Hidden Orders and
Optimal Submission Strategies
in a Dynamic Limit Order Market

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Abstract

Recent empirical evidence on traders' order submission strategies in electronic limit order markets (LOB) shows the growing use of hidden orders. This paper provides a theory of the optimal order submission strategies in an LOB, where traders can choose among limit, market and hidden orders. The dynamic model we propose allows traders to take the simultaneous three-dimensional strategic choice of price, quantity and exposure. Traders use hidden orders both to compete for the provision of liquidity and to avoid picking-off risk. The use of hidden orders increases with order size and relative depth on the opposite side of the market, while it decreases with time-to-shock. Hidden orders increase market depth but could reduce competition in the supply of liquidity, thus making the inside spread wider; hence they could be beneficial to institutional investors, but detrimental to retailers. Regulators should also be aware that hidden orders have stabilizing effects when the market is under stress.

1 Introduction

Electronic limit order markets have become the dominant market structure for trading financial securities around the world. These are order-driven markets in which traders can either supply liquidity by submitting limit orders or demand liquidity by submitting market orders. Orders posted to the limit order book (LOB) must specify a number of instructions which qualify their sign, size, and eventually their price aggressiveness and degree of disclosure.

Recent empirical evidence about traders' order submission strategies on electronic limit order books shows the growing importance of a special type of order, called "hidden" or "iceberg", that allows traders to limit their exposure by hiding a portion of the quantity they are willing to trade. Hidden orders amount to a striking proportion of trading volume: for example, they correspond to more than 44% of Euronext volume, about 28% of the Australian Stock Exchange volume, and account for more than 15% of total executions on INET and for 16% of executed shares on Xetra. Hidden orders are also extensively used at the NASDAQ and in secondary markets for treasury bonds.[1] Like limit orders, hidden orders contain an instruction about the limit price beyond which submitters are willing to trade; however, unlike limit orders, they contain a further instruction about the fraction of the order that is undisclosed to the market.

By allowing the use of hidden orders, regulators endogenously reduce the degree of pre-trade transparency and hence impact on both liquidity and price informativeness. It is therefore relevant for them to understand what are the effects on market quality of the widespread adoption of such trading facility. Also for practitioners understanding under which circumstances hidden orders are optimal order submission strategies is a crucial issue.

Despite a growing body of empirical research on hidden orders, there is little theoretical guidance on the optimal choice of order exposure. The objective of this paper is indeed to extend the existing literature on dynamic limit order markets with a theory of the optimal order submission strategies, where, beside the standard choice between limit and market orders, traders can also choose their order exposure. Traders submitting limit orders bear exposure costs that can arise from both the risk of being undercut by

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[1]See respectively Bessembinder, Panayides and Venkataraman (2008), Aitken, Berkman and Mak (2001), Hasbrouck and Saar (2004 and 2007), Frey and Sandas (2008), Tuttle (2006) and Jiang, Lo and Verdelhan (2007).

other traders competing for the provision of liquidity, and the risk of being picked-off by fast traders after an asset value shock. To capture these effects the model internalizes three elements: the strategic dynamic interaction of traders with the two sides of the LOB, asset volatility and different order sizes. Hence the model embeds both retail and institutional traders in a framework that draws from Parlour (1998) the interaction between traders' strategies and the two sides of the LOB, and from Foucault (1999) the impact of picking-off costs on traders' order placement strategies.[2]

As recent empirical evidence shows that hidden orders are predominantly used by uninformed traders,[3] in this framework hidden orders are submitted by agents who do not hold inside information, but only have a private evaluation of the asset that determines their degree of impatience; they arrive sequentially at the market and choose their optimal submission strategy contingent on the state of the LOB. A variegated spectrum of trading strategies is available to them: in addition to market and limit orders, they can opt for hidden orders, as well as marketable orders. Furthermore, they can choose their degree of price aggressiveness and hence face a simultaneous three-dimensional strategic choice of price, quantity and exposure.

The model is solved under different specifications: an initial framework is used to model competition for the provision of liquidity, and is subsequently extended to include fast traders and hence picking-off risk. The results obtained are then compared with those of a benchmark model without hidden orders. This comparison allows us to discuss the determinants of hidden orders, as well as the effects of their use on market quality.

The model shows that agents use hidden orders both to prevent incoming traders from undercutting their orders, and to reduce the losses that can arise when scalpers pick-off their orders at stale prices. In equilibrium, if traders compete for liquidity provision, they maximixe the visible part of their orders, that still prevents undercutting; if instead they seek protection from fast traders, they minimize it. Furthermore, the use of hidden orders

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[2]Goettler, Parlour and Rajan (2005, 2008) focus as well on the working of an LOB and extend Parlour's framework to model limit order trading as a stochastic sequential game with private and common value; they also introduce endogenous information acquisition. To examine the resiliency and spread dynamic of the LOB, Foucault, Kadan and Kandel (2005) include traders' waiting costs, and Rosu (2008a) considers a continuous time model with endogenous undercutting.

[3]See for example Aitken et al. (2001), Bessembinder et al. (2008) and De Winne and D'Hondt (2007).

to prevent picking-off increases with order size, and relative depth on the opposite side of the market, while it decreases with time-to-shock.

According to the model's results, hidden orders can enhance depth especially when the market is under stress; however, they can also reduce competition in the supply of liquidity, thus widening the inside spread. These results suggest that the use of hidden orders can be beneficial to institutional investors, but it could be detrimental to retailers. They also show that hidden orders can add to the stability of the market when it is under stress.

The results obtained also allow us to investigate a new component of the spread that arises from exposure costs. When traders run the risk of being exposed to a war of prices, they submit hidden orders to prevent undercutting and therefore they widen the inside spread by inducing the incoming traders to join the queue at prices away from the best bid-offer. Upon the arrival of information shocks, on the other hand, fast traders can pick-off stale prices; hence limit order submitters, who are not quick enough to cancel or update their orders, run the risk of being exposed to adverse selection costs. Clearly, the higher the expected volatility, the greater the exposure costs and the larger the spread, as traders react by submitting less aggressive limit orders. This could explain why the empirical evidence shows that spreads are larger around public announcements both in equity and in bond markets.[4]

The remainder of the paper unfolds as follows. In the next Section the related literature is discussed on hidden orders, in Section 3 the model's structure is introduced and in Section 4 the initial model with competition for liquidity provision is presented. Section 5 introduces fast traders, Section 6 discusses the effects of hidden orders on market quality, and Section 7 concludes. All the proofs are gathered in the Appendix.

2   Literature on Hidden Orders

The existing literature on hidden orders is primarily empirical with a few theoretical works. The empirical results show a large consensus that hidden orders are mainly used by uninformed traders. Aitken et al. (2001) show that in the Australian stock market there is no difference in the stock price reaction between disclosed and undisclosed limit orders, and conclude that there is no evidence that undisclosed limit orders are more frequently used by

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[4]See, for example, Pronk (2006) for equity markets, Fleming and Remolona (1999) and Jiang et al. (2007) for bond markets.

informed traders than disclosed limit orders. Similarly, Bessembinder et al. (2008) find that hidden orders bear smaller implementation shortfall costs[5] and stress that they are primarily used by uninformed traders to mitigate the option value of standing limit orders. Pardo and Pascual (2006) also conclude that the market does not attribute any unknown information content to the hidden side of liquidity, as they find that in the Spanish stock exchange hidden volume detection has no relevant impact on returns and volatility. Furthermore, De Winne and D'Hondt (2007) show that traders become significantly more aggressive when there is a signal for hidden depth at the best quotes on the opposite side of the market. They also show that traders tend to hide more when the size of their order is large relative to the prevailing displayed depth, and they conclude that traders use hidden quantity to manage both exposure and picking-off risk. Finally, Frey and Sandas (2008) show that traders bid very aggressively when they suspect hidden liquidity on the opposite side of the market, whereas they become less aggressive when competing with hidden liquidity on the same side. All these results suggest that hidden orders are used to obscure the trading strategies of large uninformed investors. More precisely, regarding their motivations behind the use of hidden orders, Bessembinder et al. (2008) and Harris (1996, 1997) show that traders are more likely to hide their orders when competition in the market is intense (i.e. the tick size is small and the trade size is large), while Aitken et al. (2001) suggest that traders use hidden orders to control the option value associated with their limit orders as they find that hidden orders' use increases with volatility.[6]

The theoretical works on the use of hidden orders are few. To our knowledge, only two models explicitly include hidden orders among the order types available to market participants. Moinas (2007) proposes a sequential signaling game where hidden orders are used by one insider to trade large volume without disseminating his private information. Yet the model does not allow uninformed traders to use hidden orders, nor informed traders to demand liq-

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[5]These costs are the weighted sum of the price impact (the appropriately signed difference between the fill price and the quote mid-point at the time of order submission) and the opportunity cost (smaller price drifts subsequent to order submission time), where the weights are the proportion of the order size that is filled and unfilled, respectively.

[6]As described in Harris (2003), large uninformed orders run the risk of being picked-off by scalpers and quote-matchers, also called parasitic traders, whose profits are increasing function of price volatility; quote matchers' profits are also decreasing in the minimum tick size.

uidity; moreover, it does not embody the interaction between the two sides of the LOB. Esser and Mönch (2007) extend the literature on optimal liquidation strategies (e.g. Bertsimas and Lo, 1998; Almgren and Chriss, 2000; Mönch, 2004) to include hidden orders: they determine the optimal limit price and pick size for an iceberg order in a static framework without any strategic interaction among traders.

3   The Framework

In this model uninformed traders use hidden orders for two reasons: the first is to compete for the provision of liquidity and thus prevent other traders from undercutting their orders; the second is to reduce the probability of being picked-off by fast traders in case of mispriced orders. We start by introducing the general features of the model which guide the choice of the optimal order submission strategy. We then investigate the two cases described above: we first model the role of hidden orders as a strategic choice aimed to prevent undercutting, and then move to a more general setting to also include hidden orders as a defensive strategy against scalpers.

3.1   The Market

A market for a risky asset takes place over a trading day that is divided into T periods: $t = 1, ..., T$. At period T, trading finishes and the risky asset pays v_T units. Two categories of risk-neutral agents are active in this market: large institutional traders, who trade j units, with $1 \leq j \leq 10$, and small retail traders, who trade α units, where, as it will be clarified later, α is the optimal undisclosed part of the hidden order. At each trading round nature chooses a large or a small trader with equal probability, and the incoming agent maximizes expected profits by choosing an optimal trading strategy that cannot be modified thereafter unless the order become mispriced. In this case the order will be cancelled. Each agent is characterized by his personal evaluation of the risky asset, β_t, that is drawn from the following uniform distribution:

$$\beta_t \sim U[\underline{\beta}, \overline{\beta}] \qquad \text{where} \quad 0 \leq \underline{\beta} \leq 1 \leq \overline{\beta} \qquad (1)$$

Notice that β_t can be interpreted as the trader's private evaluation of the

asset, or it can be considered a proxy of his degree of impatience:[7] traders with extreme β_t either value the asset very little, or very much, and they are respectively the most impatient sellers (low β_t), or the most impatient buyers (high β_t); traders holding a β_t next to 1 are instead the most patient. We also assume that the distribution of β_t is symmetric around $\beta = 1$.[8]

Upon arrival, each trader observes the LOB that is formed by a grid of six prices, three on the ask and three on the bid side of the market. It follows that the prices at which each trader can buy or sell are equal to $A_{1,2,3}$ (ask prices) and $B_{1,2,3}$ (bid prices), with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$; for simplicity, we assume that these prices are symmetric around the common value of the asset, v_t. More precisely, traders can demand liquidity over the whole price grid, whereas they can offer liquidity only at the first two levels of the book.

This is due to the fact that at A_3 and B_3 a trading crowd absorbs any amount of the risky asset demanded or offered by the incoming trader. As in Seppi (1997) and Parlour (1998), the trading crowd prevents traders from bidding prices that are too far from the inside spread, and is only a theoretical shortcut to limit the price grid. It is also assumed that the difference between the ask and the bid price is equal to the tick size, τ, that is the minimum price increment that traders are allowed to quote over the existing price.[9]

The state of the book at each period t, $b_t = [A_2, A_1, B_1, B_2]$, is characterized by both the price grid and the number of shares available at each price. It is assumed that the asset value remains constant between $t = 1$ and $t = T - 1$, while between time $T - 1$ and T a shock on the asset value occours,[10] so that v_T can either increase, remain constant, or decrease:

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[7]Notice that, as β_t is not related to the future value of the asset, it cannot be interpreted as a measure of private information.

[8]Notice that, because the tick size is assumed constant, when the common value of the asset changes due to the shock, the value of the tick size relative to the asset price changes as well. This effect slightly modifies market orders execution probability at T, and hence makes the optimal trading strategies at $t \neq T$ not perfectly symmetric around the asset value. The degree of asymmetry is however negligible.

[9]As we allow traders to submit orders on a price grid, the spread is endogenously determined and ranges from a minimum of 1 tick to a maximum of 5 ticks. Even if prices cannot take a continuous range of values as in Foucault (1999), they are still endogenous within the price grid considered.

[10]This assumption greatly simplifies the algebra and allows us to focus only on the last periods of the game. We could include an asset value shock at each trading round, but this would multiply the possible trading strategies and make the computations substantially

$$v_t = V + \varepsilon_T \quad t = T \qquad (2)$$
$$v_t = V \qquad \forall t = 1, ..., T-1$$

with:

$$\varepsilon_T = \begin{cases} +k\ \tau & with\ prob = x \\ 0 & with\ prob = (1-2x) \\ -k\ \tau & with\ prob = x \end{cases} \qquad (3)$$

where $V > 0$ is constant and assumed for simplicity equal to one. Notice that k measures the size of the asset value shock, whereas $2x$ is the probability that the shock will occur. By changing the values of these two parameters one can investigate different volatility specifications.

3.2 Order Submission Strategies

By simultaneously choosing the *sign*, the *size*, the *aggressiveness* and the *degree of exposure* of his order, a trader decides his optimal order submission strategy. The following factors drive his choice: the costs associated with each strategy, the trader's type, and the state of the book.

Trading costs Three are the costs that a trader faces when choosing an order submission strategy: execution costs, price opportunity costs and exposure costs. Execution costs are the waiting costs that traders pay whenever their orders are not immediately executed; they are generally associated with limit orders that are stored on the book and inevitably have a slower execution probability than market orders. Execution costs are hence minimized by choosing market or marketable orders that guarantee immediate execution. However, as market orders are generally executed at the top of the book, they bear higher price opportunity costs. Finally, traders face exposure costs that arise from the risk of being undercut by incoming agents competing aggressively, and/or picked-off by fast traders in case of an asset value shock**.**

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longer.

Traders' type In this model traders differ by their degree of impatience. Very impatient traders weigh a lot execution costs, whereas patient traders assign more value to price opportunity costs. It follows that facing the trade-off between execution and price opportunity costs, very impatient traders will choose market or marketable orders, whereas patient traders will also consider limit orders. But when opting for limit orders, patient traders will also have to take into account the exposure costs that arise from both competition for liquidity provision and the presence of fast traders, and eventually use hidden orders.

State of the LOB Before submitting their order, traders also take into account the state of the LOB. Due to time priority, market depth affects execution costs, and hence influences traders' order choice: higher depth on the own side increases execution costs, while the opposite occurs when the book is deep on the other side of the market, as incoming traders will submit more market than limit orders.

Table 1 presents the possible orders that a small trader (Panel A) and a large trader (Panel B) can choose. The feasibility and profitability of these strategies depend on both their type (β) and the state of the LOB at the time of the order submission (b_t).

Large Trader An aggressive large trader (Panel B) can demand liquidity by submitting a market sell order of size j (MO_jB_i) that will hit the limit buy orders with the highest precedence on the bid side. Alternatively, a large trader may opt for a marketable sell order (MRO_jB), which walks down the LOB.[11] A more patient large trader can instead choose to submit a limit sell order of size j to either A_1 or A_2 ($LO_jA_{1,2}$). This order will be executed when one or more market buy order arrive which hit the limit price after that all the other orders on the LOB with lower price and higher time priority have been executed. Alternatively, a patient large trader can decide to submit a 10-share hidden sell order and choose a visible size of α units ($HO_{10}A_{1,2}$), with $0 < \alpha < j$. The undisclosed part of the hidden order looses time priority with respect to the incoming limit orders at the same limit price, but has the

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[11]Notice that in our setting marketable orders are defined as market orders that walk up or down the book. As a marketable order generally hits different prices, we do not use an index for the level of the book as we do for the other order types.

advantage of reducing undercutting and/or picking-off costs. Finally, a large trader can decide not to trade (NTL). Similar strategies apply to the other side of the LOB. In real financial markets traders could also split their limit orders either at different prices or in different periods of time at the same price. These strategies have not been considered here as they are dominated, and later in the paper this point will be clarified.

Small Trader   To avoid trivial detection of hidden orders, small traders' order size is set equal to the visible part of the hidden order, α.[12] Aggressive small traders (Panel A) will demand liquidity by submitting a market sell order ($MO_\alpha B_i$); whereas more patient traders will act as liquidity suppliers and submit a limit sell order either to the first ($LO_\alpha A_1$), or to the second level of the LOB ($LO_\alpha A_2$). Finally, it may happen that at time t the trader does not find any profitable strategy and decides not to trade (NTS).

[**Insert Table 1 here**]

3.3   Discussion

The main objective of this paper is to investigate the role of exposure costs in securities trading, and indeed the novelty of this work is that it shows how they can be reduced by using hidden orders. To this end, it is crucial to choose a framework where traders are allowed to submit orders of different size: without trades of at least two different sizes, the discovery of a hidden quantity would be straightforward and hence hidden orders would be always dominated by limit orders. For this reason, we have modeled the market as a trading game which finishes at T, and can be solved by backward induction. Yet, the existing models with stationary equilibrium are not suitable to embed this essential feature. As Rosu (2008a) suggests, his stationary Markov equilibrium would eventually allow to include multiple submission of 1-unit orders, but not block trading. Similarly, neither Foucault (1999) nor

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[12]To prevent other market participants from easily detecting hidden depth, the equilibrium disclosed quantity of a hidden order should also be traded by small traders. Clearly, due to risk neutrality, small traders will choose the maximum possible order size; hence, to be able to investigate the choice of the optimal visible size of a hidden order, a natural assumption is indeed to allow small agents to trade up to the large traders' desired visible quantity.

Foucault et al. (2005) would be adequate to model hidden orders. In the former, not only traders cannot submit orders of different size, but they cannot even compete for the provision of liquidity as the book is always empty or full: indeed, after submission, a limit order is either executed or cancelled in the following period. The crucial assumption necessary to find a stationary solution for the latter, instead, is that traders always price improve when submitting their 1-unit orders. This rules out the possibility for an incoming trader to join the queue, and therefore it eliminates by construction all the benefits that traders can obtain by using hidden orders to reduce competition.

Our choice of a finite-horizon model that is solved by backward induction allows us to find a closed-form solution for a market in which traders' strategies include orders of different size, hidden orders and freedom to choose between price improving and joining the queue. Moreover, in this framework traders not only condition their order choice to the current state of the LOB, but also take into account the effects of their orders on the dynamic of the book, as they influence their execution probability.

4   Competition for the Provision of Liquidity

We start by considering a market where hidden orders are only used by traders to compete for the provision of liquidity. The analysis is focused on the last three periods of the trading game: Figure 1 shows the price dynamic. We use the simplest possible framework, with an asset value shock that is certain ($x = \frac{1}{2}$) and that is the smallest possible one ($k = 1$). This means that at time T the asset value will go up or down by one tick with equal probability. The ask and bid prices after a positive (negative) price change are named A_i^u (A_i^d) and B_i^u (B_i^d) respectively, with $i \in \{1, 2, 3\}$.

[**Insert Figure 1 here**]

Figure 2 shows the extensive form of the game for the case with $\alpha = 3$. Notice that at time T the incoming agent is only able to submit market or marketable orders. This is due to the fact that the market closes at T and hence the execution probability of any limit order is zero. At time $T - 1$ and $T - 2$, on the other hand, traders have an incentive to choose limit orders as well.

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[**Insert Figure 2 here**]

We assume that the market opens at $T-2$ with an empty book,[13] $b_{T-2} = [0000]$, and from period $T-2$ onwards traders' orders gradually fill the LOB. Suppose for example that nature selects a large trader at $T-2$: if he chooses to submit $LO_{10}A_2$, then at $T-1$ the book will open as $b_{T-1} = [(10)000]$, and if the trader arriving at $T-1$ chooses to undercut this order by submitting $LO_{10}A_1$, the resulting set of strategies at T in case, for instance, of a positive asset value shock will be $MO_{10}B_3$, NTL and $MO_{10}A_1$ for a large traders, and MO_3B_3, MO_3A_1 and NTS for a small one. Notice however that when agents are allowed to submit hidden orders, the depth of the book can become uncertain. For example, if the large trader arriving at $T-2$ chooses to submit a hidden order ($HO_{10}A_2$), then the book will open at $T-1$ as $b_{T-1} = [(3+7)000]$, whereas if, still at $T-2$, nature chooses a small trader who selects LO_3A_2, the opening book will be $b_{T-1} = [3000]$. In both cases the LOB at $T-1$ will show three units on A_2 and the incoming trader will be uncertain on whether the book contains any undisclosed depth beyond the visible shares: he will then assign a probability to each possible state of the LOB.

4.1 The Trader's Problem

Assume a large trader wants to evaluate the pro and cons of submitting a hidden order with α shares disclosed and $10 - \alpha$ undisclosed at A_1 or A_2. Given that the difference between A_1 and B_1 is equal to the minimum tick size, orders on the top of the book are not exposed to price competition. Therefore a hidden order posted to A_1 presents no advantages compared to a limit order on the same level of the LOB: indeed it has a lower execution probability and hence it is a dominated strategy. A hidden order on A_2, on the other hand, presents advantages and disadvantages compared j-share limit orders on A_2 or on A_1, that are the other two alternatives available to patient traders. Compared to LO_jA_2, a hidden order has the advantage of possibly inducing the next trader not to submit a large order to A_1 but rather to join the queue at A_2: this would increase the execution probability of the disclosed size, but reduce the execution probability of the undisclosed one; compared to LO_jA_1 the hidden order gains the tick size, but has a lower execution probability.

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[13]Clearly, we start from an empty book to enforce competition in prices and quantities.

This example suggests that when traders strategically choose to submit a hidden order, or any other order, they compute the execution probabilities up to time T and then compare the expected profits associated to all the available orders, conditional on the state of the LOB and, of course, their degree of impatience.

Formally, the risk-neutral large trader will choose the optimal order submission strategy, o_{L,β_t,b_t}, that maximizes his expected profits conditional on the state of the LOB, b_t, and his degree of impatience, β_t. A large seller will hence submit the order that solves:[14]

$$\max_{o_{L,\beta_t,b_t} \in [MO_jB_i, MRO_jB, LO_jA_i, HO_{10}A_2, NTL]} E\big[\pi_t(o_{L,\beta_t,b_t})\big] \tag{4}$$

where

$$\pi_t(MO_jB_i) = j(B_i^{;z} - \beta_t\, v_t)$$

$$E[\pi_t(MRO_jB)] = f_1 B_1^{;z} + f_2 B_2^{;z} \Pr_{f_2}(B_2|b_t, v_t) + (j-f_1-f_2)B_3^{;z}\left[1 - \Pr_{f_2}(B_2|b_t, v_t)\right] - j\, \beta_t\, v_t$$

$$E[\pi_t(LO_jA_i)] =$$

$$= E\left[\sum_{l=t+1}^{T}(A_i - \beta_t \widetilde{v}_l) \begin{cases} \sum_{W=0}^{j-1}\sum_{w_l=\alpha}^{j-W} w_l \Pr_{w_l}(A_i|b_l, v_l)\Pr(\sum_{m=t+1}^{l-1} w_m = W|b_t, v_t) & if\ m \le l-1 \\ \sum_{w_l=\alpha}^{j} w_l \Pr_{w_l}(A_i|b_l, v_l) & if\ m > l-1 \end{cases}\right]$$

$$E[\pi_t(HO_{10}A_2)] =$$

$$= E\left[\sum_{l=t+1}^{T}(A_2 - \beta_t \widetilde{v}_l) \begin{cases} \sum_{W=0}^{9}\sum_{w_l=\alpha}^{10-W} w_l \Pr_{w_l}(A_2|b_l, v_l)\Pr(\sum_{m=t+1}^{l-1} w_m = W|b_t, v_t) & if\ m \le l-1 \\ \sum_{w_l=\alpha}^{10} w_l \Pr_{w_l}(A_2|b_l, v_l) & if\ m > l-1 \end{cases}\right]$$

$$\pi_t(NTL) = 0$$

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[14]The optimization program of a buyer is almost symmetric; hence, when possible, we will only discuss the sell side.

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where $i \in [1,3]$, $B_i^z \in \{B_i^u, B_i^d\}$, $j \in [1,10]$, f_i is the number of shares executed at B_i with $\sum_i f_i = j$; $\text{Pr}_{f_2}(B_2|b_t, v_t)$ is the probability that f_2 hidden shares are available at B_2 at time t, $\text{Pr}_{w_l}(A_i|b_l, v_l)$ is the probability that w_l shares are executed at $t = l$.

Notice that traders do not know the exact amount of liquidity available on the LOB. For this reason, when computing the execution probabilities of hidden and limit orders, they take into account that subsequent traders will face uncertainty on the execution price of marketable orders, as is evident from the profit of MRO_jB.

The small seller will solve a similar problem:

$$\max_{o_{S,\beta_t,b_t} \in [MO_\alpha B_i, LO_\alpha A_i, NTS]} E\big[\pi_t(o_{S,\beta_t,b_t})\big] \tag{5}$$

where

$$\pi_t(MO_\alpha B_i) = \alpha(B_i^z - \beta_t\, v_t)$$

$$E[\pi_t(LO_\alpha A_i)] = E\left[\sum_{l=t+1}^{T} \alpha(A_i - \beta_t \widetilde{v}_l) \Pr_{w_l=\alpha}(A_i|b_l, v_l) \left\{ \begin{array}{ll} \prod_{m=t+1}^{l-1} \Pr_{w_m=0}(A_i|b_m, v_m) & if\ m \le l-1 \\ 1 & if\ m > l-1 \end{array} \right. \right]$$

$$\pi_t(NTS) = 0$$

where $i \in [1,3]$, $B_i^z \in \{B_i^u, B_i^d\}$, $\text{Pr}_{w_t}(A_i|b_t, v_t)$ is the probability that w shares are executed at $t = l, m$.

To determine the optimal visible size of the hidden orders, α^*, these optimization problems have been solved for all the possible values of $\alpha \in [1, 9]$.

Equilibrium definition *An equilibrium of the trading game with competition for the provision of liquidity is a set of orders o^*_{L,β_t,b_t} and o^*_{S,β_t,b_t} with $t = \{T, T-1, T-2\}$ and an optimal visible size of the hidden order, α^*, that solve Program (4) and (5), when the expected execution probabilities, $\text{Pr}_w(A_i|b_{T-l}, v_{T-l})$, and the probabilities that traders assign to the different states of the book, $\text{Pr}_{f_i}(B_i|b_{T-l}, v_{T-l})$, for $l = \{0, 1\}$, are computed assuming that traders submit the orders o^*_{L,β_t,b_t} and o^*_{S,β_t,b_t}.*

We solve the model by backward induction, assuming that the tick size is equal to $\tau = 0.1$.

4.2 Equilibrium Order Submission Strategies

We find the solution of this game by backward induction starting from the end-nodes at time T and computing the probabilities of market and marketable orders: these are the execution probabilities of limit orders placed at $T-1$ that allow us to compute the equilibrium order submission strategies at $T-1$. Given the probability of market and marketable orders at $T-1$, we can then compute the equilibrium order submission strategies at $T-2$. Finally, by solving the game for the possible values of α, we can determine the optimal size of the visible part of hidden orders.

4.2.1 Equilibrium Strategies at T

At time T small traders face no uncertainty on the execution price of market orders: as the size of their order is equal to the disclosed part of hidden orders, they are only concerned by the visible depth. It follows that a small trader will submit a market sell order if the asset price is higher than his valuation ($B_i^z \geq \beta_T v_T^z$, i.e. $\beta_T \leq B_i^z/v_T^z$, where $z = \{u, d\}$), he will submit a market buy order in the opposite case ($\beta_T v_T^z \geq A_i^z$, i.e. $\beta_T \geq A_i^z/v_T^z$) and he will not trade for intermediate values of β_T.

Differently from small traders, when computing their optimal order submission strategies, large traders have to take into account the probability of hidden depth: this means that they have to compute their execution prices as the weighted averages of all the possible execution prices given a certain visible LOB. Clearly, if there are j shares available either on the first or on the second level of the book, or if there is no depth on both levels and agents are forced to trade against the trading crowd, large traders face no uncertainty and their β_T thresholds are the same as those of retail traders, even if they will be trading j shares rather than α. However, if $f_i < j$ shares are visible on A_i and $n \geq j - f_i$ shares are available on $A_l > A_i$, large traders have the option to submit a market order of size f_i at price A_i or a marketable order of size j, whose execution price is uncertain for $j - f_i$ shares. The large trader's β_T thresholds for the ask side become the following:

$$\text{submit } MRO_j A^z \quad \text{if} \quad \beta_T \geq \frac{\Lambda_m^z}{v_T^z}$$

$$\text{submit } MO_{f_i} A_i^z \quad \text{if} \quad \frac{A_i^z}{v_T^z} \leq \beta_T < \frac{\Lambda_m^z}{v_T^z}$$

$$\text{no trade if} \quad 1 < \beta_T < \frac{A_i^z}{v_T^z}$$

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where $\Lambda_m^z = \sum \Pr_{j-f_i}(A_m^z|b_T, b_{T-1})A_m^z$, with $m^z = \{i, l\}$, is a weighted average of the possible prices and $\Pr_{j-f_i}(A_m^z|b_T, b_{T-1})$ are the probabilities that the remaining $j - f_i$ shares are executed at price A_m^z. These weights depend on the past traders' strategies at $T - 1$ and $T - 2$. For example, if a large trader arrives at T and observes $b_{T-1} = [3000]$ and $b_T = [3003]$, then in case of a positive shock the value of Λ_m^u is:

$$
\Lambda_m^u = \frac{A_1^u \Pr_{T-2}(HOS_{10}A_2 \mid b_T, b_{T-1}) + A_3^u \Pr_{T-2}(LO_3 A_2 \mid b_T, b_{T-1})}{\Pr_{T-2}(HOS_{10}A_2 \mid b_T, b_{T-1}) + \Pr_{T-2}(LO_3 A_2 \mid b_T, b_{T-1})}
$$

Once computed the ranges for β_T at T, to obtain numerical values for these thresholds it is necessary to choose a support for the probability distribution of β. We assume that β is uniformly distributed with support $[0, 2]$. Clearly, the β intervals and the execution probabilities are conditional on both the state of the LOB at time T, b_T, and the realization of v_T. As shown in the Appendix, it is straightforward to derive the orders' sybmission strategies from the thresholds obtained above.

4.2.2 Equilibrium Strategies at T-1 and at T-2

To solve the model for the equilibrium strategies at $T - 1$, we compare the profits from all the possible strategies. From this comparison we obtain both the β_{T-1} ranges and the probability that each possible order type is chosen, as well as the execution probabilities of the orders posted at $T - 2$.[15] The following Proposition summarizes the model's results.

Proposition 1 *Hidden orders are equilibrium strategies at T-2:*

- *they are posted to prevent undercutting by traders arriving at T-1*

- *traders choose the maximum disclosed size that prevents undercutting*

- *traders place hidden orders away from the spread midpoint*

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[15]For analytical convenience, we solve the model by considering only the sellers' strategies at $T - 2$. Due to the symmetry of the model, specular equilibrium strategies are obtained for buyers.

Hidden orders are indeed an optimal submission strategy for those traders arriving at the market at time $T-2$: Table 2 shows that in this period large traders submit hidden orders with probability .258. Clearly, as at time T there would be no undercutting (agents only submit market or marketable orders), hidden orders at $T-1$ are not equilibrium strategies: they loose time priority compared to limit orders without providing any advantage.

Market participants use hidden orders to prevent the incoming traders from undercutting their orders. Table 2 shows that when a 10-unit limit order is submitted to A_2, the next trader will undercut this order with probability .130 if he is a large trader ($LO_{10}A_1$) and with probability .150 if he is a retailer (LO_3A_1). When on the other hand a 10-unit hidden order is posted to the same price, large incoming traders will join the queue at A_2 with probability .136 rather than submitting their orders to A_1, and small traders will mostly join the queue rather than undercut (probabilities are .123 and .031 respectively).

When opting for a hidden order, traders have to choose the optimal proportion of the disclosed to the undisclosed part of their order. On the one hand, they would prefer to set visible the largest possible part of their order, as this would increase the execution probability; on the other hand, by increasing the visible size, they will also increase the incentive for the incoming trader to undercut and post his order to A_1. The results show that the optimal proportion of visible vs invisible size is 3 to 7 shares.

The model also suggests that traders tend to place their hidden orders away from the fundamental value of the asset that is equal to the spread midpoint. This means that when the spread is so tight that there is no room for price improvement, traders do not submit hidden orders, but rather post disclosed limit orders: it follows that they use price aggressiveness and exposure as complements. To test this prediction, price aggressiveness should be proxied by the distance between the price of the order and the spread midpoint. Furthermore, because wider spreads offer traders more opportunity to compete on prices, our results confirm a positive correlation between the use of hidden orders and the size of the spread as clearly shown in Bessembinder et al. (2008).

[**Insert Table 2 here**]

Notice finally that, due to time priority, splitting orders on the same level of the book would always be dominated by hidden orders, as the hidden part

of the order is automatically disclosed upon execution. Splitting different proportions of the order on two levels of the book would never be optimal too as it induces competitors to join the queue at the most aggressive price.

5 Hidden Orders and Picking-off Risk

In this section the model is extended to include another reason for uninformed traders to submit hidden orders: protection from picking-off by fast traders. This may happen in case of a mispriced order following an asset value shock.

In this framework it becomes relevant to consider limit orders as free options offered to market participants, and to focus on the associated exposure costs. Their option value depends on the standard factors that affect option premia: the time to maturity (i.e. the time to execution), the underlying asset volatility, and the limit price.

5.1 Extended Framework

The model used so far is modified as follows: a third category of traders, named scalpers, is introduced as well as a new distribution of the asset value shock.

5.1.1 Scalpers

In real markets scalpers are agents who trade on their own account and usually do not hold a position for more than a few minutes (Harris, 2003). Scalpers' main profits are due to gone-off prices which they quickly hound down from the book.

In this model scalpers are arbitrageurs and hence they are only interested in exploiting the free-option offered by limit order submitters when an asset value shock hits the market. What distinguishes scalpers from other traders is their speed of reaction: they are much faster than all the other market participants and hence, after the asset value shock, they can pick-off visible stale prices before limit order traders can cancel them. We assume that scalpers submit marketable limit orders with a limit price equal to the highest stale price, and with size equal to the visible mispriced quantity. Indeed if they submitted orders of larger size, they would run the risk, and therefore face the costs, of taking a position on the LOB, as the unexecuted quantity

would turn into a limit order. Hence we assume that scalpers can pick-off large mispriced limit orders, but they cannot generally hound the invisible part of mispriced hidden orders.[16] This implies that a large trader can reduce his losses by submitting hidden orders: he will loose time priority on the hidden part of his order, but he will also reduce the costs of being picked-off by scalpers by cancelling the undisclosed depth.

5.1.2 Asset Value Shock

The variance of the asset value is increased to $k = 2$. Indeed, to discuss the role of hidden orders as defensive strategy against scalpers, we need to allow orders to be eventually mispriced on both the first and the second level of the LOB. Otherwise, with a small asset value shock ($k = 1$), orders submitted to the second level would never be mispriced and therefore they wouldn't bear any exposure cost.

We also assume that v_T, the asset value at T, can either remain constant, increase, or decrease with equal probability ($x = \frac{1}{3}$). In this case the asset value shock has to be uncertain: when traders know that the shock occurs with probability 1, they loose all incentives to submit limit orders due to the certain losses in case of mispricing.

Figure 3 shows the evolution of the price grid over time for the new asset value shock. We denote the ask and the bid prices after a positive (or a negative) price change as A_i^U (A_i^D) and B_i^U (B_i^D) respectively, with $i \in \{1, 2, 3\}$.

[**Insert Figure 3 here**]

5.2 Equilibrium Order Submission Strategies

The extensive form of the game for the case with $\alpha = 1$ is shown in Figure 4. Notice that as before different strategies may imply the same visible LOB for traders arriving the next period. For example, if at $T - 2$ nature selects either a small trader who chooses a LO_1A_2 or a large trader who chooses a

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[16]We could technically relax this hypothesis and allow scalpers to partially hit hidden mispriced liquidity. However, conversations with practitioners informed us that scalpers are cautious when searching hidden liquidity and hence they would not be able to pick-off the whole hidden part as if it were visible.

$HOS_{10}A_2$, the opening visible book at $T - 1$ will be $b_{T-1} = [1000]$, and the incoming trader will assign a different probability to the two possible states: $b_{T-1} = [1000]$ and $b_{T-1} = [(1 + 9)000]$.

Notice that in this extended framework, traders could also reduce exposure costs by submitting hidden orders at time $T - 1$. Due to this additional source of uncertainty, we assume that traders arriving at T rationally compute the probability of hidden depth for orders submitted at $T - 1$; however, they hold adaptive expectations for orders submitted at $T - 2$ and hence assume that the probability of hidden liquidity is the same as that at $T - 1$.

[**Insert Figure 4 here**]

As in the previous case, to optimally choose their trading strategy, agents compare the expected profits associated with all the feasible sell orders (Table 1) and solve equations (4) and (5), where here $B_i^z \in \{B_i, B_i^U, B_i^D\}$.

In this framework small traders' strategies are still not influenced by the undisclosed depth and are derived as in the previous framework. As far as the large traders' strategies are concerned, however, there is a difference at time T as now traders have to take into account the effects that the hidden orders, submitted not only at $T - 2$, but also at $T - 1$, may have on the state of the book. Large traders' optimal submission strategies at time $T - 1$ and $T - 2$ are obtained as in the previous case.

The following Proposition summarizes the results.

Proposition 2 *Hidden orders are equilibrium strategies both at $T - 1$ and at $T - 2$, and are only submitted to the second level of the book. To prevent picking-off, traders choose the minimum disclosed size for their hidden orders. The following factors affect the use of hidden orders:*

- *the relative use of hidden compared to limit orders increases with the depth at the opposite side of the LOB*

- *the proportion of hidden orders decreases with time-to-shock*

- *the proportion of hidden orders increases with order size*

- *order exposure and price aggressiveness are complements*

Hidden orders are equilibrium strategies for traders both at $T-1$ and at $T-2$, and are submitted to the second level of the book. When opting for a hidden order, traders prefer to hide the largest possible amount ($\alpha^* = 1$), in order to increase protection from scalpers. Notice that also in this extended version of the model (Table 3), when traders choose hidden orders, they reduce competition from incoming limit order submitters.[17]

Finally, even in this framework, splitting orders is a dominated strategy: indeed when traders split their orders on the two levels of the book, they face higher exposure costs, whereas when they split their orders on the same level of the book, they loose on time priority compared to hidden orders.

[**Insert Table 3 here**]

Market Depth As Table 3 (Panel A) shows, being hidden orders just special types of limit orders, the larger the depth on the own side, the lower the execution probability and the fewer the hidden orders submitted to the LOB. This is evident if one compares the probabilities of hidden orders for the two states of the LOB, $b_{T-1} = [0000]$ and $b_{T-1} = [(1+9)000]$, and it is consistent with the results obtained by Bessembinder et al. (2008).

Notice also that hidden orders are used more intensively when the book is full or partially full on the opposite side. Table 4 shows that the relative use of hidden orders increases from .155 to .474 moving from $b_{T-1} = [0000]$ to $b_{T-1} = [(10)000]$. Indeed, larger depth on the sell side increases market buy orders' profitability at the expense of limit orders, but does not affect hidden buy orders since the latter are submitted by very patient traders. Moreover, hidden buy orders become also attractive for those traders who are very patient and, if the book were empty, would be on the sell side of the market. This result is consistent with the empirical evidence reported by both Pardo and Pascual (2006) and De Winne and D'Hondt (2007).

[**Insert Table 4 here**]

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[17]Indeed comparing the two states of the LOB at $T-1$, with $1+9$ and 10 shares posted to A_2 respectively, one can observe that when at $T-2$ a trader submits a hidden order, the next trader will join the queue at A_2, whereas when he submits a $LO_{10}A_2$, the incoming agent will undercut by submitting a limit order to A_1.

Time-to-Shock and Volatility We find that, all else equal, hidden orders are used more intensively at $T-1$ than at $T-2$ (Table 3, Panel A). When agents perceive less urgent the need to prevent their orders from being picked-off, they submit hidden orders with a lower probability. Therefore, the shorter the time to the asset value shock, the higher the probability of hidden orders. This result explains the recent empirical evidence from the US bond market: Jiang et al. (2007) show that the relative use of hidden orders significantly increases right before macroeconomic news announcements.

Notice that in this model volatility and time to shock are perfectly negatively correlated, hence we can also interpret the increase in the use of hidden orders at time T-2 as a reaction to the higher volatility that traders submitting hidden orders perceive. Furthermore, as the existing empirical evidence shows that the spread usually widens with volatility, on the empirical ground we expect that hidden orders are used more extensively when the spread is wider. This prediction is confirmed by Bessembinder et al. (2008).

Order Size Recent empirical evidence shows that the use of hidden orders increases with order size (Aitken et al., 2001; Bessembinder et al. (2008)). To analyze this effect, we compare the results obtained for $j \leq 10$ with those derived by assuming that, all else equal, large traders submit orders of smaller size ($j \leq 2$). Table 4 reports the proportion of hidden orders over the total limit orders submitted to A_2 for different states of the LOB. The results show that hidden orders are always used more intensively when the maximum order size is ten shares. For example, when the book is $b_{T-1} = [0000]$ this ratio is .081 for $j \leq 10$, and .075 for $j \leq 2$. The intuition here is straightforward: the larger the order size, the higher the exposure costs and hence traders' incentive to use hidden orders.

Order Exposure and Price Aggressiveness The model shows that when the book is empty at $T-1$, traders submit both limit and hidden orders only to A_2 (Table 3, Panel A); traders do not submit hidden orders to the first level of the book as the higher execution probability is more than compensated by the higher exposure costs and the lower sell price. When instead the book is full at A_2 ($b_{T-1} = [(10)000]$) and traders are forced to submit more aggressive orders to A_1, they submit limit rather than hidden orders. This suggests that, also in this framework, hidden orders are usually

submitted at less aggressive prices.[18] This result is consistent with the empirical evidence from De Winne and D'Hondt (2007), who find that hidden orders are generally posted less aggressively than limit orders.

6   Market Quality

In light of the growing use of hidden orders in electronic trading platforms, it becomes relevant from a regulatory viewpoint to investigate whether market participants benefit from hidden orders, and whether such orders improve market quality. To assess the effects of hidden orders on market quality, we solve again both the initial and the extended version of the model where, all else equal, traders are not allowed to use hidden orders (benchmark case). We measure the changes in market quality by comparing both the expected depth and the inside spread (effective and weighted) across the models and the benchmark. The results obtained are reported in Table 5 and summarized in the following Proposition.

Proposition 3 *The use of hidden orders affects market quality:*

- *when large traders use hidden orders to compete for the provision of liquidity, depth increases, but the inside spread widens*

- *when large traders use hidden orders to prevent picking-off, their participation to the market increases, depth is enhanced, and the increase in the expected inside spread is negligible*

Table 5 shows that with hidden orders the depth on the top of the book increases.

When traders compete for the provision of liquidity, the presence of hidden orders implies that depth concentrates at a single price and therefore it is greater at the BBO. For instance, in the benchmark model traders arriving at $T - 1$ undercut the existing orders by choosing $LO_{10}A_1$ when the book is

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[18]This is due to the fact that when there is a large amount of shares visible at A_2, hidden orders submitted to A_1 bear the same exposure costs as limit orders, but have lower execution probability. In fact, when quote-matchers observe a large mispriced order on A_2, even if only one share is visible at A_1, they submit a marketable order for the whole visible mispriced quantity: their order will walk up the book and eventually hit the undisclosed quantity at A_1.

full at A_2, whereas they join the queue at A_2 when the book opens at $T-1$ with only 3 visible shares.

When hidden orders are used to prevent picking-off, the probability that large traders refrain from trading decreases (Table 3, Panel A): for example, when the book is full or partially full at A_2, no-trading is no longer an equilibrium strategy at $T-1$. Due this effect depth at the BBO increases.

[**Insert Figure 5 here**]

These findings are consistent with the results obtained by Anand and Weaver (2004) who investigate the effect of the introduction of hidden orders at the Toronto Stock Exchange and show that the depth at the inside increases significantly when traders are allowed to use hidden orders. Analogously, Bessembinder and Venkataraman (2004) find evidence that the use of hidden orders increases market depth as they show that at the Paris Bourse the implicit transaction costs of blocks decrease due to the presence of hidden orders.

Table 5 also shows that the use of hidden orders could widen the best bid-offer.To explain this result one should consider that the use of hidden orders produces two opposite effects on the inside spread: it prevents traders from undertaking a war of price, thus widening the spread at the top of the book, and it increases traders' participation to the market, which tightens the spread. Under the first model, where traders use hidden orders to compete for liquidity provision, the only effect at work is the first one, and the spread increases. In the extended model, on the other hand, both effects are present and almost offset each other. Table 5 also reports similar results obtained for the expected weighted inside spread that we have computed by weighting each spread realization with the associated depth.

These results have relevant regulatory implications. As we have shown that hidden orders enhance market depth, the widespread adoption of such trading facility can be beneficial to institutional investors, and therefore it can be promoted for wholesale markets. Nonetheless, our results also suggest that hidden orders can make the inside spread wider, with the consequence that their extensive use could be detrimental to retail traders. However, when the market is under stress, as it happens in proximity of asset value shocks, hidden orders can be beneficial to both wholesalers and retailers. Under these circumstances, we have shown that hidden orders increase depth and therefore can add to market stability.

7   Concluding Remarks

A growing body of empirical evidence shows that hidden orders are widely used in electronic limit order platforms around the world. Hence, it becomes important to understand whether the intensive use of these orders is beneficial to market participants and/or to market quality. The empirical evidence shows that hidden orders are mostly used by large uninformed traders, but there is no theory to investigate how hidden orders can be used to control exposure costs, which factors determine their use, and how such orders affect market liquidity and traders' profits.

In this paper a theory of hidden orders is presented to discuss agents' optimal trading strategies in an LOB where traders are allowed to choose among hidden orders and a wide variety of other order types. The attractiveness of hidden orders is related to the exposure costs that can arise when traders run the risk of being undercut by other more aggressive competitors; all the more, in case of an asset value shock, traders are also exposed to the risk of being picked-off by fast traders. As a result, the dynamic model we propose embeds both retailers and institutional investors as well as a price-grid and an asset value shock right before the end of the game; the choice of this framework allows us to draw conclusions both on the determinants of hidden orders, and on the effects of hidden orders on market quality.

In accordance with the existing empirical evidence, the results show that hidden orders are indeed equilibrium strategies when posted by uninformed traders: uninformed agents use hidden orders both to prevent incoming traders from undercutting their orders, and to reduce the losses that can arise when scalpers pick-off their orders at stale prices. According to the results obtained, the proportion of hidden orders increases with order size, and relative depth on the opposite side of the market, while it decreases with time-to-shock.

The use of hidden orders is not only relevant from the viewpoint of a trader's optimal submission strategies, but also, and maybe more importantly, it is an instrument that market regulators can use to fine-tune the optimal degree of pre-trade transparency. By allowing traders to use hidden orders, the regulatory authority decreases market transparency as investors, by observing the screen, are not necessarily informed of the exact depth offered at the posted quotes. It becomes therefore relevant to understand whether there are any benefits, in terms of enhanced market quality, that can validate the use of hidden orders. This important issue in market design

is addressed in this paper by comparing a model with hidden orders to a benchmark model without undisclosed depth.

The results show that when hidden orders are used by traders to compete for the provision of liquidity, depth at the BBO increases since traders concentrate their orders at a single price; this effect could however widen the inside spread. When hidden orders are also used to prevent picking-off just before an asset value shock occurs, they increase market depth and have a negligible effect on the inside spread. It follows that, when evaluating the viability of hidden orders, market regulators should consider on the one hand that hidden orders can be beneficial to institutional investors, but detrimental to retailers, and on the other hand that hidden orders can be viewed as stabilizing devices as they provide additional depth when the market is under stress.

These findings are consistent with the existing empirical evidence on hidden orders from different financial markets,[19] and also respond to various issues raised by recent theoretical and empirical research. For example, Rosu (2008b) shows how the presence of informed traders in LOBs cannot justify the spread increase observed before public announcements in both equity and bond markets. This paper offers an alternative explanation to this puzzle as it suggests that, due to exposure costs, traders become less aggressive in proximity of asset value shocks and tend to post their limit orders away from the BBO. We therefore suggest to include among the estimated components of the bid-ask spread the option premium due to exposure costs. This component depends on the state of the book, the time of the day and the volatility of the asset.

Finally, an interesting extension of this model would be to include completely invisible orders, that are available in many platforms such as INET;[20] this would allow researchers to investigate further the optimal regulation of undisclosed depth and would certainly be an interesting topic for future research.

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[19] See, for example, Aitken et al. (2001), Anand and Weaver (2004), Bessembinder et al. (2008), De Winne and D'Hondt (2007), Frey and Sandas (2008) and Pardo and Pascual (2006).

[20] For example, Hasbrouck and Saar (2007) show that 37% of limit orders placed on INET are very short-lived and aggressively priced as in search of completely undisclosed orders. Such a behaviour suggests that invisible orders are used by agents to compete for the provision of liquidity.

8 Appendix

Proof of Proposition 1

We solve the model by backward induction, starting from $t = T$. Due to risk neutrality, large traders' profits from market orders are maximized for $j = 10$. For example, if we consider the strategy $MO_j B_i$, traders' profits will be $j(B_i - \beta_t v_t)$: the larger j, the larger the profits. The same reasoning applies to limit or hidden orders: an extra unit posted on the book will only induce additional gains in case of execution. Hence, from now onwards we assume that j is equal to its maximum possible value given the depth of the LOB.

Period T

The thresholds for period T are derived in Section 4.2.1.Given our assumption that $\beta \backsim U[0, 2]$, it is simple to derive the probabilities of the trading strategies; an example, if $b_T^u = [0\alpha00]$ and there is no uncertainty on the state of the LOB, agents' trading strategies at T are:

$$
\begin{aligned}
\Pr_T (MO_\alpha B_3^u \mid b_T^u) &= \Pr (S) \left(\frac{\beta_{MO_\alpha B_3^u, NTS^u}}{2} \right) = (1/2)\frac{(B_3^u/v_T)}{2} = (1/8)(2 - 3d) \\
\Pr_T (MO_{10} B_3^u \mid b_T^u) &= \Pr_T (MO_\alpha B_3^u \mid b_T^u) = (1/8)(2 - 3d) \\
\Pr_T (MO_\alpha^s A_1^u \mid b_T^u) &= \Pr (S) \left(\frac{2 - \beta_{MO_\alpha^s A_1^u, NTS^u}}{2} \right) = (1/2)(\frac{2 - A_1^u/v_T}{2}) = \frac{2+d}{8(1+d)} \\
\Pr_T (MO_\alpha A_1^u \mid b_T^u) &= \Pr (L) \left(\frac{\beta_{MRO_{10}A^u, MO_\alpha A_1^u} - \beta_{MO_\alpha A_1^u, NTL^u}}{2} \right) = \\
&= (1/2)(\frac{A_3^u/v_T^u - A_1^u/v_T^u}{2}) = \frac{2 - 3d}{8(1+d)} \\
\Pr_T (MRO_{10} A^u \mid b_T^u) &= \Pr (L) \left(\frac{2 - \beta_{MRO_{10}A^u, MO_\alpha A_1^u}}{2} \right) = (1/2)(\frac{2 - A_3^u/v_T^u}{2}) = \frac{d}{2(1+d)} \\
\Pr_T (NTS^u \mid b_T^u) &= 2 \left[1 - \Pr_T (MO_\alpha B_3^u) - \Pr_T (MO_\alpha^s A_1^u) \right] = \frac{4 + 12d + 7d^2}{8(1+d)} \\
\Pr_T (NTL^u \mid b_T^u) &= \Pr_T (NTS^u \mid b_T^u) = \frac{4 + 12d + 7d^2}{8(1+d)}
\end{aligned}
$$

where $\beta_{..}$ represents the threshold between two trading strategies. We indicate with $MO_\alpha^s A_i^u$ and $MO_\alpha A_i^u$ the order submitted by a small or a large trader respectively, when both traders optimally choose a market order of size α.

Period $T-1$

To obtain agents' optimal submission strategies at $T-1$, we first consider the possible states of the LOB. To simplify the analysis, we only examine sellers' strategies at $T-2$; hence at $T-1$ the bid side of the LOB is always empty. The possible states of the LOB at $T-1$ are summarized in the following Table:

LOB at $T-1$						
A_2	0	10	0	α	$\alpha + (10 - \alpha)$	0
A_1	0	0	10	0	0	α
B_1	0	0	0	0	0	0
B_2	0	0	0	0	0	0

Notice that at $T-1$ traders could face uncertainty on the state of the book: if for example the visible book is $b_{T-1} = [\alpha 000]$, they are uncertain whether the book is indeed $b_{T-1} = [\alpha 000]$ or $b_{T-1} = [(\alpha + (10 - \alpha))000]$. If instead the book shows ten shares on A_2, $b_{T-1} = [(10)000]$, there is no uncertainty.

<u>Case 1: Small Trader</u>

The small trader solves problem (5), presented in Section 4.1. His available strategies at $T-1$ depend on the initial state of the book. As we focus on the ask strategies at $T-2$, $MO_\alpha B_{1,2}$ is never feasible and hence is omitted, while $MO_\alpha B_3$, $LO_\alpha B_1$ and $LO_\alpha B_2$ are always available. The trader also has the option not to trade, NTS.

We consider the book $b_{T-1} = [\alpha 000]$ as an example, the other cases can be derived similarly. As explained before, traders are unable to differentiate between $b_{T-1} = [\alpha 000]$ and $b_{T-1} = [(\alpha + (10 - \alpha))000]$, and hence they will compute the expected state of the LOB when choosing their strategies. The available strategies are: $MO_\alpha B_3$, $LO_\alpha A_1$, $LO_\alpha A_2$, $LO_\alpha B_1$, $LO_\alpha B_2$ and $MO_\alpha A_2$. The profits from these strategies are the following:

$$\pi_{T-1}(MO_\alpha B_3) = \alpha(B_3 - \beta_{T-1} v_{T-1})$$
$$\pi_{T-1}(MO_\alpha A_2) = \alpha(\beta_{T-1} v_{T-1} - A_2)$$
$$E[\pi_{T-1}(LO_\alpha A_i)] = E\left[\alpha(A_i - \beta_{T-1}\widetilde{v}_T) \Pr_{w_T=\alpha}(A_i|b_T, v_T)\right]$$
$$E[\pi_{T-1}(LO_\alpha B_i)] = E\left[\alpha(\beta_{T-1}\widetilde{v}_T - B_i) \Pr_{w_T=\alpha}(B_i|b_T, v_T)\right]$$

As an example, we explicit the profit formula for $\pi_{T-1}(LO_\alpha A_1)$:

$$E[\pi_{T-1}(LO_\alpha A_1)] = \frac{1}{2}\alpha(A_1 - \beta_{T-1}v_T^d)\left[\frac{1}{2}\left(\Pr_T(MRO_{10}A^d| b_T^d) + \Pr_T(MO_\alpha A_2^d | b_T^d)\right) +\right.$$
$$\left. +\frac{1}{2}\Pr_T(MO_\alpha^s A_2^d | b_T^d)\right]$$
$$\text{where } b_T^d = [\alpha 000]$$

The equilibrium intervals of β_{T-1} for a small trader are obtained by comparing these profits and by finding the ranges of β_{T-1} associated with his optimal trading strategies. Results for the equilibrium states of the LOB are presented in Table 2 (Panel B) for $\alpha = 3$.

Case 2: Large Trader

The large trader solves problem (4), presented in Section 4.1. Notice that $MRO_{10}B$ and $MO_{10}B_{1,2}$ are not feasible strategies and hence we omit them; while $MO_{10}B_3$, $LO_{10}B_1$ and $LO_{10}B_2$ are always feasible strategies. For the remaining strategies, their feasibility depends on the state of the LOB.
As an example we focus again on the case $b_{T-1} = [\alpha 000]$ where the feasible strategies are: $MO_{10}B_3$, $LO_{10}A_1$, $LO_{10-\alpha}A_2$, $LO_{10}B_1$, $LO_{10}B_2$ and $MRO_{10}A$. The profits from these strategies are the following:

$$\pi_{T-1}(MO_{10}B_3) = 10(B_3 - \beta_{T-1}v_{T-1})$$
$$E[\pi_{T-1}(MRO_{10}A)] = 10\beta_{T-1}v_{T-1} - \alpha A_2 - (10-\alpha)A_2 \Pr_{10-\alpha}(A_2|b_{T-1},v_{T-1})$$
$$-(10-\alpha)A_3\left[1 - \Pr_{10-\alpha}(A_2|b_{T-1},v_{T-1})\right]$$
$$E[\pi_{T-1}(LO_jA_i)] = E\left[(A_i - \beta_{T-1}\tilde{v}_T)\sum_{w_T=\alpha}^{j} w_T \times \Pr_{w_T}(A_i|b_T,v_T)\right] \quad j = \{10-\alpha, 10\}$$
$$E[\pi_{T-1}(LO_{10}B_i)] = E\left[(\beta_{T-1}\tilde{v}_T - B_i)\sum_{w_T=\alpha}^{10} w_T \times \Pr_{w_T}(B_i|b_T,v_T)\right]$$

where in the case of LO_jA_i, $j = 10$ for A_1 and $j = 10 - \alpha$ for A_2. As an example, we explicit the profit formula for $\pi_{T-1}(LO_{10-\alpha}A_2)$:

$$E[\pi_{T-1}(LO_{10-\alpha}A_2)] = \frac{1}{2}(A_2 - \beta_{T-1}v_T^d)\left[\frac{1}{2}(10-\alpha)\Pr_T(MO_{10}A_3^d \mid b_T^d)\right]$$

$$+\frac{1}{2}(A_2 - \beta_{T-1}v_T^u) \times \left[\frac{1}{2}(10-\alpha)\Pr_T(MO_{10}A_1^u \mid \widetilde{b}_T^u)\right]$$

$$\text{where } b_T^d = [0000], \, b_T^u = [0(10)00] \text{ or } b_T^u = [0(10+(10-\alpha))00]$$

To obtain the equilibrium β_{T-1} intervals for a large trader associated with his optimal strategies, we compare these profits. Results for the equilibrium states of the LOB for the case with $\alpha = 3$ are presented in Table 2 (Panel A). Out of equilibrium states for both large and small traders are available at the authors on request.

Period $T-2$

We compute and compare the profits associated with the trader's strategies on the ask side at $T-2$, and assume that the initial book is empty. Strategies on the bid side are qualitatively similar, due to the symmetry of our modelization.

Case 1: Small Trader

The small trader solves again problem (5). The profits for the feasible strategies on the ask side ($MO_\alpha B_3$, $LO_\alpha A_1$ and $LO_\alpha A_2$) are the following:

$$\pi_{T-2}(MO_\alpha B_3) = \alpha(B_3 - \beta_{T-2}v_{T-2})$$

$$E[\pi_{T-2}(LO_\alpha A_i)] = E\left[\alpha(A_i - \beta_{T-1}v_{T-1})\Pr_{w_{T-1}=\alpha}(A_i|b_{T-1}, v_{T-1})\right.$$

$$\left. +\alpha(A_i - \beta_{T-1}\widetilde{v}_T)\Pr_{w_{T-1}=0}(A_i|b_{T-1}, v_{T-1})\Pr_{w_T=\alpha}(A_i|b_T, v_T)\right]$$

As an example, we explicit the profit formula for $\pi_{T-2}(LO_\alpha A_2)$, where we only consider the equilibrium strategies at $T-1$:

$$E[\pi_{T-2}(LO_\alpha A_2)] =$$

$$\alpha(A_2 - \beta_{T-1}v_{T-1})\left[\tfrac{1}{2}\Pr_{T-1}(MRO_{10}A \mid \widetilde{b}_{T-1}) + \tfrac{1}{2}\Pr_{T-1}(MO_\alpha A_2 \mid \widetilde{b}_{T-1})\right]$$

$$+ \left\{\left[\tfrac{1}{2}\left(\Pr_{T-1}(MO_{10}B_3 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(LO_{10}B_2 \mid \widetilde{b}_{T-1})\right)\right.\right.$$

$$\left.+ \tfrac{1}{2}\left(\Pr_{T-1}(MO_\alpha B_3 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(LO_\alpha B_2 \mid \widetilde{b}_{T-1})\right)\right] \times \left[\tfrac{1}{2}\gamma^1_{b^u_T} + \tfrac{1}{2}\gamma^2_{b^d_T}\right]$$

$$+ \tfrac{1}{2}\Pr_{T-1}(LO_{10-\alpha}A_2 \mid \widetilde{b}_{T-1}) \times \left[\tfrac{1}{2}\alpha(A_2 - \beta_{T-1}v^u_T)\left[\tfrac{1}{2}\Pr_T(MO_{10}A^u_1 \mid \widetilde{b}^u_T) + \frac{1}{2}\Pr_T(MO_\alpha A^u_1 \mid \widetilde{b}^u_T)\right] + \tfrac{1}{2}\gamma^2_{b^d_T}\right]$$

$$+ \tfrac{1}{2}\Pr_{T-1}(LO_\alpha A_1 \mid \widetilde{b}_{T-1})\left[\tfrac{1}{2}\gamma^1_{b^u_T} + \tfrac{1}{2}\alpha(A_2 - \beta_{T-1}v^d_T)\left[\tfrac{1}{2}\left(\Pr_T(MRO_{10}A^d \mid \widetilde{b}^d_T) + \Pr_T(MRO_{2\alpha}A^d \mid \widetilde{b}^d_T)\right)\right]\right]$$

$$+ \tfrac{1}{2}\Pr_{T-1}(LO_\alpha A_2 \mid \widetilde{b}_{T-1})\left[\tfrac{1}{2}\gamma^2_{b^d_T} + \tfrac{1}{2}\alpha(A_2 - \beta_{T-1}v^u_T)\right.$$

$$\left.\left.\left[\tfrac{1}{2}\left(\Pr_T(MRO_{10}A^u \mid \widetilde{b}^u_T) + \Pr_T(MO_{2\alpha}A^u_1 \mid \widetilde{b}^u_T)\right) + \tfrac{1}{2}\Pr_T(MO_\alpha A^u_1 \mid \widetilde{b}^u_T)\right]\right]\right\}$$

where $b_{T-1} = [\alpha 000]$ or $b_{T-1} = [(\alpha + (10 - \alpha))000]$, $\gamma^1_{b^u_T}$ and $\gamma^2_{b^d_T}$ are defined as follows:

$$\gamma^1_{b^u_T} = \alpha(A_2 - \beta_{T-1}v^u_T)\left[\tfrac{1}{2}\left(\Pr_T(MRO_{10}A^u \mid \widetilde{b}^u_T) + \Pr_T(MO_\alpha A^u_1 \mid \widetilde{b}^u_T)\right) + \tfrac{1}{2}\Pr_T(MO^s_\alpha A^u_1 \mid \widetilde{b}^u_T)\right]$$

$$\gamma^2_{b^d_T} = \alpha(A_2 - \beta_{T-1}v^d_T)\left[\tfrac{1}{2}\Pr_T(MO_{10}A^d_3 \mid \widetilde{b}^d_T) + \tfrac{1}{2}\Pr_T(MO_\alpha A^d_3 \mid \widetilde{b}^d_T)\right]$$

As for $T-1$, it is straightforward to derive the β_{T-2} intervals associated with small traders' optimal strategies. Results are presented for $\alpha = 3$ in Table 2 (Panel B).

Case 2: Large Trader

The large trader solves again problem (4) and the available strategies on the ask side are $MO_{10}B_3$, $LO_{10}A_1$, $LO_{10}A_2$, $HO_{10}A_2$. The profits from these strategies are the following:

$$\pi_{T-2}(MO_{10}B_3) = 10(B_3 - \beta_{T-2}v_{T-2})$$

$$E[\pi_{T-2}(LO_{10}A_i)] = E\left[(A_i - \beta_{T-1}v_{T-1})\sum_{w_{T-1}=\alpha}^{10} w_T \times \Pr_{w_T}(A_i|b_{T-1}, v_{T-1}) + \right.$$
$$\left. + (A_i - \beta_{T-1}\widetilde{v}_T)\sum_{W=0}^{9}\sum_{w_T=\alpha}^{10-W} w_T \Pr_{w_T}(A_i|b_T, v_T)\Pr(w_{T-1} = W|b_{T-1}, v_{T-1})\right]$$

$$E[\pi_{T-2}(HO_{10}A_2)] = E\left[(A_2 - \beta_{T-1}v_{T-1})\sum_{w_{T-1}=\alpha}^{10} w_T \times \Pr_{w_T}(A_2|b_{T-1}, v_{T-1}) + \right.$$
$$\left. + (A_2 - \beta_{T-1}\widetilde{v}_T)\sum_{W=0}^{9}\sum_{w_T=\alpha}^{10-W} w_T \Pr_{w_T}(A_2|b_T, v_T)\Pr(w_{T-1} = W|b_{T-1}, v_{T-1})\right]$$

As an example, we explicit the profit formula for $\pi_{T-2}(HO_{10}A_2)$:

$$E[\pi_{T-2}(HO_{10}A_2)] =$$

$$\tfrac{1}{2}(A_2 - \beta_{T-1}v_{T-1})10\Pr_{T-1}(MRO_{10}A\,|\widetilde{b}_{T-1}) + \frac{1}{2}\Pr_{T-1}(MO_\alpha A_2\,|\widetilde{b}_{T-1})\left[\alpha(A_2 - \beta_{T-1}v_{T-1}) + \right.$$

$$\tfrac{1}{2}\gamma^3_{b^u_T,10-\alpha} + \tfrac{1}{2}\gamma^4_{b^d_T,10-\alpha}\Big] + \left\{\left[\tfrac{1}{2}\left(\Pr_{T-1}(MO_{10}B_3\,|\,\widetilde{b}_{T-1}) + \Pr_{T-1}(LO_{10}B_2\,|\,\widetilde{b}_{T-1})\right)\right.\right.$$

$$+ \tfrac{1}{2}\left(\Pr_{T-1}(MO_\alpha B_3\,|\,\widetilde{b}_{T-1}) + \Pr_{T-1}(LO_\alpha B_2\,|\,\widetilde{b}_{T-1})\right)\Big] \times \left[\tfrac{1}{2}\gamma^3_{b^u_T,10} + \tfrac{1}{2}\gamma^4_{b^d_T,10}\right]$$

$$+ \tfrac{1}{2}\Pr_{T-1}(LO_{10-\alpha}A_2\,|\,\widetilde{b}_{T-1}) \times \left[\tfrac{1}{2}\,\alpha(A_2 - \beta_{T-1}v^u_T)\left[\tfrac{1}{2}\Pr_T(MO_{10}A^u_1\,|\,\widetilde{b}^u_T) + \frac{1}{2}\Pr_T(MO_\alpha A^u_1\,|\widetilde{b}^u_T)\right] + \right.$$

$$\tfrac{1}{2}\gamma^4_{b^d_T,\alpha}\Big] + \tfrac{1}{2}\Pr_{T-1}(LO_\alpha A_1\,|\,\widetilde{b}_{T-1})\left[\tfrac{1}{2}\,\gamma^3_{b^u_T,10} + \tfrac{1}{2}(A_2 - \beta_{T-1}v^d_T)\right.$$

$$\left[\tfrac{1}{2}\left((10-\alpha)\Pr_T(MRO_{10}A^d|\,\widetilde{b}^d_T) + \alpha\Pr_T(MRO_{2\alpha}A^d|\widetilde{b}^d_T)\right)\right]\Big]$$

$$+ \tfrac{1}{2}\Pr_{T-1}(LO_\alpha A_2\,|\,\widetilde{b}_{T-1})\left[\tfrac{1}{2}\gamma^4_{b^d_T,10-\alpha} + \tfrac{1}{2}(A_2 - \beta_{T-1}v^u_T)\right.$$

$$\left[\tfrac{1}{2}\left((10-\alpha)\Pr_T(MRO_{10}A^u|\,\widetilde{b}^u_T) + \alpha\Pr_T(MO_{2\alpha}A^u_1\,|\,\widetilde{b}^u_T)\right) + \tfrac{1}{2}\alpha\Pr_T(MO_\alpha A^u_1\,|\,\widetilde{b}^u_T)\right]\Big]\right\}$$

where $b_{T-1} = [\alpha 000]$ or $b_{T-1} = [(\alpha + (10 - \alpha))000]$, $\gamma^3_{b^u_T,j}$ and $\gamma^4_{b^d_T,j}$ are defined as follows:

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$$\gamma^3_{b^u_T, j} = (A_2 - \beta_{T-1} v^u_T) \left[\frac{1}{2} \left(j \Pr_T(MRO_{10} A^u | \widetilde{b}^u_T) + \alpha \Pr_T(MO_\alpha A^u_1 | \widetilde{b}^u_T) \right) + \frac{1}{2} \alpha \Pr_T(MO^s_\alpha A^u_1 | \widetilde{b}^u_T) \right]$$

$$\gamma^4_{b^d_T, j} = (A_2 - \beta_{T-1} v^d_T) \left[\frac{1}{2} j \Pr_T(MO_{10} A^d_3 | \widetilde{b}^d_T) + \frac{1}{2} \alpha \Pr_T(MO_\alpha A^d_3 | \widetilde{b}^d_T) \right]$$

Results are presented for $\alpha = 3$ in Table 2 (Panel A).

Optimal exposure size (α^*)

We solve the model for different values of α. When α shares are visible at A_2, for $\alpha > 3$ incoming traders at $T - 1$ prefer to undercut and post at A_1, hence hidden orders do not protect from competition and are not optimal strategies; for $\alpha \leq 3$, incoming traders at $T - 1$ will join the queue at A_2. However, the profits of hidden orders increase with the size of the visible part, as time priority is preserved on the visible shares. As a result, the optimal disclosed size is the largest one compatible with traders joining the queue at $T - 1$: $\alpha^* = 3$.

Proof of Proposition 2

We only provide a sketch of the proof, since there are many similarities with the proof of Proposition 1. Compared to the basic model, there are two main differences: first, the asset value shock has now a larger size ($k = 2$) and v_T can also remain constant ($x = 1/3$); second, scalpers are active in the market. Their presence implies that hidden orders could be an optimal strategy also at $T - 1$. Notice further that in this framework a 10-unit limit order does not necessarily dominate an α-unit one since the latter has possibly higher profits due to both higher execution probability and lower losses in case of mispricing. Nevertheless, it is possible to show that α-unit limit orders are always dominated by hidden orders, hence we will assume again that j is equal to the maximum possible value given the depth of the LOB.

Period T

The thresholds and the order submission probabilities at T are derived as in Proposition 1. Notice however that in this case, due to the assumption of $k = 2$ and to the presence of scalpers, if a shock occurs, traders arriving at T will always observe an empty order book: the existing orders will be either

mispriced and picked-off or too far away from the new asset value. The only other relevant difference is that here an additional case has to be considered for large traders, as now there are incentives to submit hidden orders on both A_1 and A_2. So, if there are $f_i < j$ visible shares on A_i for both $i = 1$ and $i = 2$, with $f_1 + f_2 < j$, the large trader's β_T thresholds for the ask side will be the following:

$$
\begin{aligned}
&\text{submit } MRO_j A && \text{if} && \beta_T \geq \tfrac{\Lambda_y}{v_T}\\
&\text{submit } MRO_{f_1+f_2} A && \text{if} && \tfrac{\Lambda_m}{v_T} \leq \beta_T < \tfrac{\Lambda_y}{v_T}\\
&\text{submit } MO_{f_1} A_1 && \text{if} && \tfrac{A_1}{v_T} \leq \beta_T < \tfrac{\Lambda_m}{v_T}\\
&\text{no trade} && \text{if} && 1 \leq \beta_T < \tfrac{A_1}{v_T}
\end{aligned}
$$

where $\Lambda_m = \sum \Pr_{j-f_1}(A_m | b_T, b_{T-1}) A_m$, with $m = \{1, 2\}$, and $\Lambda_y = \sum \Pr_{j-f_2}(A_y | b_T, b_{T-1}) A_y$, with $y \in \{1, 2, 3\}$.

Period $T - 1$

Notice that we now add to the possible initial states of the LOB an additional strategy as traders can submit hidden orders at A_1. So, it is possible to have uncertainty on two levels of the LOB. The possible states of the LOB at $T-1$ are summarized in the following Table:

LOB at $T - 1$						
A_2	0	10	0	α	$\alpha + (10 - \alpha)$	0	0
A_1	0	0	10	0	0	α	$\alpha + (10 - \alpha)$
B_1	0	0	0	0	0	0	0
B_2	0	0	0	0	0	0	0

Case 1: Small Trader

The trader still solves problem (5). In order to enlighten the differences between the two models, we consider again the book $b_{T-1} = [\alpha 000]$ as an example. The available strategies are the same as in the proof of Proposition 1 ($MO_\alpha B_3$, $LO_\alpha A_1$, $LO_\alpha A_2$, $LO_\alpha B_1$, $LO_\alpha B_2$ and $MO_\alpha A_2$) and traders are again unable to differentiate between $b_{T-1} = [\alpha 000]$ and $b_{T-1} = [(\alpha + (10 - \alpha))000]$; they will hence compute the expected state of the LOB. The general profit

formulas are unchanged and hence omitted; however, to underline the differences between the two models, we explicit the profit formula for $\pi_{T-1}(LO_\alpha A_1)$:

$$E[\pi_{T-1}(LO_\alpha A_1)] = \frac{1}{3}\alpha(A_1 - \beta_{T-1}v_T^U) +$$

$$+\frac{1}{3}\alpha(A_1 - \beta_{T-1}v_T)\left[\frac{1}{2}\left(\Pr_T(MRO_{10}A \mid \widetilde{b}_T) + \Pr_T(MO_\alpha A_2 \mid \widetilde{b}_T)\right) + \frac{1}{2}\Pr_T(MO_\alpha^s A_2 \mid \widetilde{b}_T)\right]$$

where $b_T = [\alpha 000]$ or $b_T = [(\alpha + (10 - \alpha))000]$. Notice that in case of a positive asset value shock the order will be picked-off with probability one by scalpers, while in case of a negative asset value shock the limit order will never be executed since too far from the BBO. Results are presented in Table 3 (Panel B) for $\alpha = 1$.

Case 2: Large Trader

The large trader solves again problem (4), however there are two differences with the previous model. First, both $HO_{10}A_1$ and $HO_{10}A_2$ are now included among the available strategies; second, the profits from $MRO_j B$ are different as there can be hidden liquidity on the first level of the LOB. The large trader's problem is modified as follows (formulas that are unchanged are omitted):

$$\max_{o_{L,\beta_t} \in [MO_j B_i, MRO_j B, LO_j A_i, HO_{10}A_i, NTL]} E[\pi_t(o_{L,\beta_t})] \qquad (4a)$$

where:

$$\pi_t(MO_j B_i) = ...; \quad E[\pi_t(LO_j A_i)] = ...; \quad E[\pi_t(HO_{10}A_i)] = ...; \quad \pi_t(NTL) = 0$$

$$E[\pi_t(MRO_j B)] = f_1 B_1^{;z}\Pr_{f_2}(B_1|b_t, v_t) + f_2 B_2^{;z}\Pr_{f_2}(B_2|b_t, v_t)$$

$$+ (j - f_1 - f_2)B_3^{;z}\left[1 - \Pr_{f_2}(B_2|b_t, v_t)\right] - j\,\beta_t\,v_t$$

As an example, we focus again on the case $b_{T-1} = [\alpha 000]$. The feasible strategies are the same as in the proof of Proposition 1, with the addition of $HO_{10}A_1$ and $HO_{10-\alpha}A_2$. We focus here only on the profits from these additional strategies and refer to the proof of Proposition 1 for the others:

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$$E[\pi_{T-1}(HO_j A_i)] = E\left[(A_i - \beta_{T-1}\widetilde{v}_T) \sum_{w_T=\alpha}^{j} w_T \Pr_{w_T}(A_i|b_T, v_T)\right]$$

As an example, we explicit again the profit formula for $\pi_{T-1}(LO_{10-\alpha}A_2)$:

$$E[\pi_{T-1}(LO_{10-\alpha}A_2)] = \frac{1}{3}(A_2 - \beta_{T-1}v_T^U) + \frac{1}{3}(A_2 - \beta_{T-1}v_T) \times \left[\frac{1}{2}(10-\alpha)\Pr_{T}(MO_{10}A_2 \,|\widetilde{b}_T)\right]$$

where $b_T = [(10)000]$ or $b_T = [(10+(10-\alpha))000]$. Results are presented in Table 3 (Panel A) for $\alpha = 1$.

Period $T-2$

As in the basis model, we compute and compare the profits associated to traders strategies on the ask side at $T-2$, and assume that the initial book is empty.

Case 1: Small Trader

The small trader solves problem (5). The available strategies are the same as in the proof of Proposition 1, and given the similarities between the two models, we omit to present the profit formulas. Results are presented for $\alpha = 1$ in Table 3 (Panel B).

Case 2: Large Trader

The large trader solves again problem (4a). To enlighten the differences with the model without scalpers, we explicit the profit formula for $\pi_{T-2}(HO_{10}A_2)$:

$$E[\pi_{T-2}(HO_{10}A_2)] =$$

$$\frac{1}{2}\Pr_{T-1}(MRO_{10}A \mid \widetilde{b}_{T-1})10(A_2 - \beta_{T-1}v_{T-1}) + \frac{1}{2}\Pr_{T-1}(MO_\alpha A_2 \mid \widetilde{b}_{T-1})\left[\alpha(A_2 - \beta_{T-1}v_{T-1}) + \gamma^5_{b_T,10-\alpha}\right]$$

$$+\left\{\left[\frac{1}{2}\left(\Pr_{T-1}(MO_{10}B_3 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(LO_{10}B_2 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(HO_{10}B_2 \mid \widetilde{b}_{T-1})\right)\right.\right.$$

$$\left.+\frac{1}{2}\left(\Pr_{T-1}(MO_\alpha B_3 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(LO_\alpha B_2 \mid \widetilde{b}_{T-1}) + \Pr_{T-1}(NTS \mid \widetilde{b}_{T-1})\right)\right] \times \gamma^5_{b_T,10}$$

$$+\frac{1}{2}\Pr_{T-1}(LO_{10-\alpha}A_2 \mid \widetilde{b}_{T-1})\left[\frac{1}{3}\alpha(A_2 - \beta_{T-1}v_T)\left[\frac{1}{2}\Pr_{T}(MO_{10}A_2 \mid \widetilde{b}_T) + \frac{1}{2}\Pr_{T}(MO_\alpha A_2 \mid \widetilde{b}_T)\right] + \gamma^7\right]$$

$$+\frac{1}{2}\Pr_{T-1}(HO_{10-\alpha}A_2 \mid \widetilde{b}_{T-1})\gamma^6_{b_T} + \frac{1}{2}\Pr_{T-1}(LO_\alpha A_2 \mid \widetilde{b}_{T-1})\gamma^6_{b_T} + \frac{1}{2}\Pr_{T-1}(LO_\alpha A_1 \mid \widetilde{b}_{T-1})$$

$$\left[\frac{1}{3}(A_2 - \beta_{T-1}v_T)\left[\frac{1}{2}\left((10-\alpha)\Pr_{T}(MRO_{10}A \mid \widetilde{b}_T) + \alpha\Pr_{T}(MRO_{2\alpha}A \mid \widetilde{b}_T)\right)\right] + \gamma^7\right]\right\}$$

where $b_{T-1} = [(\alpha + (10 - \alpha))000]$ or $b_{T-1} = [\alpha000]$, where $\gamma^5_{b_T,j}$, $\gamma^6_{b_T}$ and γ^7 are defined as follows:

$$\gamma^5_{b_T,j} = \frac{1}{3}(A_2 - \beta_{T-1}v_T)\frac{1}{2}\left[j\Pr_{T}(MRO_{10}A \mid \widetilde{b}_T) + \alpha(\Pr_{T}(MO_\alpha A_2 \mid \widetilde{b}_T) + \Pr_{T}(MO^s_\alpha A_1 \mid \widetilde{b}_T))\right] + \gamma^7$$

$$\gamma^6_{b_T} = \frac{1}{3}(A_2 - \beta_{T-1}v_T)\frac{1}{2}\left[(10-\alpha)\Pr_{T}(MRO_{10}A \mid \widetilde{b}_T) + \alpha(\Pr_{T}(MO_{2\alpha}A_2 \mid \widetilde{b}_T) + \Pr_{T}(MO_\alpha A_2 \mid \widetilde{b}_T))\right] + \gamma^7$$

$$\gamma^7 = \frac{1}{3}\alpha(A_2 - \beta_{T-1}v^U_T)$$

Results are presented for $\alpha = 1$ in Table 3 (Panel A).

Optimal exposure size (α^*)

We solve the model for different values of α. Since now traders are mainly concerned about picking-off by scalpers, it is easy to show that it is optimal to hide as much as possible. Hence, $\alpha^* = 1$.

Proof of Proposition 3

The Proposition is obtained through a straightforward comparison of the results obtained for the benchmark and the model with hidden orders, respec-

tively. Clearly, the benchmarks are simplified versions of the models solved in Proposition 1 and 2, where hidden orders are not available strategies.
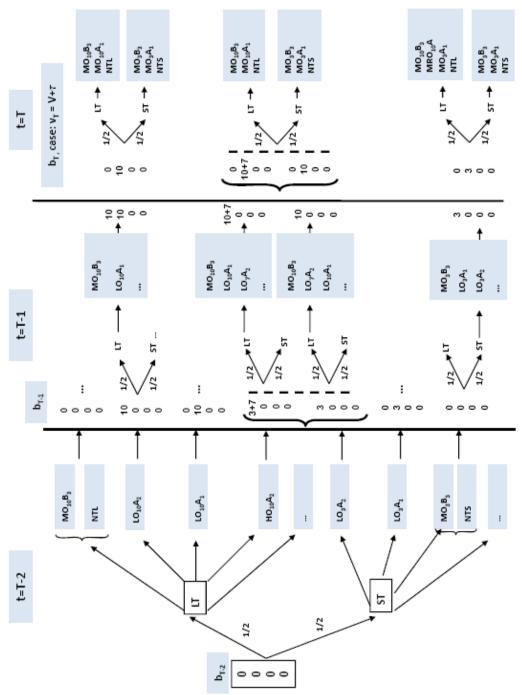
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	T-2	T-1	T	
$1+ (7/2)\tau$			A_3^u	
$1+ (5/2)\tau$	A_3	A_3	A_2^u	
$1 + (3/2)\tau$	A_2	A_2	A_1^u	A_3^d
$1 + (1/2)\tau$	A_1	A_1	B_1^u	A_2^d
$1 -(1/2)\tau$	B_1	B_1	B_2^u	A_1^d
$1 - (3/2)\tau$	B_2	B_2	B_3^u	B_1^d
$1 - (5/2)\tau$	B_3	B_3		B_2^d
$1 - (7/2)\tau$				B_3^d

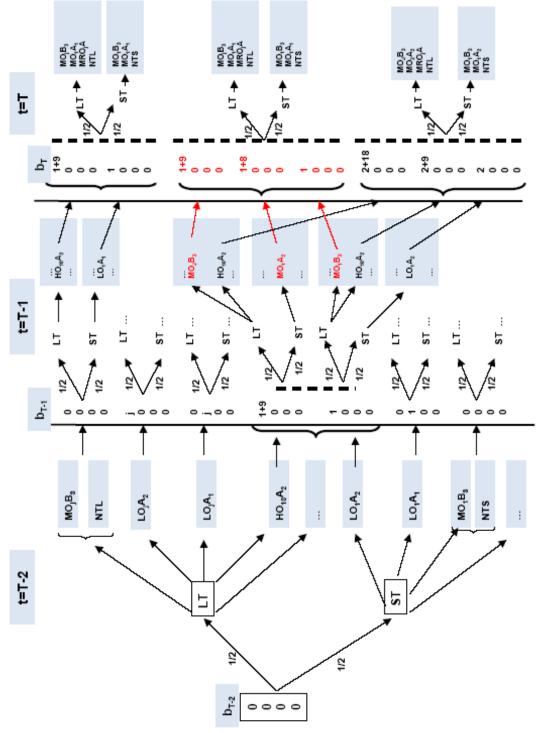$v^u =1+2\tau$

$v = 1$

$v^d =1-2\tau$

Figure 1 This Figure shows the price grid for $k = 1$. The ask prices are equal to $A_{1,2,3}$ and the bid prices are equal to $B_{1,2,3}$, with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v, that at time T can take values v, v^u and v^d respectively.

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Figure 2 This Figure shows the extensive form of the game for $j = 10$ and $\alpha = 3$. At $T - 2$ the book opens empty, $b_{T-2} = [0000]$; nature chooses with equal probability a large trader (LT) or a small trader (ST) who decides his optimal submission strategy among all the feasible orders (Table 1). If for example at $T - 2$ a LT chooses $LO_{10}A_2$, at $T - 1$ the book will be $b_{T-1} = [(10)000]$; if then another LT arrives who, still as an example, chooses $LO_{10}A_1$, then at T the book will open equal to $b_T = [0(10)00]$ so that the incoming LT will submit either $MO_{10}B_3$, $MO_{10}A_1$, or he will not trade (NTL); the ST instead will choose among MO_3B_3 and MO_3A_1, or decide not to trade (NTS). On the other hand, if at T-2 a LT chooses $HO_{10}A_2$, traders arriving at time $T - 1$ and T will be uncertain about the actual depth of the book .

	T-2	T-1	T		
$1+ (9/2)\tau$				A_3^U	
$1+ (7/2)\tau$				A_2^U	
$1+ (5/2)\tau$	A_3	A_3	A_3	A_1^U	
$1 + (3/2)\tau$	A_2	A_2	A_2	B_1^U	
$1 + (1/2)\tau$	A_1	A_1	A_1	B_2^U	A_3^D
$1 -(1/2)\tau$	B_1	B_1	B_1	B_3^U	A_2^D
$1 - (3/2)\tau$	B_2	B_2	B_2		A_1^D
$1 - (5/2)\tau$	B_3	B_3	B_3		B_1^D
$1 - (7/2)\tau$					B_2^D
$1 - (9/2)\tau$					B_3^D

Left-axis labels: $v^U =1+2\tau$ (at row $1 + (3/2)\tau$), $v=1$ (at row $1 + (1/2)\tau$), $v^D =1-2\tau$ (at row $1 - (3/2)\tau$).

Figure 3 This Figure shows the price grid for $k = 2$. The ask and bid prices are equal to $A_{1,2,3}$ and $B_{1,2,3}$ respectively with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v, that at time T can take values v, v^U and v^D.

Figure 4 This Figure shows the extensive form of the game for $j = 10$ and $\alpha = 1$. At $T - 2$ the book opens empty, $b_{T-2} = [0000]$; nature chooses with equal probability a large trader (LT) or a small trader (ST) who decides his optimal submission strategy among all the feasible orders (Table 1). The Figure reports eight of the equilibrium game paths, which end into three sets of books at T. Because each of these books may contain hidden depth, before submitting his order, the incoming trader at T has to estimate the probability of observing the hidden liquidity, conditional on the previous states of the LOB. For example, the first two books from above, that appear at T and derive from the same book at $T - 1$ ($b_{T-1} = [0000]$), differ as the visible unit on A_2 can either come from an LO_1A_2 submitted at $T - 1$ by a ST, or from a $HO_{10}A_2$ submitted at $T - 1$ by a LT. Notice that the second sets of states of the LOB is complicated by the fact that they can derive from two different books at $T - 1$ ($b_{T-1} = [1000]$ or $b_{T-1} = [(1+9)000]$) and hence are characterized by a further degree of uncertainty. It follows that traders at T have to make inference both on the other traders' strategies at $T - 1$, and at $T - 2$.

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TABLE 1: ORDER SUBMISSION STRATEGIES

Panel A: Small Trader		Panel B: Large Trader	
($i \in [1,3]$ for MO, and $i \in [1,2]$ for LO)		($j \in [1,10]$, $i \in [1,3]$ for MO, and $i \in [1,2]$ for LO and HO)	
Market Sell Order	$MO_\alpha B_i$	Market Sell Order	$MO_j B_i$
		Marketable Sell Order	$MRO_j B$
Limit Sell Order	$LO_\alpha A_i$	Limit Sell Order	$LO_j A_i$
		Hidden Sell Order	$HO_{10} A_i$
No Trade	NTS	No Trade	NTL
		Hidden Buy Order	$HO_{10} B_i$
Limit Buy Order	$LO_\alpha B_i$	Limit Buy Order	$LO_j B_i$
		Marketable Buy Order	$MRO_j A$
Market Buy Order	$MO_\alpha A_i$	Market Buy Order	$MO_j A_i$

Table 1: Order Submission Strategies. This Table presents the possible orders that a small trader (Panel A) and a large trader (Panel B) can choose upon arrival at the market. By assumption, small traders can only trade α shares, whereas large traders can submit orders of size up to ten shares. On the sell side (the buy side is symmetrical) small traders can submit a market sell order ($MO_\alpha B_i$) to one of the three levels of the LOB. Small traders can also opt to submit a limit sell order to the first ($LO_\alpha A_1$) or to the second level of the ask side of the LOB ($LO_\alpha A_2$), and they can also decide not to trade (NTS). A large trader can submit a market sell order ($MO_j B_i$) of size j to one of the three levels of the book, or he can hit the buy side by submitting a marketable sell order ($MRO_j B$). A large trader can also choose to submit a limit sell order of size j to either A_1 or A_2 ($LO_j A_i$), or he can decide to disclose only α units and submit a hidden sell order to the first or to the second level of the ask side ($HO_{10} A_i$). By assumption, hidden orders are of size 10. Finally, a large trader can decide not to trade (NTL).

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Table 2 - Order Submission Probabilities (Competition for liquidity provision)					
Panel A	$b_{T-1(T-2)} = [0000]$		$b_{T-1} = [(10)000]$	$b_{T-1} = [(3+7)000]$	$b_{T-1} = [3000]$
Large Trader	Benchmark	Hidden	Benchmark	Hidden	Benchmark
$MO_{10}B_3$	.302(.224)	.302(.242)	.346	.341	.341
$HO_{10}A_2$		(.258)			
$LO_{10}A_2$	.197(.276)	.197			
LO_7A_2			.130	.136	.136
$LO_{10}A_1$					
$LO_{10}B_2$	.201	.201	.154	.184	.202
$MO_{10}A_1$					
$MRO_{10}A$				.339	.321
$MO_{10}A_2$			.370		
$MO_{10}A_3$	.300	.300			
Panel B	$b_{T-1(T-2)} = [0000]$		$b_{T-1} = [(10)000]$	$b_{T-1} = [(3+7)000]$ & $b_{T-1} = [3000]$	
Small Trader	Benchmark	Hidden	Benchmark	Benchmark $\equiv$ Hidden	
MO_3B_3	.233(.072)	.233(.065)	.327	.323	
LO_3A_2	.265(.428)	.265(.435)		.123	
LO_3A_1			.150	.031	
LO_3B_2	.272	.272	.206	.206	
MO_3A_1					
MO_3A_2			.317	.317	
MO_3A_3	.230	.230			

Table 2 Order Submission Probabilities. This Table reports both large (Panel A) and small traders' (Panel B) submission probabilities for the orders listed in column 1 for both the benchmark (no hidden orders) and the hidden orders case. In equilibrium the large trader's order size is 10 or 7 shares, whereas the small traders' order size is $\alpha^* = 3$ shares. Execution probabilities are reported for the equilibrium states of the book listed in row 2, and for both period $T - 1$ and $T - 2$ (in parenthesis). For example, when the book opens at $T - 1$ with 3 shares visible at A_2, e.g. $b_{T-1} = [(3 + 7)000]$ or $b_{T-1} = [3000]$ for the framework with hidden orders and $b_{T-1} = [3000]$ for the benchmark model, large sellers submit market orders at B_3, $MO_{10}B_3$, with probability .341, while small sellers submit 3-share market orders with probability .323; more patient traders join the queue at A_2 with probability .136 in case of a large trader (LO_7A_2) and .123 for a small trader (LO_3A_2).

Table 3 - Order Submission Probabilities (Extended framework)

Panel A	$b_{T-1(T-2)} = [0000]$		$b_{T-1} = [(10)000]$		$b_{T-1} = [1000]$		$b_{T-1} = [(1+9)000]$
Large Trader	Benchmark	Hidden	Benchmark	Hidden	Benchmark	Hidden	Hidden
$MO_{10}B_3$	.265 (.169)	.265 (.170)	.301	.301	.288		.288
$HO_{10}A_2$		.019 (.009)					.013
$LO_{10,9}A_2$	.228 (.331)	.216 (.321)			.206		.193
LO_1A_2	.007						
$LO_{10}A_1$			.151	.151			
LO_1A_1			.01	.005			
NTL			.032		.001		
LO_1B_2	.009		.015		.014		
$LO_{10}B_2$	.178	.158	.102	.081	.17		.15
$HO_{10}B_2$		.029		.073			.034
$MRO_{10}A$					.321		.322
$MO_{10}A_2$			.389	.389			
$MO_{10}A_3$	.313	.313					

Table 3 (Panel A) Order Submission Probabilities. This Table reports the submission probabilities for the orders listed in column 1, for both the benchmark and the hidden orders case. These probabilities are computed for the four equilibrium states of the book listed in row 2, and for both periods $T-1$ and $T-2$ (in parenthesis). For example, the second column shows that when the book is empty, $b_{T-1}=[0000]$, large sellers submit market orders at B_3, $MO_{10}B_3$, with probability .265 at $T-1$ and .169 at $T-2$, in the benchmark model, whereas the corresponding probabilities for the model with hidden orders are .265 at $T-1$ and .170 at $T-2$.

Table 3 - Order Submission Probabilities (Extended framework)

Panel B	$b_{T-1} = [0000]$	$b_{T-1} = [(10)000]$	$b_{T-1} = [1000]$	$(b_{T-1} = [1000])$ & $(b_{T-1} = [(1+9)000])$
Small Trader			Benchmark ≡ Hidden	
MO_1B_3	.222 (0)		.264	.265
LO_1A_2	.278 (.5)			.200
LO_1A_1			.198	.029
NTS			.032	.001
LO_1B_2	.215		.137	.136
MO_1A_2			.369	.369
MO_1A_3	.285			

Table 3 (Panel B) Order Submission Probabilities. This Table reports the small traders' submission probabilities for the orders listed in column 1. Results for small traders in the benchmark model are the same as in the model with hidden orders. These probabilities are computed for the four states of the book listed in row 2, and for both periods $T-1$ and $T-2$ (in parenthesis). For example, the second column shows that when the book is empty, $b_{T-1} = [0000]$, small sellers submit limit orders to A_2, LO_1A_2, with probability .278 at $T-1$ and with probability .5 at $T-2$.

Table 4	-Hidden Orders and Order Size- $T-1$ $(T-2)$					
b_{T-1}	[0000]		[2000]	[(10)000]	[0200]	[0(10)00]
	$j=2$	$j=10$	$j=2$	$j=10$	$j=2$	$j=10$
Ask Side						
HO_jA_2	.019 (.011)	.019 (.009)				
LO_jA_2	.233 (.384)	.216 (.321)				
$\frac{HO_jA_2}{HO_jA_2+LO_jA_2}$	.075 (.027)	.081 (.027)				
Bid Side						
HO_jB_2	.028	.029	.044	.073	.045	.077
LO_jB_2	.170	.158	.090	.081	.010	.004
$\frac{HO_jB_2}{HO_jB_2+LO_jB_2}$	.141	.155	.328	.474	.818	.951

Table 4 Hidden Orders and Order Size. This Table reports the order submission probabilities of both hidden orders (HO_jA_2 and HO_jB_2) and limit orders (LO_jA_2 and LO_jB_2) posted at A_2 and B_2; it also reports the proportion of hidden orders over the total depth, disclosed and undisclosed, at the second level of the book, for both the model with $j=10$ and with $j=2$. Comparisons between the two models are computed for the five states of the book listed in row 2, for periods $T-1$ and $T-2$ (in parenthesis). For example, when the book is empty ($b_{T-1}=[0000]$), large hidden orders on A_2 (HO_jA_2) for $j=2$ are submitted at $T-1$ with probability .019 and at $T-2$ with probability .011. The same submission probabilities for $j=10$ are equal to .019 and .009 for $T-1$ and $T-2$ respectively.

Table 5 Estimated Depth and Inside Spread			
	Depth at the BBO	Inside Semi-Spread	Weighted Semi-Spread
Competition Framework			
Hidden	1.874	.0956	.8079
Benchmark	1.770	.0935	.7713
Extended Framework			
Hidden	1.695	.0899	.6932
Benchmark	1.677	.0897	.6900

Table 5 Estimated Depth and Inside Spread. This Table reports the estimated depth at the BBO (column two) and the best semi-spread, quoted and weighted by the associated depth (columns three and four); the latter is computed as the difference between the best ask and the spread midpoint ($V = 1$). All the indicators are reported both for the basic and the extended model, under the two cases with and without hidden orders.