional mean and variance of returns implied by the model, and then maximizing the univariate Gaussian likelihood using these two conditional moments. These return-based log-likelihood results are reported in the row labeled "Maximized on Returns". They indicate that the Heston-Nandi GARCH model is superior to the raw RV model, but that it is dominated by the other three models.

Table 2 presents an analysis similar to Table 1, but now 60-minute intervals are used for constructing realized volatility. The estimate of n in the GRV model is somewhat higher than in Table 1, at 0.50, and the estimate of n in the GERV model is substantially higher, at 0.35. All estimates of γ_1 and γ_2 are again positive, and the estimates of ρ are very similar to Table 1. Overall the estimates of the persistence for all models are also comparable to those in Table 1. The log-likelihood comparison between the models also yields similar results, although the evidence in favor of modeling using the expectation of realized volatility is even stronger, with the much more parsimonious ERV model outperforming the GRV model.

Overall the estimation based on returns and realized volatility yields two important conclusions. First, both the GARCH volatility and the realized volatility dynamic contribute to the modeling of daily index returns. Second, modeling the expectation of realized volatility is superior to modeling realized volatility directly.

3.2 Dynamic Model Properties

Figures 2-5 report on various dynamic properties of the five models we have estimated. For brevity we only report on the models estimated on 5-minute realized variance. Figure 2 plots the daily conditional volatility $\sqrt{\bar{h}_t} = \sqrt{Var_t\left(R_{t+1}\right)}$ for each of the four new models when estimated on 5-minute realized variance. Not surprisingly, all four models track the market volatility during the 1983-2006 period in a similar way. Notice however, that the pure RV model in the top-left panel tends to exhibit much more high-frequency variation in the conditional volatility. Figure 3 confirms this by plotting the model-implied conditional volatility of variance defined as the square root of $Var_t\left(\bar{h}_{t+1}\right)$. The volatility of variance is generally higher in the RV model than in the other models and it also tends to show more high-frequency moments inherited from $\bar{h}_t$.

Figure 4 plots the model-implied conditional correlation between return and variance, defined as

$$Corr_t\left(R_{t+1}, \bar{h}_{t+1}\right) = \frac{Cov_t\left(R_{t+1}, \bar{h}_{t+1}\right)}{\sqrt{Var_t\left(\bar{h}_{t+1}\right)\bar{h}_t}}.$$

Figure 4 shows that the models differ considerably in this regard. The conditional correlation is roughly constant over time in each model. Notice that it is around -10% in the RV and ERV models, around -30% in the GRV model and around -50% in the GERV model. Recall that the GRV and GERV models have two sources of the leverage effect: γ_1 from the GARCH part and $\rho\gamma_2$ from the RV part. Clearly allowing for the GARCH to play a role in the models increases the estimated leverage effect. This is important to keep in mind when analyzing the models' ability to fit options which we turn to next.

Before turning to option valuation we plot in Figure 5 the conditional volatility, conditional volatility of variance, and conditional correlation of return and variance for the benchmark HN GARCH model. When comparing with the RV models in Figure 2-4 we see that the conditional volatility is much less volatile in the GARCH model and that the conditional correlation with returns is much larger in magnitude and fluctuates much more over time in GARCH compared with the RV models.

4 Risk Neutralization and Option Valuation

In this section we use the return processes defined above to derive option valuation formulas, using the models' conditional moment generating functions. We first consider the GRV model and subsequently the GERV model. Recall that the RV and ERV models are special cases of these.

4.1 The GRV Model

4.1.1 Moment Generating Function

The appendix demonstrates that the one-period conditional moment generating function for the GRV model is of the form

$$\begin{aligned} & E_t\left[\exp\left(u_1 R_{t+1} + u_2 h_{t+1} + u_3 RV_{t+1}\right)\right] \\ = \ & \exp\left(A_1\left(u_1, u_2, u_3\right) h_t + A_2\left(u_1, u_2, u_3\right) RV_t + B\left(u_1, u_2, u_3\right)\right). \end{aligned}$$

This allows us to find $E_t\left[\exp\left(u\sum_{j=1}^{M} R_{t+j}\right)\right]$. Since the model is affine, we conjecture that the multi-period conditional moment generating function is of the form

$$\Psi_{t,t+M}(u) \equiv E_t\left[\exp\left(u\sum_{j=1}^{M} R_{t+j}\right)\right] = \exp\left(C_1(u,M)\,h_t + C_2(u,M)\,RV_t + D(u,M)\right).$$

The appendix proves this conjecture.

4.1.2 Risk Neutralization

The pricing kernel provided in Christoffersen, Elkamhi, Feunou and Jacobs (2009) is defined via

$$Z_{t+1} = \frac{\exp\left(\nu_{1,t}\varepsilon_{1,t+1} + \nu_{2,t}\varepsilon_{2,t+1}\right)}{E_t\left[\exp\left(\nu_{1,t}\varepsilon_{1,t+1} + \nu_{2,t}\varepsilon_{2,t+1}\right)\right]}.$$

Given that $\varepsilon_{1,t+1}$ and $\varepsilon_{2,t+1}$ are bivariate standard normal with correlation ρ we have

$$Z_{t+1} = \exp\left(\nu_{1,t}\varepsilon_{1,t+1} + \nu_{2,t}\varepsilon_{2,t+1} - \frac{\nu_{1,t}^2}{2} - \frac{\nu_{2,t}^2}{2} - \nu_{1,t}\nu_{2,t}\rho\right).$$

We need to impose that

$$E_t^Q\left[\exp\left(R_{t+1}\right)\right] = \exp(r).$$

The risk-neutral expected compound return is

$$E_t^Q\left[\exp\left(R_{t+1}\right)\right] = E_t\left[Z_{t+1}\exp\left(R_{t+1}\right)\right] = \exp\left(r + \lambda\bar{h}_t + \left(\nu_{1,t} + \nu_{2,t}\rho\right)\sqrt{\bar{h}_t}\right).$$

Setting this to the risk-free rate gives

$$\begin{aligned} E_t^Q\left[\exp\left(R_{t+1}\right)\right] &= \exp(r) \Leftrightarrow \lambda\bar{h}_t + \left(\nu_{1,t} + \nu_{2,t}\rho\right)\sqrt{\bar{h}_t} = 0 \\ &\Leftrightarrow \nu_{1,t} + \nu_{2,t}\rho = -\lambda\sqrt{\bar{h}_t}. \end{aligned}$$

For the bivariate shocks we have the risk-neutral expectation

$$\begin{aligned} E_t^Q\left[\exp\left(u_1\varepsilon_{1,t+1} + u_2\varepsilon_{2,t+1}\right)\right] &= E_t\left[Z_{t+1}\exp\left(u_1\varepsilon_{1,t+1} + u_2\varepsilon_{2,t+1}\right)\right] \\ &= \exp\left(u_1\left(\nu_{1,t} + \nu_{2,t}\rho\right) + u_2\left(\nu_{2,t} + \nu_{1,t}\rho\right) + \frac{u_1^2}{2} + \frac{u_2^2}{2} + u_1u_2\rho\right). \end{aligned}$$

Under the risk-neutral probability measure we know that $\varepsilon_{1,t+1}^* = \varepsilon_{1,t+1} - \left(\nu_{1,t} + \nu_{2,t}\rho\right)$ and $\varepsilon_{2,t+1}^* = \varepsilon_{2,t+1} - \left(\nu_{2,t} + \nu_{1,t}\rho\right)$, are bivariate standard normal with correlation ρ. We can therefore rewrite the model as follows

$$\begin{aligned} R_{t+1} &= r + \lambda\bar{h}_t - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon_{1,t+1}. \\ &= r + \lambda\bar{h}_t - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\left(\varepsilon^*_{1,t+1} + \nu_{1,t} + \nu_{2,t}\rho\right) \\ &= r + \lambda\bar{h}_t + \sqrt{\bar{h}_t}\left(\nu_{1,t} + \nu_{2,t}\rho\right) - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon^*_{1,t+1} \\ &= r - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon^*_{1,t+1} \end{aligned}$$

which holds because

$$\lambda\bar{h}_t + \sqrt{\bar{h}_t}\left(\nu_{1,t} + \nu_{2,t}\rho\right) = 0.$$

The dynamic of the GARCH component of the volatility can be rewritten in term of the risk-neutral shock as follows

$$\begin{aligned} h_{t+1} &= \omega_1 + \beta_1 h_t + \alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 \\ &= \omega_1 + \beta_1 h_t + \alpha_1\left(\varepsilon^*_{1,t+1} - \gamma^*_1\sqrt{\bar{h}_t}\right)^2 \end{aligned}$$

with $\gamma^*_1 = \gamma_1 - \lambda$.

The dynamic of the RV-component of the volatility can be rewritten in term of risk-neutral shock as follows

$$\begin{aligned} RV_{t+1} &= \omega_2 + \beta_2 RV_t + \alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2 \\ &= \omega_2 + \beta_2 RV_t + \alpha_2\left(\varepsilon^*_{2,t+1} - \gamma^*_{2t}\sqrt{\bar{h}_t}\right)^2 \end{aligned}$$

with $\gamma^*_{2t} = \gamma_2 - \frac{\nu_{2,t}+\nu_{1,t}\rho}{\sqrt{\bar{h}_t}}$.

In order to keep model affine under the risk-neutral probability measure, we assume

$$\nu_{2,t} + \nu_{1,t}\rho = \chi\sqrt{\bar{h}_t},$$

where χ is a given constant. In that case we have

$$\nu_{2,t} = \chi\sqrt{\bar{h}_t} - \nu_{1,t}\rho$$

and

$$\gamma^*_{2t} = \gamma^*_2 = \gamma_2 - \chi.$$

In summary, the dynamic under the risk-neutral probability measure is

$$\begin{aligned}
R_{t+1} &= r - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon^*_{1,t+1} \\
\bar{h}_t &= nh_t + (1-n)RV_t \\
h_{t+1} &= \omega_1 + \beta_1 h_t + \alpha_1\left(\varepsilon^*_{1,t+1} - \gamma^*_1\sqrt{\bar{h}_t}\right)^2 \\
RV_{t+1} &= \omega_2 + \beta_2 RV_t + \alpha_2\left(\varepsilon^*_{2,t+1} - \gamma^*_{2t}\sqrt{\bar{h}_t}\right)^2
\end{aligned}$$

where $\varepsilon^*_{1,t+1}$ and $\varepsilon^*_{2,t+1}$, are bivariate standard normal with correlation ρ under Q.

4.1.3 Option Valuation

Using the results in Section 4.1.1, we can show that the price at time t of a European call option with payoff $(S_{t+M} - X)^+$ at time $t+M$ is given by

$$C_t = \exp(-rM)S_t P_{1,t} - \exp(-rM)XP_{2,t}, \tag{12}$$

where the probabilities are defined via Fourier inversion of the conditional characteristic function

$$\begin{aligned}
P_{1,t} &= \frac{\exp(rM)}{2} + \int_0^{+\infty} \mathrm{Re}\left[\frac{\exp\left(\Psi^Q_{t,t+M}(1+iu) - iu\ln\left(\frac{X}{S_t}\right)\right)}{\pi iu}\right] du, \\
P_{2,t} &= \frac{1}{2} + \int_0^{+\infty} \mathrm{Re}\left[\frac{\exp\left(-iu\ln\left(\frac{X}{S_t}\right) + \Psi^Q_{t,t+M}(iu)\right)}{\pi iu}\right] du.
\end{aligned}$$

The risk-neutral conditional characteristic function is defined using its physical counterpart as

$$\Psi^Q_{t,t+M}(u) = C^*_1(u,M)\,h_t + C^*_2(u,M)\,RV_t + D^*(u,M).$$

The price of a European put option can be computed using put-call parity.

4.2 The GERV Model

4.2.1 Moment-Generating Function

For the GERV model the appendix shows that the one-period conditional moment generating function is also of the exponentially affine form

$$\begin{aligned} & E_t\left[\exp\left(u_1 R_{t+1} + u_2 h_{t+1} + u_3 m_{t+1}\right)\right] \\ = & \exp\left(A_1\left(u_1, u_2, u_3\right) h_t + A_2\left(u_1, u_2, u_3\right) m_t + B\left(u_1, u_2, u_3\right)\right). \end{aligned}$$

Since the model is affine, we conjecture that the multiperiod conditional moment generating function is of the following form

$$\begin{aligned} \Psi_{t,t+M}(u) & \equiv E_t\left[\exp\left(u\sum_{j=1}^{M} R_{t+j}\right)\right] \\ & = \exp\left(C_1(u,M)\, h_t + C_2(u,M)\, m_t + D(u,M)\right) \end{aligned}$$

which is derived in the appendix.

4.2.2 Risk Neutralization

The risk-neutralization and the pricing kernel is very similar to the one provide in the GRV model. Let us just write the dynamics of the realized variance RV_{t+1}, as it is the only difference between the two models.

$$\begin{aligned} RV_{t+1} & = m_t + \alpha_2\left[\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2 - \left(1+\gamma_2^2\bar{h}_t\right)\right] \\ & = m_t + \alpha_2\left[\left(\varepsilon^*_{2,t+1} + \nu_{2,t} + \nu_{1,t}\rho - \gamma_2\sqrt{\bar{h}_t}\right)^2 - \left(1+\gamma_2^2\bar{h}_t\right)\right] \\ & = m_t + \alpha_2\left[\left(\varepsilon^*_{2,t+1} - \gamma^*_{2t}\sqrt{\bar{h}_t}\right)^2 - \left(1+\gamma_2^2\bar{h}_t\right)\right] \end{aligned}$$

with

$$\gamma^*_{2t} = \gamma_2 - \frac{\nu_{2,t} + \nu_{1,t}\rho}{\sqrt{\bar{h}_t}}.$$

Again, to keep the model affine under Q, we will impose a constant γ^*_{2t}. This is done by assuming

$$\nu_{2,t} + \nu_{1,t}\rho = \chi\sqrt{\bar{h}_t}$$

where χ is a given constant. In that case we have

$$\nu_{2,t} = \chi\sqrt{\bar{h}_t} - \nu_{1,t}\rho$$

and thus

$$\gamma_{2t}^* = \gamma_2 - \chi.$$

We now have

$$\begin{aligned} RV_{t+1} &= m_t + \alpha_2 \left[\left(\varepsilon_{2,t+1}^* - \gamma_2^* \sqrt{\bar{h}_t} \right)^2 - \left(1 + \gamma_2^2 \bar{h}_t \right) \right] \\ &= m_t + \alpha_2 \left(\gamma_2^{*2} - \gamma_2^2 \right) \bar{h}_t + \alpha_2 \left[\left(\varepsilon_{2,t+1}^* - \gamma_2^* \sqrt{\bar{h}_t} \right)^2 - \left(1 + \gamma_2^{*2} \bar{h}_t \right) \right]. \end{aligned}$$

There is one notable difference between the physical (P) and the risk-neutral (Q) measures. Note that

$$\begin{aligned} E_t^Q \left[RV_{t+1} \right] &= m_t + \alpha_2 \left(\gamma_2^{*2} - \gamma_2^2 \right) \bar{h}_t \\ &= E_t \left[RV_{t+1} \right] + \alpha_2 \left(\gamma_2^{*2} - \gamma_2^2 \right) \bar{h}_t, \end{aligned}$$

and thus

$$E_t^Q \left[RV_{t+1} \right] - E_t \left[RV_{t+1} \right] = \alpha_2 \left(\gamma_2^{*2} - \gamma_2^2 \right) \bar{h}_t,$$

which means that the difference between γ_2^* and γ_2 measures the variance premium.

In summary, the dynamic under the risk-neutral probability measure is

$$\begin{aligned} R_{t+1} &= r - \frac{1}{2}\bar{h}_t + \sqrt{\bar{h}_t}\varepsilon_{1,t+1}^* \\ \bar{h}_t &= n h_t + (1-n) m_t \end{aligned}$$

with

$$\begin{aligned} h_{t+1} &= \omega_1 + \beta_1 h_t + \alpha_1 \left(\varepsilon_{1,t+1}^* - \gamma_1^* \sqrt{\bar{h}_t} \right)^2 \\ RV_{t+1} &= m_t^* + \alpha_2 \left[\left(\varepsilon_{2,t+1}^* - \gamma_2^* \sqrt{\bar{h}_t} \right)^2 - \left(1 + \gamma_2^{*2} \bar{h}_t \right) \right] \\ m_t &= \omega_2 + \theta m_{t-1} + \beta_2 RV_t \\ m_t^* &= m_t + \alpha_2 \left(\gamma_2^{*2} - \gamma_2^2 \right) \bar{h}_t, \end{aligned}$$

where $\varepsilon_{1,t+1}^*$ and $\varepsilon_{2,t+1}^*$, are bivariate standard normal with correlation ρ under Q.

4.2.3 Option Valuation

Using these results, the price at time t of a European call option with payoff $(S_{t+M} - X)^+$ at time $t + M$ is given by

$$C_t = \exp(-rM)S_t P_{1,t} - \exp(-rM)XP_{2,t}, \tag{13}$$

where again the probabilities can be computed using Fourier inversion of the risk-neutral conditional characteristic function which is defined by

$$\Psi^Q_{t,t+M}(u) = C_1^*(u,M)\,h_t + C_2^*(u,M)\,m_t + D^*(u,M).$$

As before put options can be valued using put-call parity.

5 Option Valuation: Empirical Findings

We now discuss the option fit of the four proposed models, and compare it with the fit of the benchmark GARCH model. We first discuss the option data used in our empirical analysis. Then we estimate the models on option data using Nonlinear Least Squares (NLS).

5.1 Option Data

We use closing prices on European S&P500 index options from OptionMetrics observed from January 1, 1996 through December 31, 2004. In order to ensure that the contracts we use are liquid, we only rely on out-of-the-money options with maturity between 15 and 180 days. For each maturity on each Wednesday we retain only the seven most liquid strike prices. We restrict attention to Wednesday data. This enables us to study a fairly long time-period while keeping the size of the data set manageable. Our sample has 10,138 options.

Table 3 describes key features of the data. The top panel of Table 3 sorts the data by six moneyness categories and reports the number of contracts, the average option price, the average Black-Scholes implied volatility, and the average bid-ask spread in dollars. Moneyness is defined as the implied index futures price, F, divided by the option strike price X. The implied volatility row shows that deep out-of-the-money puts, those with $F/X > 1.06$ are relatively expensive. The implied volatility for those options is 25.73% compared with 19.50% for at-the-money options. The data thus display the well-known smirk pattern across moneyness.

The bottom panel sorts the data by maturity reported in calendar days. The implied volatility row shows that the term structure of volatility is roughly flat on average during the period.

5.2 Estimating Model Parameters from Option Prices

We estimate the GERV, GRV, ERV, and RV models, as well as the benchmark Heston-Nandi GARCH model by minimizing implied volatility root mean squared error (IVRMSE). We refer to Broadie, Chernov, and Johannes (2007) for a discussion on the benefits of using the IVRMSE metric for comparing option pricing models. For the computation of the IVRMSE, we invert each computed option price O_j from the model using the Black-Scholes formula to get the implied volatilities $IV\left(O_j, X_j, M_{j,} S_{j,} r_j\right)$. With N denoting the total number of options in the sample, the IVRMSE is then computed as

$$IVRMSE \equiv \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\sigma_j^{BS} - IV\left(O_j, X_j, M_j, S_j, r_j\right)\right)^2}.$$

Tables 4 and 5 contain the results of NLS estimation. Table 4 uses realized volatility constructed using 5-minute intervals, and Table 5 uses 60-minute intervals.

The third row from the bottom in Table 4 indicates the IVRMSE metric. The IVRMSE for the most general GERV model is 3.067, compared with 3.904 for the benchmark GARCH model. This is an improvement of 21.44% on a percentage basis, which is impressive. The IVRMSE of the ERV model is 3.485. The GERV model therefore outperforms this model by almost 12%, which indicates that the GARCH-type volatility dynamic contributes to option valuation as does the RV based dynamic. This conclusion is reinforced by comparing the IVRMSE of the GRV model, 3.503, with the IVRMSE of the RV model, 3.923.

Another obvious conclusion from the IVRMSEs is the superiority of the modeling approach based on expected realized volatility. The IVRMSE of the GERV model, 3.067, is 12.44% lower than the IVRMSE of the GRV model, 3.503. Moreover, the IVRMSE of the ERV model, 3.485, is 11.16% lower than the IVRMSE of the RV model, 3.923.

It is also important to keep this in mind when comparing the performance of the GARCH model, which exclusively uses exponentially weighted lagged squared returns, with the performance of models based exclusively on current realized volatility. The IVRMSE of the GARCH model, 3.904, is lower than the IVRMSE of the RV model, but substantially higher than the IVRMSE of the ERV model.

Comparing parameter estimates with the QMLE estimates in Table 1 based on returns and realized volatility, the estimate of n in the GERV model is substantially higher, while the estimate of n in the GRV model is lower. All estimates of γ_1 and γ_2 are positive in all cases, as expected, but the estimate of γ_2 in the GERV model is surprisingly small. Estimates of ρ are much higher. Also, processes and components are more persistent compared to Table 1. In the case of the GERV model, both components are now very persistent.

Table 5 contains results for realized volatility constructed using 60-minute intervals. The ranking of the models is similar to Table 4, but the overall conclusion is that the models containing realized volatility perform somewhat worse compared to Table 4. For the purpose of option valuation, modeling realized volatility using higher frequency data therefore seems preferable. Parameter estimates are also largely consistent with those in Table 4, although the estimate of γ_2 in the GERV model is now larger, and the persistence of the volatility process associated with lagged squared returns for the GERV model is less persistent compared to Table 4.

Overall the results from model estimation based on option data confirm the main conclusions from QMLE estimation on realized volatility and returns in Section 3. First, realized volatility contains important information that is not contained in lagged squared returns. Second, modeling expected realized volatility is superior to modeling realized volatility directly.

5.3 Model Fit by Moneyness, Maturity, and Volatility Level

We now dissect the overall IVRMSE results reported in Tables 4 and 5 sorting the data by moneyness and maturity, as in Table 3, as well as by the level of market volatility captured by the VIX volatility index obtained from cboe.com. Table 6 contains the results for the 5–minute RV estimates and Table 7 has the results for the 60-minute intervals. In Tables 8 and 9 we study the extent to which the IVRMSE results may be driven by bias defined as market IV less model IV.

Consider first Panel A of Table 6 which reports the IVRMSE for the 5-minute RV models by moneyness bins corresponding to those used in Table 3. We also report results for the benchmark HN GARCH model. Looking across columns we see that the GERV model which had the lowest overall IVRMSE in Table 4 indeed has the lowest IVRMSE in each of the six moneyness categories considered. The benefits offered by the GERV model is therefore not restricted to any particular subset of strike prices. The performance of the GRV and ERV models is also quite robust across strikes. Notice also that all models tend to perform worst for deep out-of-the-money put options ($F/X > 1.06$) which have the highest average implied volatility (see Table 3).

Consider now Panel B in Table 6 which reports the IVRMSE across maturity categories. Again we see that the GERV model performs the best in all six maturity categories. All models have relatively more difficulty fitting the very short maturity and the longest-maturity options.

Panel C reports the IVRMSE across VIX levels. The GERV model is now best in five of the six categories. When VIX is in the 20-25% range the ERV model is slightly better. Perhaps not surprisingly, all models have most difficulty fitting options when the level of market volatility is high. The performance of the HN and RV models is particularly poor when VIX is above 35%.

Table 7 reports the IVRMSE by moneyness, maturity and VIX level for the models using

60-minute RVs. The patterns from Table 7 remain. The GERV performs the best in 17 out of 18 categories with the exception again being when the VIX is between 20 and 25%.

The mean squared error criterion is a sum of two components: Bias squared and variance. We therefore next consider how the IV bias in the different models may shed light on the general model performance across categories as well as on the relative performance across models.

Table 8 contains the results for the 5-minute RVs and thus relates to the IVRMSE numbers in Table 6. Considering first moneyness in Panel A it is clear that all models underprice deep out–of-the-money puts which to some extend drives the poor IVRMSE performance for this category. However all the RV-based models perform better than the benchmark HN model where the bias (market less model IV) is close to +2.5%. Notice also that all models tend to overprice at-the-money options. This bias is smallest for the GERV model which performed best in terms of IVRMSE. The GERV model does tend to underprice deep out-of-the-money calls, however. These results suggest the role for augmenting the models with non-normal shocks or perhaps a richer specification of the leverage effect.

Panel B shows that all models tend to underprice long-maturity options (except for GRV) and overprice short-maturity options. This pattern is particularly pronounced for the GERV and ERV models and suggest that richer volatility dynamics may be needed. Panel C in Table 8 shows that not surprisingly all models underprice options when VIX is high and overprice options when VIX is low. The ERV and GERV models perform relatively better than other models in this regard but the bias is still substantial for extreme levels of the VIX. Table 9 using 60-day RVs confirm the results from Table 8.

5.4 Model Fit Over Time

In Figures 6 and 7 we complement the results in Tables 6-9 by plotting the weekly IVRMSE and weekly IV bias over time.

Figure 6 shows that the RV model in particular and the GRV and ERV models to a lesser extent exhibit a tendency for the IVRMSE to spike up to 10% or higher in certain weeks which of course will drive the overall IVRMSE higher in those models. The GERV model on the other hand only shows one such dramatic spike towards the end of 1998. The weekly IVRMSE for the HN model is shown in dots in Figure 6. It is clear that the HN model also shows evidence of weekly spikes but perhaps less so than the RV and GRV models. Figure 6 also shows that towards the end of the sample, the IVRMSE rises in the RV, GRV and HN models but much less so in the ERV and GERV models. The overall volatility level decreases at the end of the sample which is captured best by the models incorporating expected realized volatility.

Figure 7 tries to understand the weekly IVRMSE patters in Figure 6 but plotting the corresponding weekly IV bias (average market IV less model IV) over time. Clearly some of the spikes in the IVRMSE in the RV model is explained by spikes in the IV bias. It appears that when realized volatility spikes, the RV model prices spike up too much causing a large negative IV bias in the RV model. The GRV model inherits some of these negative spikes where as the ERV and particularly GERV models show less evidence of negative bias spikes. Notice that all models tend to overprice options at the beginning of the sample. At the end of the sample the HN, RV and GRV model tend to overprice the options as well but the ERV and GERV models fare much better. The improvement in weekly IVRMSE in these models during this period found in Figure 6 seems at least partly to be explained by a smaller (absolute) bias.

Overall Tables 6-9 suggest that the superior IVRMSE performance in the GERV model arises from a superior fit across moneyness, maturity and VIX categories. Figures 6-7 suggest that the dynamic properties of the GERV model are the most appropriate.

6 Summary and Conclusions

We develop a class of affine discrete-time models that allow for option valuation in closed form. The models' volatility dynamic contains both a GARCH component and a realized volatility component. We find that incorporating realized volatility leads to a better fit on returns and realized volatility, and that it significantly reduces the pricing errors of the benchmark Heston-Nandi GARCH model. Modeling expected realized volatility is superior to directly modeling realized volatility. Higher conditional moments for our newly proposed models are very different from those of the benchmark GARCH model.

There are several promising avenues for future research. First, the rich literature on realized volatility estimation should provide useful guidance for choosing a more robust estimators than the simple 5-minute and 60-minute RVs currently used. Our results suggest that richer volatility dynamics may be needed. Extending our models to allow for longer lags in RV and squared return could prove valuable. Finally, the leverage effect is modeled differently in the GARCH and RV dynamics. Exploring further the optimal strategy for modeling is key asymmetry is likely to yield important benefits.

7 Appendix

In this appendix we first derive the variance persistence properties and the stationarity conditions of the GRV and GERV models. We also provide some detail on our variance and equity premium estimation strategy. Subsequently, we derive the moment generating functions needed for option valuation.

7.1 Persistence, Stationarity and Targeting: The GRV Model

Note first that

$$E_t\left[h_{t+1}\right] = \omega_1 + \alpha_1 + \left(\beta_1 + \alpha_1\gamma_1^2 n\right) h_t + \alpha_1\gamma_1^2(1-n)RV_t. \tag{14}$$

Therefore from (5) and (14)

$$E_t\begin{pmatrix} h_{t+1} \\ RV_{t+1} \end{pmatrix} = \phi_0 + \phi_1 \begin{pmatrix} h_t \\ RV_t \end{pmatrix},$$

where

$$\begin{aligned} \phi_0 &= (\omega_1 + \alpha_1, \omega_2 + \alpha_2)' \\ \phi_1 &= \begin{bmatrix} \beta_1 + \alpha_1\gamma_1^2 n & \alpha_1\gamma_1^2(1-n) \\ \alpha_2\gamma_2^2 n & \beta_2 + \alpha_2\gamma_2^2(1-n) \end{bmatrix}. \end{aligned}$$

If the largest eigenvalue of ϕ_1 has a modulus smaller than one we have

$$E\left[\begin{pmatrix} h_{t+1} \\ RV_{t+1} \end{pmatrix}\right] = \left(I_2 - \phi_1\right)^{-1}\phi_0. \tag{15}$$

We can therefore derive the following necessary conditions for stationarity

$$\beta_1 < 1, \quad \beta_2 < 1, \quad |\gamma_1| < \sqrt{\frac{1-\beta_1}{\alpha_1 n}}, \quad |\gamma_2| < \sqrt{\left(\frac{1-\beta_2}{1-\beta_1}\right)\left(\frac{1-(\beta_1+\alpha_1\gamma_1^2 n)}{\alpha_2(1-n)}\right)},$$

which we impose when estimating the models.

We also use variance and equity premium targeting in estimation, which greatly facilitates estimation. In variance targeting we set the model-implied long-run mean of the variance, $\bar{h}$, equal to the sample variance of returns.[4] Using (15) and imposing $E\left[h_{t+1}\right] = E\left[RV_{t+1}\right]$ and

[4] For examples of variance targeting see Engle and Mezrich (1996). For the use of the variance targeting technique in option valuation see Christoffersen, Dorion, Jacobs and Wang (2009).

$E\left[h_{t+1}\right]=\bar{h}$ we get

$$\begin{aligned}\omega_1 &= \left(1-\left(\beta_1+\alpha_1\gamma_1^2\right)\right)\bar{h}-\alpha_1\\ \omega_2 &= \left(1-\left(\beta_2+\alpha_2\gamma_2^2\right)\right)\bar{h}-\alpha_2.\end{aligned} \tag{16}$$

We impose (16) in estimation, which makes the numerical problem better behaved, and allows us to estimate two fewer parameters, ω_1 and ω_2. In the empirical results below we report ω_1 and ω_2 as implied by the other parameter estimates and (16).

To further facilitate estimation, we use equity premium targeting for λ. We have

$$E\left(R_{t+1}-r\right)=\left(\lambda-\frac{1}{2}\right)E\left(\bar{h}_t\right),$$

and since we also impose variance targeting, we have $E\left(h_t\right)=E\left(RV_t\right)=E\left(\bar{h}_t\right)=\bar{h}$, therefore

$$E\left(R_{t+1}-r\right)=\left(\lambda-\frac{1}{2}\right)\bar{h},$$

which will be set to the sample mean of excess returns to provide an estimate of λ.

7.2 Persistence, Stationarity and Targeting: The GERV Model

In order to derive stationarity conditions and impose variance targeting in the GERV model, note that

$$\begin{aligned}E_t\left[h_{t+1}\right] &= \omega_1+\alpha_1+\left(\beta_1+\alpha_1\gamma_1^2 n\right)h_t+\alpha_1\gamma_1^2(1-n)m_t\\ E_t\left(m_{t+1}\right) &= \omega_2+\left(\theta+\beta_2\right)m_t.\end{aligned}$$

The vector $\left(h_{t+1},m_{t+1}\right)'$ therefore follows

$$E_t\begin{pmatrix}h_{t+1}\\ m_{t+1}\end{pmatrix}=\phi_0+\phi_1\begin{pmatrix}h_t\\ m_t\end{pmatrix},$$

where

$$\begin{aligned}\phi_0 &= \left(\omega_1+\alpha_1,\omega_2\right)'\\ \phi_1 &= \begin{bmatrix}\beta_1+\alpha_1\gamma_1^2 n & \alpha_1\gamma_1^2(1-n)\\ 0 & \beta_2+\theta\end{bmatrix}.\end{aligned}$$

The stationarity conditions are therefore $\beta_1 + \alpha_1\gamma_1^2 n < 1$ and $\beta_2 + \theta < 1$. Under these two conditions we can compute the unconditional mean

$$E\left[\begin{pmatrix} h_{t+1} \\ RV_{t+1} \end{pmatrix}\right] = (I_2 - \phi_1)^{-1}\phi_0. \tag{17}$$

For variance targeting we impose $E[h_{t+1}] = E[RV_{t+1}]$ and $E[h_{t+1}] = \bar{h}$ to get

$$\begin{aligned}
\omega_2 &= (1 - (\beta_2 + \theta))\bar{h} \\
\omega_1 &= \left(1 - \left(\beta_1 + \alpha_1\gamma_1^2\right)\right)\bar{h} - \alpha_1.
\end{aligned}$$

We also use equity premium targeting as in the GRV model.

7.3 Moment Generating Function of the GRV Model

Using (1), (3) and (4) we have

$$\begin{aligned}
&E_t\left[\exp\left(u_1 R_{t+1} + u_2 h_{t+1} + u_3 RV_{t+1}\right)\right] \qquad (18)\\
&= E_t\left[\exp\begin{pmatrix} u_1\left(r + \left(\lambda - \frac{1}{2}\right)nh_t + \left(\lambda - \frac{1}{2}\right)(1-n)RV_t + \sqrt{\bar{h}_t}\varepsilon_{1,t+1}\right) \\ +u_2\left(\omega_1 + \beta_1 h_t + \alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2\right) \\ +u_3\left(\omega_2 + \beta_2 RV_t + \alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right) \end{pmatrix}\right].
\end{aligned}$$

Collecting terms we get

$$\begin{aligned}
&E_t\left[\exp\left(u_1 R_{t+1} + u_2 h_{t+1} + u_3 RV_{t+1}\right)\right] \qquad (19)\\
&= \exp\begin{pmatrix} u_1\left(r + \left(\lambda - \frac{1}{2}\right)nh_t + \left(\lambda - \frac{1}{2}\right)(1-n)RV_t\right) \\ +u_2\left(\omega_1 + \beta_1 h_t\right) + u_3\left(\omega_2 + \beta_2 RV_t\right) \end{pmatrix} \qquad (20)\\
&\times E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + u_3\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right]. \qquad (21)
\end{aligned}$$

Now consider the conditional expectation in (19)

$$\begin{aligned}
&E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + u_3\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right] \qquad (22)\\
&= E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2\right)E_t\left[\exp\left(u_3\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right]\right].
\end{aligned}$$

We first consider the second component of (22). Since $\varepsilon_{2,t+1}|\varepsilon_{1,t+1} \sim N\left(\rho\varepsilon_{1,t+1}, 1-\rho^2\right)$ we have

$$\begin{aligned} & E_t\left[\exp\left(u_3\alpha_2\left(\varepsilon_{2,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right] \\ = \; & E_t\left[\exp\left(u_3\alpha_2\left(1-\rho^2\right)\left(\frac{\varepsilon_{2,t+1}-\rho\varepsilon_{1,t+1}}{\sqrt{1-\rho^2}}+\frac{\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}}{\sqrt{1-\rho^2}}\right)^2\right)|\varepsilon_{1,t+1}\right]. \end{aligned}$$

Furthermore for $Z \sim N(0,1)$ we have

$$E\left[\exp\left(a(Z+b)^2\right)\right] = \exp\left(-\frac{1}{2}\ln\left(1-2a\right)+\frac{ab^2}{1-2a}\right). \tag{23}$$

Therefore

$$\begin{aligned} & E_t\left[\exp\left(u_3\alpha_2\left(\varepsilon_{2,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right] \\ = \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_3\alpha_2\left(1-\rho^2\right)\right)+\frac{u_3\alpha_2\left(1-\rho^2\right)\left(\frac{\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}}{\sqrt{1-\rho^2}}\right)^2}{1-2u_3\alpha_2\left(1-\rho^2\right)}\right) \\ = \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_3\alpha_2\left(1-\rho^2\right)\right)+\frac{u_3\alpha_2\left(\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2}{1-2u_3\alpha_2\left(1-\rho^2\right)}\right). \end{aligned}$$

Using these results in (22)

$$\begin{aligned} & E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2+u_3\alpha_2\left(\varepsilon_{2,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right] \\ = \; & E_t\left[\exp\left(\begin{array}{c} u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2 \\ -\frac{1}{2}\ln\left(1-2u_3\alpha_2\left(1-\rho^2\right)\right)+\frac{u_3\alpha_2\left(\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2}{1-2u_3\alpha_2(1-\rho^2)} \end{array}\right)\right] \\ = \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_3\alpha_2\left(1-\rho^2\right)\right)\right) \qquad (24) \\ & \times E_t\left[\exp\left(\begin{array}{c} u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2 \\ +u_3^*\alpha_2\left(\varepsilon_{1,t+1}-\gamma_2^*\sqrt{\bar{h}_t}\right)^2 \end{array}\right)\right], \qquad (25) \end{aligned}$$

where $u_3^* = \frac{u_3\rho^2}{1-2u_3\alpha_2(1-\rho^2)}$ and $\gamma_2^* = \frac{\gamma_2}{\rho}$. Rewriting (24)

$$\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + u_3^*\alpha_2\left(\varepsilon_{1,t+1} - \gamma_2^*\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= \ & \exp\left(\left(u_2\alpha_1\gamma_1^2 + u_3^*\alpha_2\gamma_2^{*2} - (u_2\alpha_1 + u_3^*\alpha_2)\gamma_3^2\right)\bar{h}_t + (u_2\alpha_1 + u_3^*\alpha_2)\left(\varepsilon_{1,t+1} - \gamma_3\sqrt{\bar{h}_t}\right)^2\right),
\end{aligned}$$

where $\gamma_3 = \frac{u_2\alpha_1\gamma_1 + u_3^*\alpha_2\gamma_2^* - \frac{1}{2}u_1}{u_2\alpha_1 + u_3^*\alpha_2}$. Again using (23) we therefore have

$$\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + u_3^*\alpha_2\left(\varepsilon_{1,t+1} - \gamma_2^*\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= \ & \exp\left(\begin{array}{c} \left(u_2\alpha_1\gamma_1^2 + u_3^*\alpha_2\gamma_2^{*2} - (u_2\alpha_1 + u_3^*\alpha_2)\gamma_3^2\right)\bar{h}_t \\ -\frac{1}{2}\ln\left(1 - 2(u_2\alpha_1 + u_3^*\alpha_2)\right) + \frac{(u_2\alpha_1 + u_3^*\alpha_2)\gamma_3^2}{1-2(u_2\alpha_1 + u_3^*\alpha_2)}\bar{h}_t \end{array}\right) \\
= \ & \exp\left(\begin{array}{c} \left(u_2\alpha_1\gamma_1^2 + u_3^*\alpha_2\gamma_2^{*2} - (u_2\alpha_1 + u_3^*\alpha_2)\gamma_3^2 + \frac{(u_2\alpha_1 + u_3^*\alpha_2)\gamma_3^2}{1-2(u_2\alpha_1 + u_3^*\alpha_2)}\right)\bar{h}_t \\ -\frac{1}{2}\ln\left(1 - 2(u_2\alpha_1 + u_3^*\alpha_2)\right) \end{array}\right) \\
= \ & \exp\left(\begin{array}{c} \left(u_2\alpha_1\gamma_1^2 + u_3^*\alpha_2\gamma_2^{*2} + \frac{2(u_2\alpha_1 + u_3^*\alpha_2)^2\gamma_3^2}{1-2(u_2\alpha_1 + u_3^*\alpha_2)}\right)\bar{h}_t \\ -\frac{1}{2}\ln\left(1 - 2(u_2\alpha_1 + u_3^*\alpha_2)\right) \end{array}\right) \\
= \ & \exp\left[\begin{array}{c} \left(u_2\alpha_1\gamma_1^2 + u_3^*\alpha_2\gamma_2^{*2} + \frac{2(u_2\alpha_1\gamma_1 + u_3^*\alpha_2\gamma_2^* - \frac{1}{2}u_1)^2}{1-2(u_2\alpha_1 + u_3^*\alpha_2)}\right)\bar{h}_t \\ -\frac{1}{2}\ln\left(1 - 2(u_2\alpha_1 + u_3^*\alpha_2)\right) \end{array}\right] \\
\equiv \ & \exp\left[a(u_1, u_2, u_3)\bar{h}_t + b(u_1, u_2, u_3)\right]. \qquad (26)
\end{aligned}$$

Now using (26) in (19) and (18) we obtain

$$\begin{aligned}
& E_t\left[\exp\left(u_1R_{t+1} + u_2h_{t+1} + u_3RV_{t+1}\right)\right] \\
= \ & \exp\left(A_1(u_1, u_2, u_3)h_t + A_2(u_1, u_2, u_3)RV_t + B(u_1, u_2, u_3)\right).
\end{aligned}$$

where

$$\begin{aligned}
A_1(u_1, u_2, u_3) &= u_1\left(\lambda - \frac{1}{2}\right)n + u_2\beta_1 + na(u_1, u_2, u_3) \\
A_2(u_1, u_2, u_3) &= u_1\left(\lambda - \frac{1}{2}\right)(1-n) + u_3\beta_2 + (1-n)a(u_1, u_2, u_3) \\
B(u_1, u_2, u_3) &= u_1r + u_2\omega_1 + u_3\omega_2 + b(u_1, u_2, u_3) - \frac{1}{2}\ln\left(1 - 2u_3\alpha_2\left(1-\rho^2\right)\right).
\end{aligned}$$

Since the model is affine, we conjecture that the multi-period conditional moment generating function is of the form

$$\Psi_{t,t+M}(u) \equiv E_t\left[\exp\left(u\sum_{j=1}^{M}R_{t+j}\right)\right] = \exp\left(C_1(u,M)h_t + C_2(u,M)RV_t + D(u,M)\right).$$

This gives

$$\begin{aligned}
\Psi_{t,t+M}(u) &= E_t\left[E_{t+1}\left[\exp\left(u\sum_{j=1}^{M+1}R_{t+j}\right)\right]\right] \\
&= E_t\left[\exp(uR_{t+1})E_{t+1}\left[\exp\left(u\sum_{j=2}^{M+1}R_{t+j}\right)\right]\right] \\
&= E_t\left[\exp(uR_{t+1})E_{t+1}\left[\exp\left(u\sum_{k=1}^{M}R_{t+1+k}\right)\right]\right] \\
&= E_t\left[\exp(uR_{t+1} + C_1(u,M)h_{t+1} + C_2(u,M)RV_{t+1} + D(u,M))\right] \\
&= \exp\left(\begin{array}{c} A_1(u, C_1(u,M), C_2(u,M))h_t + A_2(u, C_1(u,M), C_2(u,M))RV_t \\ +B(u, C_1(u,M), C_2(u,M)) + D(u,M) \end{array}\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
C_1(u,M+1) &= A_1(u, C_1(u,M), C_2(u,M)) \\
C_2(u,M+1) &= A_2(u, C_1(u,M), C_2(u,M)) \\
D(u,M+1) &= B(u, C_1(u,M), C_2(u,M)) + D(u,M)
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
C_1(u,1) &= A_1(u,0,0) \\
C_2(u,1) &= A_2(u,0,0) \\
D(u,1) &= B(u,0,0)
\end{aligned}$$

7.4 Moment Generating Function of the GERV Model

Using (7), (8), (10), and (9) we have

$$\begin{aligned}
& E_t\left[\exp\left(u_1R_{t+1} + u_2h_{t+1} + u_3m_{t+1}\right)\right] \\
= \; & E_t\left[\exp\left(\begin{array}{c} u_1\left(E_t\left[R_{t+1}\right] + \sqrt{h_t}\varepsilon_{1,t+1}\right) \\ +u_2\left(\omega_1 + \beta_1h_t + \alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{h_t}\right)^2\right) + u_3\left(\omega_2 + (\beta_2+\theta)m_t + \beta_2 z_{t+1}\right) \end{array}\right)\right].
\end{aligned}$$

Rewriting, we get

$$\begin{aligned}
& E_t\left[\exp\left(u_1 R_{t+1} + u_2 h_{t+1} + u_3 m_{t+1}\right)\right] \\
= \; & \exp\left(u_2\omega_1 + u_3\omega_2 + u_1 E_t\left[R_{t+1}\right] + u_3\left(\beta_2+\theta\right) m_t + u_2\beta_1 h_t - \alpha_2\beta_2 u_3(1+\gamma_2^2\bar{h}_t)\right) \times \\
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + \alpha_1 u_2\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + \alpha_2\beta_2 u_3\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right]. \quad (27)
\end{aligned}$$

Denoting $\beta_2 u_3 = u_2^*$ the second part of (27) can be written as

$$\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + \alpha_1 u_2\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2 + \alpha_2 u_2^*\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right] \qquad (28) \\
= \; & E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1} + u_2\alpha_1\left(\varepsilon_{1,t+1} - \gamma_1\sqrt{\bar{h}_t}\right)^2\right) E_t\left[\exp\left(u_2^*\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right]\right].
\end{aligned}$$

Using the fact that $\varepsilon_{2,t+1}|\varepsilon_{1,t+1} \sim N\left(\rho\varepsilon_{1,t+1}, 1-\rho^2\right)$

$$\begin{aligned}
& E_t\left[\exp\left(u_2^*\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right] \\
= \; & E_t\left[\exp\left(u_2^*\alpha_2\left(1-\rho^2\right)\left(\frac{\varepsilon_{2,t+1} - \rho\varepsilon_{1,t+1}}{\sqrt{1-\rho^2}} + \frac{\rho\varepsilon_{1,t+1} - \gamma_2\sqrt{\bar{h}_t}}{\sqrt{1-\rho^2}}\right)^2\right)|\varepsilon_{1,t+1}\right],
\end{aligned}$$

and using (23) we can rewrite as

$$\begin{aligned}
& E_t\left[\exp\left(u_2^*\alpha_2\left(\varepsilon_{2,t+1} - \gamma_2\sqrt{\bar{h}_t}\right)^2\right)|\varepsilon_{1,t+1}\right] \\
= \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right) + \frac{u_2^*\alpha_2\left(1-\rho^2\right)\left(\frac{\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}}{\sqrt{1-\rho^2}}\right)^2}{1-2u_2^*\alpha_2\left(1-\rho^2\right)}\right) \\
= \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right) + \frac{u_2^*\alpha_2\left(\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2}{1-2u_2^*\alpha_2\left(1-\rho^2\right)}\right).
\end{aligned}$$

This gives for (28)

$$
\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2+u_2^*\alpha_2\left(\varepsilon_{2,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= \; & E_t\left[\exp\left(\begin{array}{c} u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2 \\ -\frac{1}{2}\ln\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right)+\frac{u_2^*\alpha_2\left(\rho\varepsilon_{1,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2}{1-2u_2^*\alpha_2\left(1-\rho^2\right)} \end{array}\right)\right] \\
= \; & \exp\left(-\frac{1}{2}\ln\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right)\right) \qquad (29) \\
& E_t\left[\exp\left(\begin{array}{c} u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2 \\ +u_4^*\alpha_2\left(\varepsilon_{1,t+1}-\gamma_2^*\sqrt{\bar{h}_t}\right)^2 \end{array}\right)\right],
\end{aligned}
$$

where $u_4^* = \frac{u_2^*\rho^2}{1-2u_2^*\alpha_2(1-\rho^2)}$ and $\gamma_2^* = \frac{\gamma_2}{\rho}$. The second part of (29) can be written as

$$
\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2+u_4^*\alpha_2\left(\varepsilon_{1,t+1}-\gamma_2^*\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= \; & \exp\left(\left(u_2\alpha_1\gamma_1^2+u_4^*\alpha_2\gamma_2^{*2}-(u_2\alpha_1+u_4^*\alpha_2)\gamma_3^2\right)\bar{h}_t+(u_2\alpha_1+u_4^*\alpha_2)\left(\varepsilon_{1,t+1}-\gamma_3\sqrt{\bar{h}_t}\right)^2\right),
\end{aligned}
$$

where $\gamma_3 = \frac{u_2\alpha_1\gamma_1+u_4^*\alpha_2\gamma_2^*-\frac{1}{2}u_1}{u_2\alpha_1+u_4^*\alpha_2}$. Using (23) we have

$$
E_t\left[\exp\left(u_1^*\left(\varepsilon_{1,t+1}-\gamma_3\sqrt{\bar{h}_t}\right)^2\right)\right] = \exp\left(-\frac{1}{2}\ln\left(1-2u_1^*\right)+\frac{u_1^*\gamma_3^2\bar{h}_t}{1-2u_1^*}\right),
$$

where $u_1^* = u_2\alpha_1+u_4^*\alpha_2$. Therefore

$$
\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2+u_4^*\alpha_2\left(\varepsilon_{1,t+1}-\gamma_2^*\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= \; & \exp\left(\left(u_2\alpha_1\gamma_1^2+u_4^*\alpha_2\gamma_2^{*2}-u_1^*\gamma_3^2\right)\bar{h}_t-\frac{1}{2}\ln\left(1-2u_1^*\right)+\frac{u_1^*\gamma_3^2\bar{h}_t}{1-2u_1^*}\right).
\end{aligned}
$$

This gives for (29)

$$
\begin{aligned}
& E_t\left[\exp\left(u_1\sqrt{\bar{h}_t}\varepsilon_{1,t+1}+u_2\alpha_1\left(\varepsilon_{1,t+1}-\gamma_1\sqrt{\bar{h}_t}\right)^2+u_2^*\alpha_2\left(\varepsilon_{2,t+1}-\gamma_2\sqrt{\bar{h}_t}\right)^2\right)\right] \\
= & \exp\left(\left(u_2\alpha_1\gamma_1^2+u_4^*\alpha_2\gamma_2^{*2}-u_1^*\gamma_3^2\right)\bar{h}_t-\frac{1}{2}\ln\left(1-2u_1^*\right)-\frac{1}{2}\ln\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right)+\frac{u_1^*\gamma_3^2\bar{h}_t}{1-2u_1^*}\right) \\
= & \exp\left(\left(u_2\alpha_1\gamma_1^2+u_4^*\alpha_2\gamma_2^{*2}-u_1^*\gamma_3^2+\frac{u_1^*\gamma_3^2}{1-2u_1^*}\right)\bar{h}_t-\frac{1}{2}\ln\left[\left(1-2u_1^*\right)\left(1-2u_2^*\alpha_2\left(1-\rho^2\right)\right)\right]\right) \\
= & \exp\left[a\left(u_1,u_2,u_3\right)\bar{h}_t+b\left(u_1,u_2,u_3\right)\right].
\end{aligned}
$$

Collecting these results gives

$$
\begin{aligned}
& E_t\left[\exp\left(u_1R_{t+1}+u_2h_{t+1}+u_3m_{t+1}\right)\right] \\
= & \exp\left(\begin{array}{c} u_2\omega_1+u_3\omega_2+u_1E_t\left[R_{t+1}\right]+u_3\left(\beta_2+\theta\right)m_t+u_2\beta_1h_t \\ -\alpha_2\beta_2u_3(1+\gamma_2^2\bar{h}_t)+a\left(u_1,u_2,u_3\right)\bar{h}_t+b\left(u_1,u_2,u_3\right)\end{array}\right) \\
= & \exp\left(\begin{array}{c} u_2\omega_1+u_3\omega_2+u_1\left(r+\left(\lambda-\frac{1}{2}\right)nh_t+\left(\lambda-\frac{1}{2}\right)\left(1-n\right)m_t\right)+u_3\left(\beta_2+\theta\right)m_t+u_2\beta_1h_t \\ -\alpha_2\beta_2u_3(1+\gamma_2^2\bar{h}_t)+a\left(u_1,u_2,u_3\right)\bar{h}_t+b\left(u_1,u_2,u_3\right)\end{array}\right) \\
= & \exp\left(A_1\left(u_1,u_2,u_3\right)h_t+A_2\left(u_1,u_2,u_3\right)m_t+B\left(u_1,u_2,u_3\right)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
A_1\left(u_1,u_2,u_3\right) &= u_1\left(\lambda-\frac{1}{2}\right)n+u_2\beta_1-\alpha_2\gamma_2^2\beta_2u_3n+a\left(u_1,u_2,u_3\right)n \\
A_2\left(u_1,u_2,u_3\right) &= u_1\left(\lambda-\frac{1}{2}\right)\left(1-n\right)+u_3\left(\beta_2+\theta\right)-\alpha_2\gamma_2^2\beta_2u_3\left(1-n\right)+a\left(u_1,u_2,u_3\right)\left(1-n\right) \\
B\left(u_1,u_2,u_3\right) &= u_1r+u_2\omega_1+u_3\omega_2-\alpha_2\beta_2u_3+b\left(u_1,u_2,u_3\right).
\end{aligned}
$$

Since the model is affine, we conjecture that

$$
\begin{aligned}
& E_t\left[\exp\left(u\sum_{j=1}^{M}R_{t+j}\right)\right] \\
= & \exp\left(C_1\left(u,M\right)h_t+C_2\left(u,M\right)m_t+D(u,M)\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& E_t\left[\exp\left(u\sum_{j=1}^{M+1}R_{t+j}\right)\right] = E_t\left[E_{t+1}\left[\exp\left(u\sum_{j=1}^{M+1}R_{t+j}\right)\right]\right] \\
= \;& E_t\left[\exp(uR_{t+1})E_{t+1}\left[\exp\left(u\sum_{j=2}^{M+1}R_{t+j}\right)\right]\right] \\
= \;& E_t\left[\exp(uR_{t+1})E_{t+1}\left[\exp\left(u\sum_{k=1}^{M}R_{t+1+k}\right)\right]\right] \\
= \;& E_t\left[\exp(uR_{t+1} + C_1(u,M)\,h_{t+1} + C_2(u,M)\,m_{t+1} + D(u,M))\right] \\
= \;& \exp\left(\begin{array}{c} A_1(u,0,C_1(u,M),C_2(u,M))\,h_t + A_2(u,0,C_1(u,M),C_2(u,M))\,RV_t \\ +B(u,0,C_1(u,M),C_2(u,M)) + D(u,M) \end{array}\right).
\end{aligned}
$$

This yields

$$
\begin{aligned}
C_1(u,M+1) &= A_1(u,0,C_1(u,M),C_2(u,M)) \\
C_2(u,M+1) &= A_2(u,0,C_1(u,M),C_2(u,M)) \\
D(u,M+1) &= B(u,0,C_1(u,M),C_2(u,M)) + D(u,M),
\end{aligned}
$$

with the following initial conditions

$$
\begin{aligned}
C_1(u,1) &= A_1(u,0,0,0) \\
C_2(u,1) &= A_2(u,0,0,0) \\
D(u,1) &= B(u,0,0,0)\,.
\end{aligned}
$$

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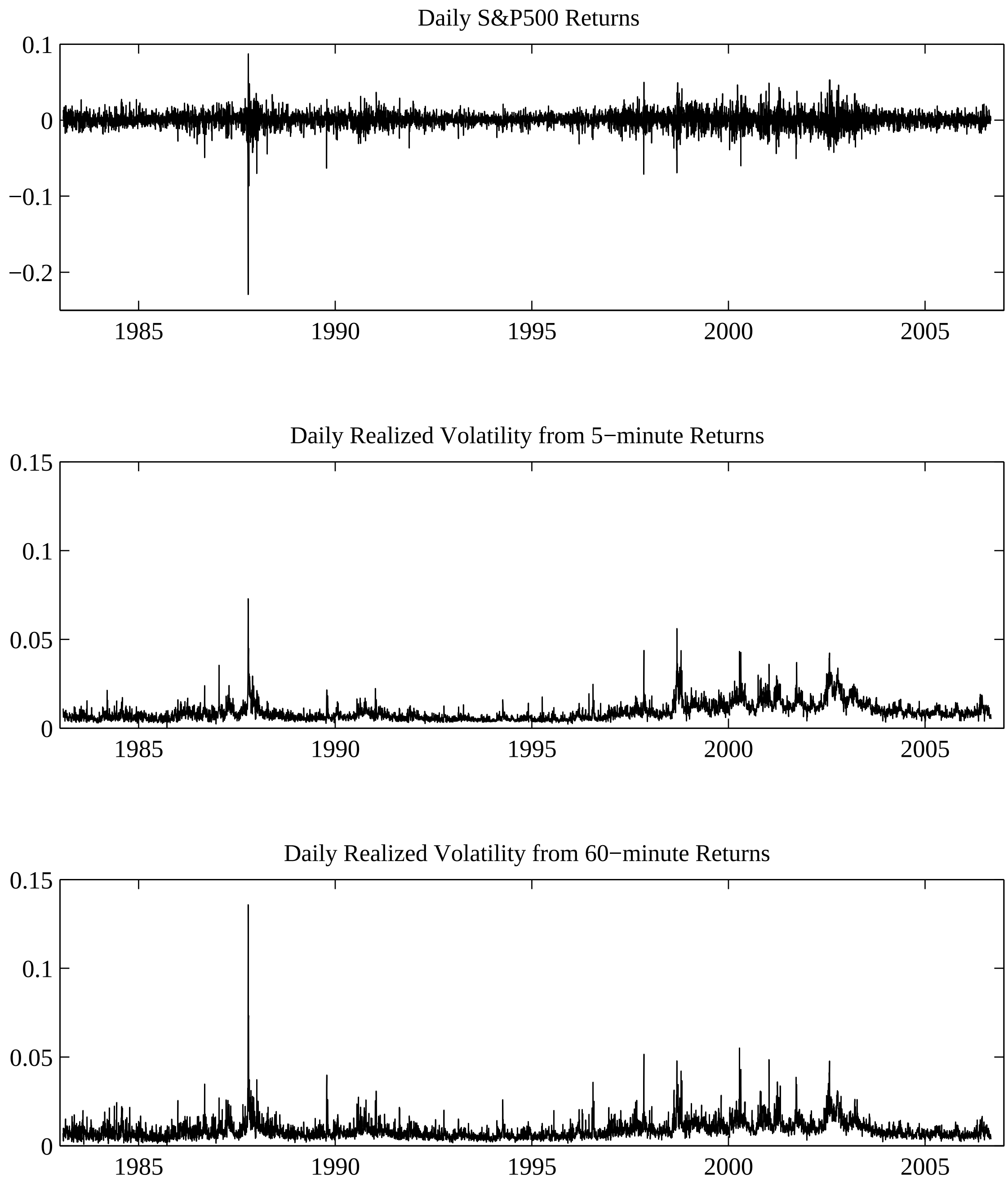
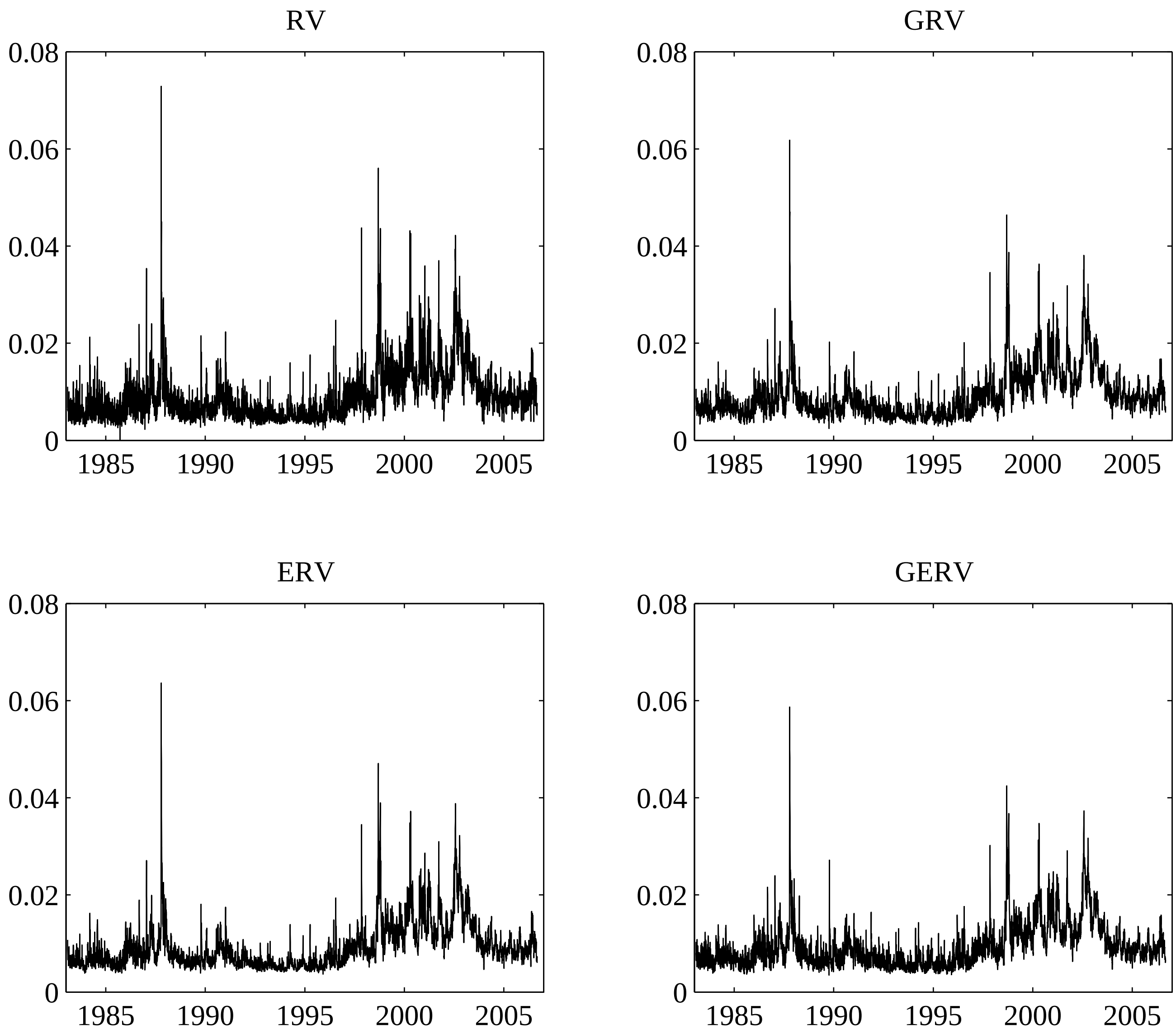
Figure 1: Daily Returns and Daily Realized Volatilities from Intraday Data. 1983-2006



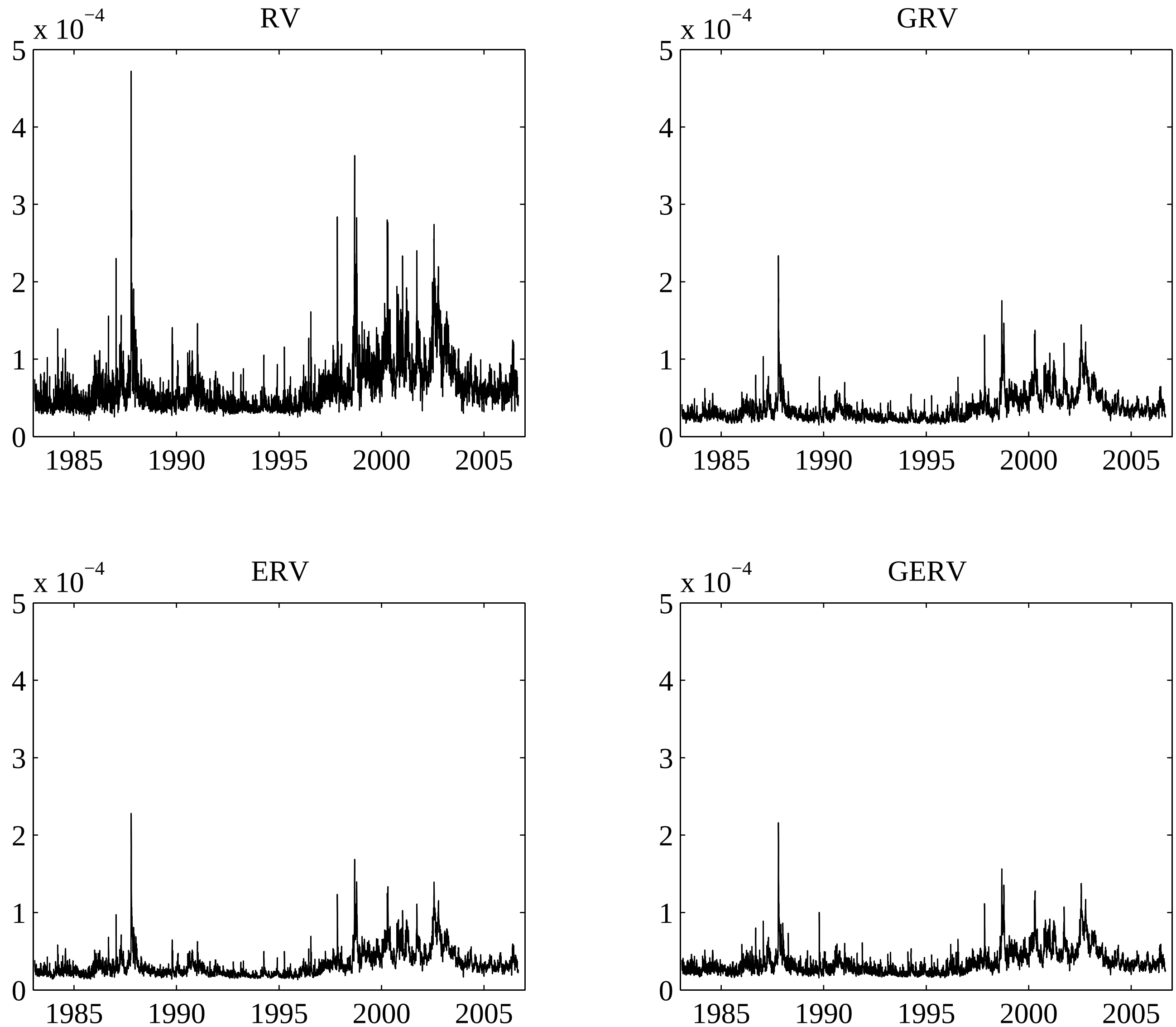
Notes to Figure: In the top panel we plot the daily returns on the S&P500 index from February 2, 1983 to August 31, 2006. In the middle and bottom panels we plot the square root of the realized variance, RV_t, using sum of squared of 5-minute and 60-minute intraday returns, respectively. The RV measures have been rescaled to match the unconditional variance of daily returns.

Figure 2: Daily Conditional Volatility from 5-min RV Models. 1983-2006



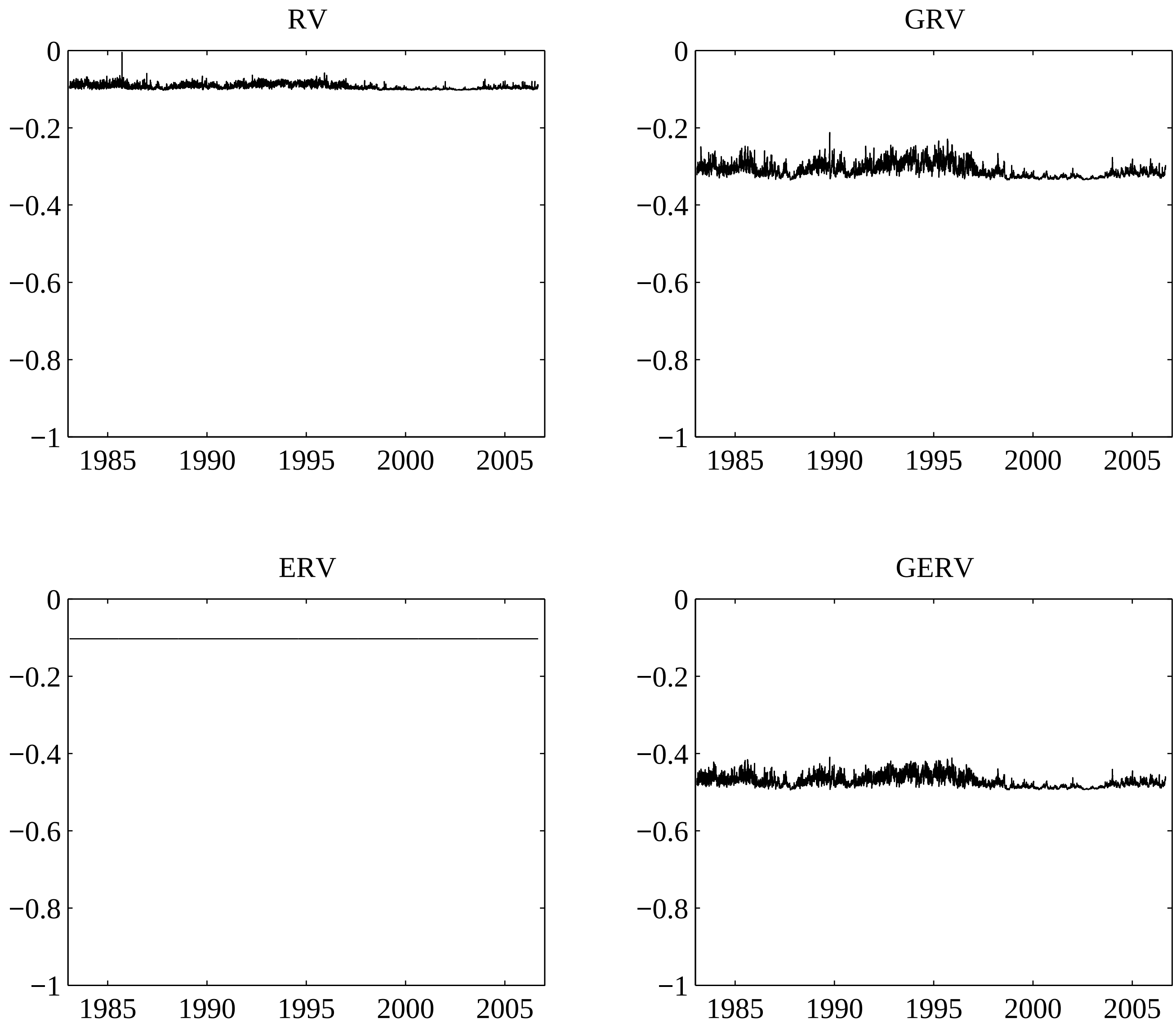
Notes to Figure: We plot the square root of the daily conditional variance, $\bar{h}_t = Var_t(R_{t+1})$ from each of the four models that incorporate daily realized variance.

Figure 3: Daily Conditional Volatility of Variance in 5-min RV Models. 1983-2006



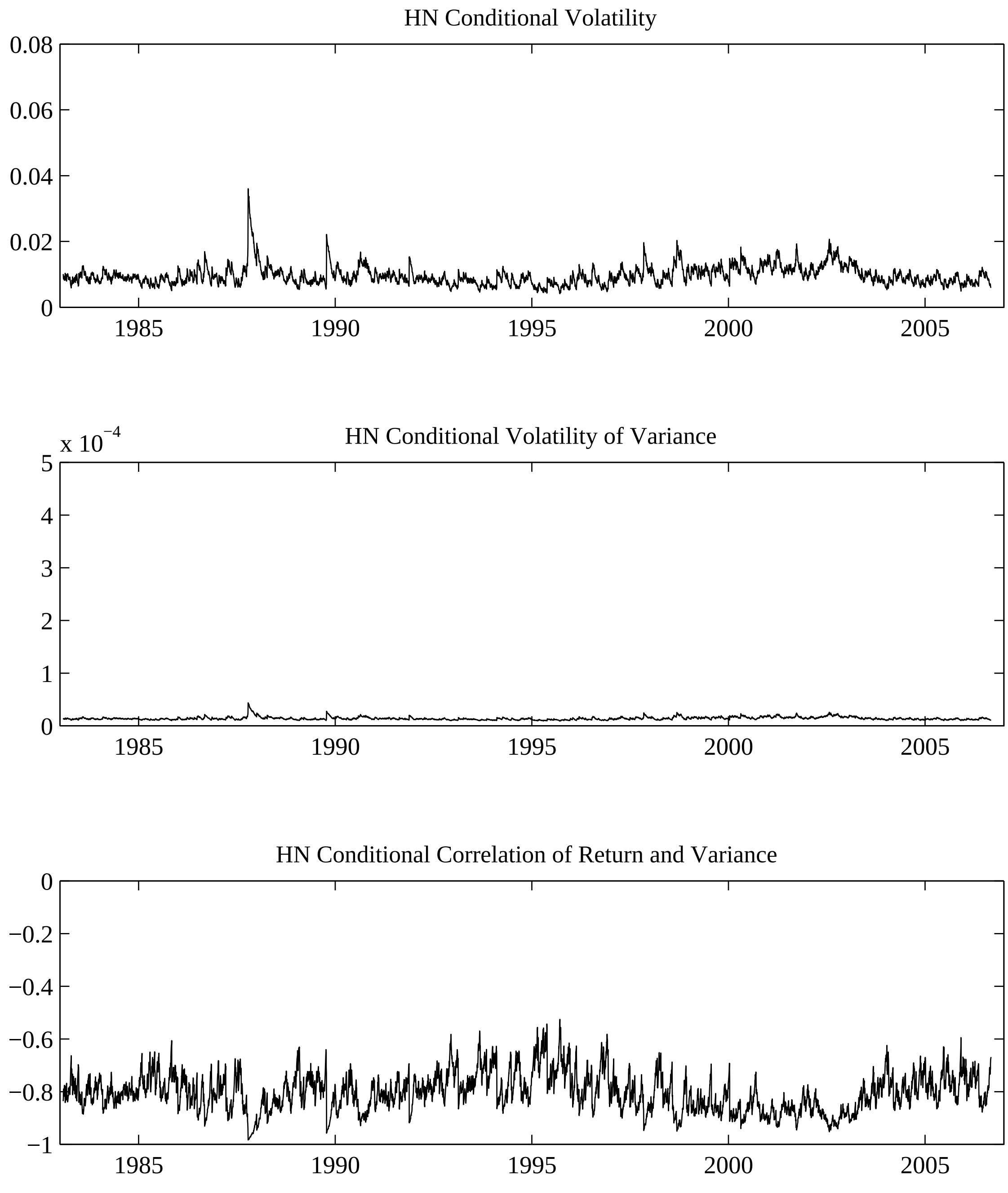
Notes to Figure: We plot the square root of the daily conditional variance of variance, $Var_t\left(\bar{h}_{t+1}\right)$ from each of the four models that incorporate daily realized variance.

Figure 4: Daily Correlation of Return and Variance in 5-min RV Models. 1983-2006



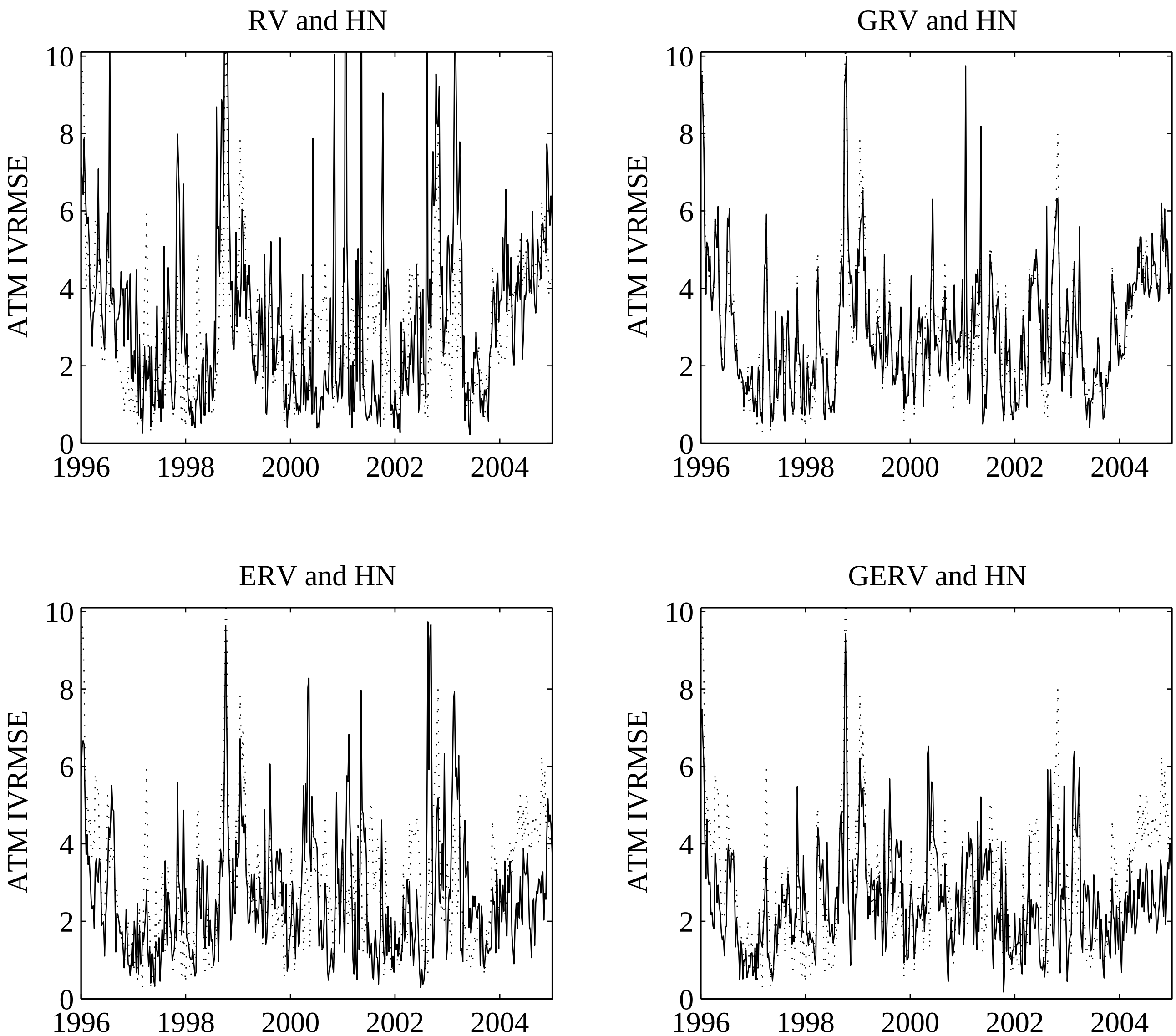
Notes to Figure: We plot the daily conditional correlation of return and variance, $Corr_t\left(R_{t+1}, \bar{h}_{t+1}\right)$ from each of the four models that incorporate daily realized variance.

Figure 5: Conditional Daily Moments in the HN GARCH Model. 1983-2006.



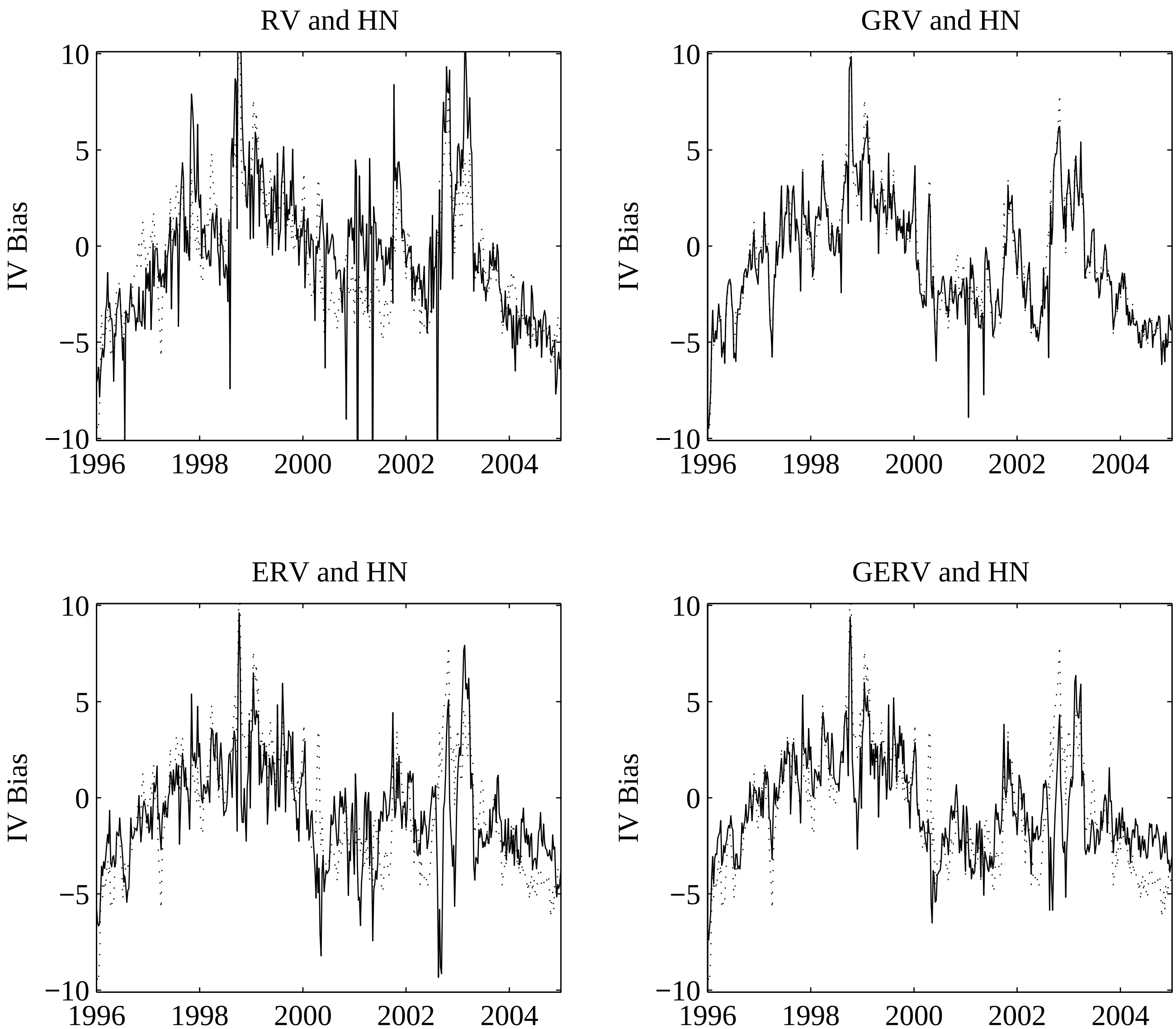
Notes to Figure: We plot the square root of the daily conditional variance, the square root of the conditional variance of variance, and the conditional correlation between return and variance for the HN GARCH model.

Figure 6: Weekly IVRMSE from at-the-money Options. 5-min RV. 1996-2004



Notes to Figure: On each Wednesday we plot the implied volatility root mean squared error (IVRMSE) for options that are at-the-money. The solid line in each panel corresponds to an RV-based model and the dashed line corresponds to the HN GARCH based model.

Figure 7: Weekly IV Bias from at-the-money Options. 5-min RV. 1996-2004



Notes to Figure: On each Wednesday we plot the implied volatility bias (IV Bias) defined as market IV less model IV for options that are at-the-money. The solid line in each panel corresponds to an RV-based model and the dashed line corresponds to the HN GARCH based model.

Table 1: QML Estimates on Daily Returns and 5 min RVs. 1983-2006.

Parameters	GARCH	RV	GRV	ERV	GERV
n	1.0000	0.0000	0.4497	0.0000	0.2017
			(5.90E-02)		(1.10E-01)
λ	2.6722	2.6722	2.6722	2.6722	2.6722
α_1	4.532E-06		2.048E-06		2.210E-05
	(4.56E-07)		(1.30E-06)		(1.47E-05)
β_1	0.8806		0.4778		0.1093
	(9.79E-03)		(9.24E-02)		(1.52E-01)
γ_1	130.7		495.7		176.2
	(1.14E+01)		(1.62E+02)		(6.50E+01)
ω_1	2.417E-10		1.327E-09		-6.078E-11
α_2		1.193E-05	9.820E-06	4.846E-07	8.095E-07
		(9.07E-07)	(1.17E-06)	(2.11E-08)	(3.71E-08)
β_2		1.124E-05	2.081E-05	5.408E-01	4.802E-01
		(1.42E-06)	(8.58E-06)	(9.42E-02)	(8.44E-02)
θ				0.4371	0.5067
				(1.00E-01)	(8.80E-02)
γ_2		271.2	304.3	6839.4	4005.1
		(9.63E+00)	(1.99E+01)	(6.19E+02)	(2.28E+02)
ω_2		1.333E-06	5.471E-10	2.388E-06	1.415E-06
ρ		0.1028	0.1080	0.1033	0.1010
		(1.61E-02)	(1.78E-02)	(1.78E-02)	(1.64E-02)
Volatility Persistence					
From Returns	0.9581		0.7041		0.2477
From RV		0.8773	0.5003	0.9779	0.9869
Log Likelihoods					
Joint Returns and RV		68,100	69,340	69,294	69,416
Marginal for Returns		18,564	19,445	19,457	19,517
Marginal for RV		49,473	49,826	49,768	49,835
Maximized on Returns	19,589	18,564	19,678	19,602	19,678

Notes: We estimate five models using daily close-to-close returns and realized variance data for the S&P500 index, for the period February 2, 1983 to August 31, 2006. Realized variance is constructed using five-minute intervals. Standard errors are indicated in parentheses. The parameters are estimated using variance targeting to 16.5% per year and equity premium targeting to 6% per year, thus certain parameters do not have standard errors.

Table 2: QML Estimates on Daily Returns and 60 min RVs. 1983-2006.

Parameters	GARCH	RV	GRV	ERV	GERV
n	1.0000	0.0000	0.4979	0.0000	0.3549
			(5.91E-02)		(8.29E-02)
λ	2.6722	2.6722	2.6722	2.6722	2.6722
α_1	4.532E-06		5.966E-06		1.674E-05
	(4.56E-07)		(1.38E-06)		(6.42E-06)
β_1	0.8806		0.0680		0.0370
	(9.79E-03)		(1.27E-01)		(1.16E-01)
γ_1	130.7		375.3		219.7
	(1.14E+01)		(1.90E+02)		(4.97E+01)
ω_1	2.417E-10		3.935E-06		3.029E-10
α_2		3.224E-05	2.785E-05	1.202E-06	1.228E-06
		(5.54E-06)	(1.08E-06)	(1.16E-07)	(5.64E-08)
β_2		2.179E-06	9.917E-06	5.768E-01	5.384E-01
		(1.48E-06)	(2.32E-06)	(9.41E-02)	(1.32E-01)
θ				0.3962	0.4426
				(9.23E-02)	(1.37E-01)
γ_2		147.5	163.2	5024.6	4737.8
		(1.81E+01)	(1.79E+01)	(1.00E+03)	(6.99E+02)
ω_2		1.063E-10	3.182E-11	2.918E-06	2.047E-06
ρ		0.1212	0.1163	0.1122	0.1005
		(2.21E-02)	(1.76E-02)	(2.04E-02)	(1.61E-02)
Volatility Persistence					
From Returns	0.9581		0.4865		0.3237
From RV		0.7017	0.3727	0.9730	0.9811
Log Likelihoods					
Joint Returns and RV		64,174	65,351	65,762	65,969
Marginal for Returns		18,558	19,445	19,490	19,524
Marginal for RV		45,537	45,832	46,191	46,381
Maximized on Returns	19,589	18,558	19,695	19,655	19,699

Notes: We estimate five models using daily close-to-close returns and realized variance data for the S&P500 cash index, for the period February 2, 1983 to August 31, 2006. Realized variance is constructed using sixty-minute intervals. Standard errors are indicated in parentheses. The parameters are estimated using variance targeting to 16.5% per year and equity premium targeting to 6% per year, thus certain parameters do not have standard errors.

Table 3: S&P500 Index Option Data. 1996-2004.

By Moneyness	F/X<0.96	0.96<F/X<0.98	0.98<F/X<1.02	1.02<F/X<1.04	1.04<F/X<1.06	F/X>1.06	All
Number of Contracts	1,162	961	3,294	1,325	951	2,445	10,138
Average Price	17.41	22.37	30.59	25.55	22.13	17.51	23.69
Average Implied Volatility	19.63	18.71	19.50	21.23	22.22	25.73	21.42
Average Bid-Ask Spread	1.187	1.378	1.572	1.400	1.298	1.154	1.361
By Maturity	**DTM<30**	**30<DTM<60**	**60<DTM<90**	**90<DTM<120**	**120<DTM<150**	**DTM>150**	**All**
Number of Contracts	695	3,476	2,551	1,063	1,332	1,021	10,138
Average Price	12.11	17.97	24.47	27.51	31.20	35.36	23.69
Average Implied Volatility	20.69	21.04	21.62	21.86	21.75	21.87	21.42
Average Bid-Ask Spread	0.830	1.184	1.452	1.539	1.588	1.610	1.361

Notes: We use Wednesday closing out of the money (OTM) call and put option data from OptionMetrics from January 1, 1996 through December 31, 2004. F denotes the implied Futures price of the S&P500 index, X denotes the strike price, and DTM denotes the number of calendar days to maturity. The average bid-ask spread is reported in dollars.

Table 4: NLS Estimates on Options using 5 min RVs. 1996-2004.

Parameters	GARCH	RV	GRV	ERV	GERV
n	1.0000	0.0000	0.1232	0.0000	0.6841
			(7.60E-05)		(1.32E-03)
α_1	1.950E-06		1.349E-06		1.057E-06
	(2.34E-07)		(1.81E-09)		(3.95E-08)
β_1	0.9021		0.9724		0.9916
	(6.91E-03)		(3.43E-05)		(3.13E-04)
γ_1	205.6		112.0		0.0012
	(6.06E+00)		(9.22E-03)		(1.05E-04)
ω_1	2.105E-10		3.065E-13		-3.960E-20
α_2		7.768E-06	3.356E-06	4.189E-05	5.246E-06
		(2.31E-09)	(4.22E-09)	(2.40E-08)	(5.03E-07)
β_2		0.0000	0.0000	0.0848	0.2166
		(0.00E+00)	(0.00E+00)	(6.23E-05)	(1.72E-03)
θ				0.8870	0.7737
				(9.65E-05)	(2.03E-03)
γ_2		347.6	538.5	523.1	4013.0
		(5.51E-02)	(3.48E-01)	(4.73E-01)	(6.33E+01)
ω_2		9.126E-14	8.015E-14	3.554E-06	1.216E-06
ρ		0.6346	1.0000	0.5825	0.4935
		(5.21E-05)	(0.00E+00)	(1.82E-04)	(3.23E-03)
IVRMSE(%)	3.904	3.923	3.503	3.485	3.067
IVRMSE Ratio	1.000	1.005	0.897	0.893	0.786
Volatility Persistence					
From Returns	0.9845		0.9745		0.9916
From RV		0.9384	0.8535	0.9718	0.9904

Notes: We estimate five models using option data for the period January 1, 1996 to December 31, 2004. Realized variance is constructed using five-minute intervals. Standard errors, computed using the outer product of the gradient, are indicated in parentheses. The parameters are estimated using variance targeting to 17.3% per year and equity premium targeting to 6% per year.

Table 5: NLS Estimates on Options using 60 min RVs. 1996-2004.

Parameters	GARCH	RV	GRV	ERV	GERV
n	1.0000	0.0000	0.8634	0.0000	0.8422
			(2.33E-03)		(1.16E-03)
α_1	1.950E-06		2.412E-06		5.701E-06
	(2.34E-07)		(8.16E-08)		(6.12E-08)
β_1	0.9021		0.8671		0.6533
	(6.91E-03)		(2.32E-03)		(7.47E-04)
γ_1	205.6		217.2		230.0
	(6.06E+00)		(2.08E+00)		(1.14E+00)
ω_1	2.105E-10		2.894E-10		7.638E-10
α_2		9.097E-06	1.024E-06	4.155E-05	1.470E-05
		(2.27E-09)	(4.57E-08)	(3.18E-08)	(8.51E-07)
β_2		0.0000	0.8180	0.0400	0.0288
		(0.00E+00)	(2.40E-03)	(7.12E-05)	(5.12E-04)
θ				0.9468	0.9688
				(1.05E-04)	(5.86E-04)
γ_2		319.4	412.1	770.7	4016.6
		(4.29E-02)	(6.77E+00)	(9.93E-01)	(4.18E+01)
ω_2		1.242E-12	4.437E-10	1.661E-06	3.044E-07
ρ		0.5583	0.7726	0.5551	0.3737
		(4.56E-05)	(4.44E-03)	(2.59E-04)	(1.76E-03)
IVRMSE	3.904	4.330	3.833	3.349	3.142
IVRMSE Ratio	1.000	1.109	0.982	0.858	0.805
Volatility Persistence					
From Returns	0.9845		0.9653		0.9072
From RV		0.9279	0.8417	0.9868	0.9976

Notes: We estimate five models using option data for the period January 1, 1996 to December 31, 2004. Realized variance is constructed using sixty-minute intervals. Standard errors, computed using the outer product of the gradient, are indicated in parentheses. The parameters are estimated using variance targeting to 17.3% per year and equity premium targeting to 6% per year.

Table 6: In-Sample IVRMSE (%) by Moneyness, Maturity, and VIX Level. 5-min RV.

Panel A. IVRMSE by Moneyness

Model	F/X<0.96	0.96<F/X<0.98	0.98<F/X<1.02	1.02<F/X<1.04	1.04<F/X<1.06	F/X>1.06
HN	3.5535	3.3616	3.3777	3.3808	3.4910	5.1428
RV	3.7865	3.4900	3.7696	3.7673	3.7609	4.4579
GRV	3.1645	3.0431	3.2956	3.2435	3.2599	4.2427
ERV	3.3211	3.0465	3.3742	3.3761	3.4350	3.9184
GERV	2.8368	2.4940	2.8074	2.9409	3.0588	3.7125
Average	3.3325	3.0870	3.3249	3.3417	3.4011	4.2949

Panel B. IVRMSE by Maturity

Model	DTM<30	30<DTM<60	60<DTM<90	90<DTM<120	120<DTM<150	DTM>150
HN	4.1795	3.8601	3.8627	3.6176	3.9104	4.2266
RV	4.4570	3.9053	3.7060	3.7303	4.0441	4.1503
GRV	3.7423	3.4589	3.3603	3.2875	3.6783	3.7988
ERV	4.0070	3.5049	3.1656	3.2490	3.6632	3.7793
GERV	3.1908	3.0019	2.8365	2.9228	3.3623	3.4683
Average	3.9153	3.5462	3.3862	3.3615	3.7316	3.8847

Panel C. IVRMSE by VIX Level

Model	VIX<15	15<VIX<20	20<VIX<25	25<VIX<30	30<VIX<35	VIX>35
HN	5.0345	3.2704	3.2199	4.1722	5.1322	7.0269
RV	5.7170	3.3897	2.6832	4.3998	5.3163	7.2152
GRV	5.0589	3.0191	2.8606	3.7862	4.2374	5.8201
ERV	5.1403	3.1490	2.6417	3.9024	4.4791	5.1772
GERV	3.7289	2.3087	2.6497	3.5782	4.1802	4.9599
Average	4.9359	3.0274	2.8110	3.9677	4.6690	6.0398

Notes: We report the IVRMSE from the models estimated in Table 4 by moneyness, maturity and VIX level.

Table 7: In-Sample IVRMSE (%) by Moneyness, Maturity, and VIX Level. 60-min RV.

Panel A. IVRMSE by Moneyness

Model	F/X<0.96	0.96<F/X<0.98	0.98<F/X<1.02	1.02<F/X<1.04	1.04<F/X<1.06	F/X>1.06
HN	3.5535	3.3616	3.3777	3.3808	3.4910	5.1428
RV	4.4823	3.8836	4.0973	4.0357	4.1669	4.9044
GRV	3.7162	3.3856	3.3831	3.3328	3.4570	4.8751
ERV	3.6827	2.9243	3.0709	3.1292	3.2274	3.8245
GERV	3.2923	2.6005	2.8427	2.9782	3.0550	3.7148
Average	3.7454	3.2311	3.3544	3.3713	3.4794	4.4923

Panel B. IVRMSE by Maturity

Model	DTM<30	30<DTM<60	60<DTM<90	90<DTM<120	120<DTM<150	DTM>150
HN	4.1795	3.8601	3.8627	3.6176	3.9104	4.2266
RV	4.4794	4.3855	4.2542	4.0137	4.2340	4.6496
GRV	4.1147	3.8049	3.7578	3.5166	3.8562	4.1860
ERV	3.7799	3.4128	3.0908	2.8987	3.4950	3.6625
GERV	3.5391	3.1232	2.8915	2.7881	3.3850	3.5150
Average	4.0185	3.7173	3.5714	3.3670	3.7761	4.0480

Panel C. IVRMSE by VIX Level

Model	VIX<15	15<VIX<20	20<VIX<25	25<VIX<30	30<VIX<35	VIX>35
HN	5.0345	3.2704	3.2199	4.1722	5.1322	7.0269
RV	5.6126	3.3383	2.7833	5.0048	6.3478	9.5626
GRV	4.9147	3.2673	3.1334	4.1924	4.9704	6.7159
ERV	4.1949	2.4487	2.7519	3.9232	5.1347	5.2613
GERV	3.7850	2.3770	2.7821	3.5022	4.4529	4.9862
Average	4.7083	2.9403	2.9341	4.1590	5.2076	6.7106

Notes: Notes: We report the IVRMSE from the models estimated in Table 5 by moneyness, maturity and VIX level.

Table 8: In-Sample Bias by Moneyness, Maturity, and VIX Level. 5-min RV.

Panel A. Bias by Moneyness

Model	F/X<0.96	0.96<F/X<0.98	0.98<F/X<1.02	1.02<F/X<1.04	1.04<F/X<1.06	F/X>1.06
HN	-0.4799	-1.3771	-1.0445	-0.0463	0.4182	2.4985
RV	0.3188	-0.8125	-1.1486	-0.9283	-0.7526	0.6220
GRV	-0.3247	-1.3550	-1.4766	-0.9241	-0.5639	1.1835
ERV	0.1096	-1.0074	-1.2768	-0.9493	-0.8336	0.3303
GERV	0.4535	-0.2700	-0.4368	-0.1436	-0.0350	0.8197
Average	0.0155	-0.9644	-1.0767	-0.5983	-0.3534	1.0908

Panel B. Bias by Maturity

Model	DTM<30	30<DTM<60	60<DTM<90	90<DTM<120	120<DTM<150	DTM>150
HN	-0.4647	-0.0857	0.2423	0.4478	0.2106	0.3623
RV	-1.8552	-0.8832	-0.2396	0.1879	0.0602	0.0702
GRV	-1.6218	-0.9191	-0.3712	0.0661	-0.1580	-0.0012
ERV	-2.2923	-1.2610	-0.3363	0.1283	0.0365	0.3523
GERV	-1.4929	-0.6126	0.2920	0.8149	0.7533	1.1380
Average	-1.5454	-0.7523	-0.0826	0.3290	0.1805	0.3843

Panel C. Bias by VIX Level

Model	VIX<15	15<VIX<20	20<VIX<25	25<VIX<30	30<VIX<35	VIX>35
HN	-4.6098	-1.9193	0.3539	1.9827	3.4822	5.7049
RV	-5.5034	-2.6994	0.0244	1.3920	3.1558	4.1616
GRV	-4.7651	-2.1949	-0.2413	0.9255	2.2825	3.7110
ERV	-4.9424	-2.4668	-0.0150	0.9571	1.7793	2.3446
GERV	-3.3138	-1.2933	0.3537	1.2448	2.2973	2.9464
Average	-4.6269	-2.1147	0.0951	1.3004	2.5994	3.7737

Notes: We report the IV Bias from the models estimated in Table 4 by moneyness, maturity and VIX level. Bias is defined as average market IV less average model IV.

Table 9: In-Sample Bias by Moneyness, Maturity, and VIX Level. 60-min RV.

Panel A. Bias by Moneyness

Model	F/X<0.96	0.96<F/X<0.98	0.98<F/X<1.02	1.02<F/X<1.04	1.04<F/X<1.06	F/X>1.06
HN	-0.4799	-1.3771	-1.0445	-0.0463	0.4182	2.4985
RV	0.5372	-0.4968	-0.6857	-0.4394	-0.2633	1.3417
GRV	-0.3359	-1.3343	-1.1322	-0.2798	0.1306	2.0618
ERV	0.1680	-0.8091	-0.9497	-0.6486	-0.5265	0.6183
GERV	0.3325	-0.3957	-0.6828	-0.4253	-0.3347	0.5721
Average	0.0444	-0.8826	-0.8990	-0.3679	-0.1151	1.4185

Panel B. Bias by Maturity

Model	DTM<30	30<DTM<60	60<DTM<90	90<DTM<120	120<DTM<150	DTM>150
HN	-0.4647	-0.0857	0.2423	0.4478	0.2106	0.3623
RV	-0.6076	-0.2336	0.2017	0.4600	0.2251	0.2613
GRV	-0.6825	-0.2829	0.0515	0.3510	0.1116	0.1931
ERV	-1.6456	-0.9715	-0.1420	0.3949	0.1072	0.7457
GERV	-1.6520	-0.8284	-0.0379	0.5019	0.4414	1.2506
Average	-1.0105	-0.4804	0.0631	0.4311	0.2192	0.5626

Panel C. Bias by VIX Level

Model	VIX<15	15<VIX<20	20<VIX<25	25<VIX<30	30<VIX<35	VIX>35
HN	-4.6098	-1.9193	0.3539	1.9827	3.4822	5.7049
RV	-5.4108	-2.6842	0.2755	2.2922	4.8359	7.1191
GRV	-4.5088	-1.9810	0.2604	1.5805	3.1278	4.8075
ERV	-3.9100	-1.7217	-0.0361	0.7733	1.9283	2.9970
GERV	-3.2249	-1.3428	0.0683	0.6705	1.9341	3.1636
Average	-4.3329	-1.9298	0.1844	1.4598	3.0617	4.7584

Notes: We report the IV Bias from the models estimated in Table 5 by moneyness, maturity and VIX level. Bias is defined as average market IV less average model IV.