Supply Function Equilibria: Step functions and continuous representations*

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Abstract

In most electricity markets generators must submit step-function offers to a uniform price auction and are most simply modelled as pure-strategy Nash equilibria of continuous supply functions (SFs), in which each supplier has a unique profit maximising choice of SF given competitors' choices. Critics argue that the discreteness and discontinuity of the required steps can rule out pure-strategy equilibria, resulting in price instability. We argue that if prices must be selected from a finite set the resulting step function converges to the continuous SF as the number of steps increases, reconciling the apparently very disparate approaches to modelling electricity markets.

Key words: Multi-unit auctions, supply function equilibria, convergence of Nash equilibria, wholesale electricity markets

JEL Classification C62, C72, D43, D44, L94

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1 INTRODUCTION

This paper fills an increasingly embarrassing gap between theory and reality in multi-bid auction markets such as electricity wholesale markets. The leading equilibrium theory underpinning market analysis and the econometric estimation of strategic bidding behaviour in electricity auctions assumes that generating companies offer a piecewise differentiable supply function, specifying the amounts they are willing to supply at each price. The market operator aggregates these supplies and clears the market at the lowest price at which supply is equal to demand – the Market Clearing Price (MCP). Generators on this theory choose their offers by optimizing against the smooth residual demand, which gives well-defined first-order conditions. In reality, wholesale markets require offers to take the form of a step function, and the resulting residual demand facing any generator is also a step function, whose derivative is zero almost everywhere.

Faced with this, economists have chosen either to model the market as a discrete unit auction, which typically leads to complex mixed strategy equilibria, or have argued that with enough steps, the residual demand can be smoothed and then treated as differentiable. The difference between these approaches appears dramatic, and it is the purpose of this paper to demonstrate that in a well-defined sense it can be legitimate to approximate step-functions by smooth differentiable functions, and hence to draw on the well-developed theory associated with continuous supply functions.

To prove this result, we develop a new discrete model with stepped offer functions, which has a pure-strategy equilibrium that converges to the equilibrium of the limit game with continuous supply functions. To our knowledge we are the first to prove convergence of equilibria in multi-unit auctions to equilibria in divisible good auctions in this rigorous manner.

1.1 Modelling electricity markets

Electricity liberalization creates electricity markets. The two key markets that we wish to model are the day-ahead market and the balancing market (in the English Electricity Pool they were combined). In most such markets there is a separate auction for each delivery period, typically an hour. Normally, the post-2001 British balancing mechanism being an exception, the markets are organized as uniform price auctions. Thus all accepted bids and offers pay or are paid the market clearing price (MCP). Rationing of excess supply at the clearing price may be necessary and so market designs must specify how rationing will take place, normally by pro-rata on-the-margin rationing (Kremer and Nyborg, 2004a). Hence, only incremental supply at the clearing price is rationed and the accepted share of each producer's incremental supply at this price is proportional to the size of its increment.

Table 1: Constraints on the supply functions in various electricity markets.

Market	*Installed capacity*	*Max steps*	*Price range*	*Price tick size*	*Quantity multiple*	*No. quantities/ No. prices*
Nord Pool spot	90,000 *MW*	64 *per bidder*	0-5,000 *NOK/MWh*	0.1 *NOK/MWh*	0.1 *MWh*	18
ERCOT balancing	70,000 *MW*	40 *per bidder*	-*$1,000/MWh*- *$1,000/MWh*	*$0.01/MWh*	0.01 *MWh*	35
PJM	160,000 *MW*	10 *per genset*	*0-$1,000/MWh*	$0.01/*MWh*	0.01 *MWh*	160
UK (NETA)	80,000 *MW*	5 *per genset*	-*£9,999/MWh*- *£9,999/MWh*	£0.01/*MWh*	0.001 *MWh*	4
Spain Intra-day market	46,000 *MW*	5 *per genset*	*Yearly cap on revenues*	€0.01/*MWh*	0.1 *MWh*	—

Producers submit non-decreasing step function offers to the auction (and in some markets agents, normally retailers, may submit non-increasing demands). With its offer the producer states how much power it is willing to generate at each price. The Amsterdam Power Exchange (APX) provides a good example and the bid and offer ladders that determine the MCP can be readily downloaded. The successive offers specify a quantity that would be available at a fixed per unit price. The smallest step in the ladder is given by the number of allowed decimals in the offer. Thus all prices and quantities in an offer have to be a multiple of the price tick size and quantity multiple, respectively. Table 1 summarizes these and other offer constraints for some of the electricity markets in U.S. and Europe. In particular it is worth noting that most electricity markets have significantly more possible quantity levels compared with possible price levels. In that sense, the quantity multiple is small relative to the price tick size.

Offers are submitted ahead of time (typically the day before) and may have to be valid for an extended period (e.g. 48 half-hour periods in the English Pool) during which demand can vary significantly. Plant may fail suddenly, requiring replacement at short notice, so the residual demand (i.e. the total demand less the supply accepted at each price from other generators) may shift suddenly with an individual failure, again increasing the range over which offers are required.

Green and Newbery (1992) argued that the natural way to model such a market was to adapt Klemperer and Meyer's (1989) supply function equilibrium (SFE) formulation, in which firms make offers before the realization of demand is revealed. Units of electricity are assumed to be divisible, so firms offer continuous supply functions (SFs) to the auction. Accordingly, residual demand is piece-wise differentiable and firms have a well-defined piece-wise continuous marginal revenue, which offers the prospect of a well-defined best response function at each point. An equilibrium is such that each firm ensures that given the supplies offered by all other firms, it maximizes its profits for each realization of demand.

With a uniform price auction and a continuous SF the effect of lowering the price to capture the marginal unit lowers the price for the large quantity of inframarginal units (the 'price' effect) while only capturing an infinitesimal sale (the 'quantity' effect). The quantity effect is small if competitors' supply functions are close to inelastic and as a result very collusive supply function equilibria can be supported.

The first order conditions for the Nash equilibrium for each demand realization satisfy a set of linked differential equations. Analytical solutions can be found for the case of equal and constant marginal costs and linear marginal costs. Closed form solutions are also available for symmetric firms and perfectly inelastic demand (Rudkevich et al, 1998; Anderson and Philpott, 2002). The literature on numerical algorithms for finding SFE of markets with asymmetric firms and general cost functions (Holmberg, 2008; Anderson and Hu, 2008) is particularly relevant to our investigation. For example, numerical instabilities often arise in computation especially when mark-ups are small (Baldick and Hogan, 2002; Holmberg, 2008). Our analysis explains this observation as the relationship between the discrete and continuous cases relies on positive outputs, so that mark-ups are strictly positive.

Green and Newbery (1992) argued that the large number of possible steps meant that, given the uncertainty about, and variability of, demand, such steps could reasonably be approximated by continuous and piecewise differentiable functions. von der Fehr and Harbord (1993), however, argued that the ladders were step functions that were not continuously differentiable, and it would be inappropriate to assume that they were. Instead, they model the electricity market as a multiple-unit uniform-price auction in which each generating set submits a single bid from a continuum of prices (although in all existing electricity markets the set of prices is finite) for its entire capacity (supplies are chosen from a discrete set). With these assumptions, competition is almost everywhere in prices, with winner takes all over the whole step. Thus the 'price' effect, which can be made infinitesimally small in their model, of stealing some market is no longer larger than the now significant 'quantity' effect. If a producer is pivotal, i.e. competitors are not able to meet maximum demand without this producer, then such Bertrand competition often destroys any pure strategy equilibrium, leaving only a mixed-strategy equilibrium in which firms randomize over a distribution of possible prices. Choosing a mixed strategy in prices means that prices will be inherently volatile or unstable, even under unchanged conditions. Solving for such mixed strategy equilibrium is extremely difficult, so the result was destructive, in the sense that existing supply function models were claimed to be flawed but suitable auction models were intractable.

If one can show that the continuous SFE model is a valid approximation, this would justify the common practice in empirical work of smoothing the residual demand, allowing a well-defined best response to be identified. Three empirical studies applying this approach to the balancing market in Texas (ERCOT) suggest that a continuous representation is consistent with profit-maximizing behaviour for the largest producers in this market (Niu et al., 2005; Hortacsu and Puller, 2008; Sioshansi and Oren, 2007). Sweeting (2007) similarly estimates best responses to smoothed residual demand schedules in the English Electricity Pool to characterize the exercise of market power.

Wolak (2003) has used observed bidding behaviour to back out the unobserved underlying cost and contract positions of generators bidding into the Australian market. He notes continuity of the SF gives a one-to-one mapping between the shocks and the market price and hence the best response does not depend on the distribution of shocks. Wolak smooths the ex post observed stepped residual demand schedule to find the best response supply, which is then compared with the actual supply (chosen before the residual demand was realized). He notes, however, that, unlike the continuous approximation, the choice of an optimal step function will depend on the distribution of the shocks. Hence, discrete models might enhance accuracy in empirical work.

1.2 Reconciling step and continuous supply functions

The central question raised by the von der Fehr and Harbord critique and the empirical applications is whether smoothing and/or increasing the number of steps in the ladder can reconcile the step function and continuous approaches to modelling electricity markets. Do markets with uncertain or variable demand and sufficiently finely graduated bidding ladders converge to supply function equilibria, or do they remain resolutely and significantly different? The central claim of this paper is that under well-defined conditions, convergence can be assured, providing an intellectually solid basis for accepting the SFE approach. As such it marks a major step forward in the theory of supply function equilibria.

Fabra et al (2006) argue that the difference between the two approaches derives from the finite benefit of infinitesimal price undercutting in the ladder model. But this argument assumes that prices can be infinitely finely varied. In practice, the price tick size cannot be less than the smallest unit of account (e.g. 1 US cent, 1 pence, normally per MWh), and might be further restricted, as in the multi-round California PX auction. In this case, the undercutting strategy is not necessarily profitable, because the price reduction cannot be made arbitrarily small. Whereas von der Fehr and Harbord (1993) considered the extreme case when the set of quantities is finite and the set of prices is infinite, this paper considers the other extreme when the set of quantities is infinite and the set of prices is finite, consistent with the practice noted in Table 1.

We show that, with sufficiently many allowed steps in the bid curves, the step function and the market-clearing price (MCP) converge to the supply functions and price predicted by the SFE model. As in Dahlquist/Lax-Richtmyer's equivalence theorem (LeVeque, 2007), convergence of first-order solutions requires that the discrete system is consistent with the continuous system – the first-order conditions of the two systems converge - and that the discrete solution is stable, i.e. the difference between the two solutions does not grow too rapidly. Moreover, solutions should exist in both the discrete and continuous system. To get convergence of equilibria in the two systems, the converging first-order solution must in addition be global profit maxima in both systems. If a producer is pivotal, the stationary solutions with the lowest mark-ups will typically not be equilibria, because such solutions give pivotal producers incentives to withhold output until the capacity constraints of the competitors bind, so that the market price can be significantly increased. Disregarding such solutions and assuming concave demand, we prove that remaining monotonically increasing first-order solutions of the discrete and continuous systems are Nash equilibria if the number of price levels in the discrete system is larger than some sufficiently large finite number. Hence, convergence of the Nash equilibria follows straightforwardly from convergence of the first-order solutions. Note that partly decreasing offers are not in the strategy set, as such offers are not allowed in electricity markets.

Dahlquist/Lax-Richtmyer's equivalence theorem is a standard technique for analyzing the convergence of numerical methods, but it seems that we are the first to apply this method as a crucial step to prove convergence of Nash equilibria. Our existence and convergence result suggest that with a negligible quantity multiple and sufficiently many steps, the stepped supply functions are deterministic (and hence so is the price for each realization, cet. par.) and a continuous SFE is a valid approximation of bidding in such electricity auctions.

Our model has parallels in the theoretical work by Anderson and Xu (2004). They analyse a duopoly model of the Australian electricity market, where each of two producers first chooses and discloses its price grid and later its offers at each price. They assume demand is random but inelastic, with an elastic outside supply at some price, P, which effectively sets a price ceiling. In the uniform-price/single-price auction, they show that, under certain conditions, the second stage has a pure strategy equilibrium in quantities, although the first stage only has mixed strategies in the choice of prices. Their second stage has similarities with our model, because prices are chosen from discrete sets in both models. On the other hand, generators' chosen price vectors generally differ as the declared prices are chosen by randomizing over a continuous range of prices. In our paper, however, the available price levels are given by the market design and accordingly are the same for all firms. Moreover, Anderson and Xu (2004) do not compare their discrete equilibrium with a continuous SFE.

Wolak (2007), in a path-breaking empirical paper, develops a similar model of the Australian market to that of Anderson and Xu, but Wolak observes the step function bids, the contract positions and the market clearing prices, and hence is able to construct the *ex post* residual demand facing any generator. Wolak applies a standard kernel smoothing function to transform the step function residual demand into a smooth function satisfying various rate restrictions, which can be differentiated to derive the marginal revenue that should be equal to the marginal cost on the maintained assumption of profit maximization. This allows the cost function to be identified, and to test whether on average there is any evidence to reject the maintained hypothesis that the generator selects stepped bids to maximize profits, given its contract position. This approach avoids the problem facing the generator of deciding the set of prices at which to offer variable quantities, where the optimal choice is likely to be a mixed strategy. Given repeated observations it is possible to test whether on average the bids were profit maximizing, without having to solve for the pure or mixed strategy optimal bids. The same model is used by Gans and Wolak (2007) to assess the impact of vertical integration between a large electricity retailer and a large electricity generator in the Australian market.

Anderson and Hu (2008) develop a numerical method for solving asymmetric supply function equilibria. To achieve this they approximate equilibria of the continuous system with piece-wise linear supply functions and discretise the demand distribution. They show that equilibria of this approximation converge to equilibria in the original continuous model. The piece-wise linear bid functions are carefully chosen to avoid the influence of kinks in the residual demand curves. These approximate bid curves are drawn so that all producers have locally well-defined derivatives in their residual demand curves for all possible discrete demand realizations. Anderson and Hu's discrete model is motivated by its computational properties. In contrast, as in real electricity markets, we deal with the worst kinks possible, i.e. steps, and we do so explicitly, because we want to prove equilibrium convergence for a more problematic case where convergence has been disputed both empirically and theoretically.

Kastl (2008) analyzes divisible-good auctions with certain demand and private values, i.e. bidders have incomplete information. This set-up, introduced by Wilson (1979), is mainly used to analyze treasury auctions. Kastl considers both uniform-price and discriminatory auctions. He assumes that both quantities and prices are chosen from continuous sets, but the maximum number of steps is restricted. He verifies consistency, i.e. that the first-order condition (the Euler condition) of the stepped bid curve converges to the first-order condition of a continuous bid-curve when the number of steps becomes unbounded. But he does not verify stability, nor that solutions exists and globally maximize agents' profits in the discrete and continuous systems, which all are necessary conditions for the convergence of Nash equilibria in the discrete system to Nash equilibria in the continuous systems.

More generally, the convergence problem under study is related to the seminal paper by Dasgupta and Maskin (1986) on games with discontinuous profits. They show that if payoffs are discontinuous, then Nash equilibria (NE) in games with finite approximations of the strategy space of a limit game may not necessarily converge to NE of the limit game. Later Simon (1987) showed that convergence may depend on how the strategy space is approximated. This intuitively explains why NE in the model by von der Fehr and Harbord (1993), in which payoffs are discontinuous, do not necessarily converge to continuous SFE, and also why it is not surprising that NE in our discrete model, in which payoffs are continuous, converge to continuous SFE. However, Dasgupta and Maskin (1986) and Simon (1987) derive their convergence results for a limit game in which the strategy space has a finite dimension, whereas our limit game has infinitely many dimensions (a continuous supply function has infinitely many price/quantity pairs). Moreover, the purpose in Dasgupta and Maskin (1986), Simon (1987), and in related papers by Bagh (2010), Gatti (2005), and Remy (1999) is different to ours. They want to use existence of equilibria in discrete approximations in order to prove existence in the limit game, whereas our intention is the opposite – to show that existence of continuous SFE, i.e. in the limit game, implies existence of discrete NE which converge to the continuous SFE as the number of steps increases. Hence we use a proof strategy that is very different from theirs. We check whether a continuous NE is robust to discrete approximations, which is related to problems of numerical analysis.

2 THE MODEL AND ANALYSIS

Consider a uniform price auction, so that all accepted offers are paid the Market Clearing Price with any excess demand or supply at the MCP rationed pro-rata on-the-margin. We calculate a pure strategy Nash equilibrium of a one-shot game, in which each risk-neutral electricity producer or Generator, i, chooses a step supply function to maximize its expected profit, $E(\pi_i)$ in (1) below. There are M price levels, $p^j,\ j = 1, 2, \ldots M$, with the price tick $\Delta p^j = p^j - p^{j-1}$, for the most part assumed equal and then denoted by Δp. The minimum quantity increment is zero - quantities can be continuously varied.

Generator i ($i = 1, .., N$) submits a supply vector $\boldsymbol{s}_i$ consisting of non-negative maximum quantities $\{s_i^1, \ldots, s_i^M\}$ it is willing to produce at each price level $\{p^1, \ldots, p^M\}$. The step length $\Delta s_i^j = s_i^j - s_i^{j-1} \geq 0$: offers must be non-decreasing in price and bounded above by the capacity $\overline{s}_i$ of Generator i. Let $\boldsymbol{s} = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N\}$ and denote competitors' collective quantity offers at price p^j as s_{-i}^j and the total market offer as s^j. In the continuous model the set of individual supply functions is $\{s_i(p)\}_{i=1}^N$. The cost function of firm i, $C_i(s_i)$, is an increasing, convex and twice continuously differentiable function up to the capacity constraint $\overline{s}_i$. Costs are common

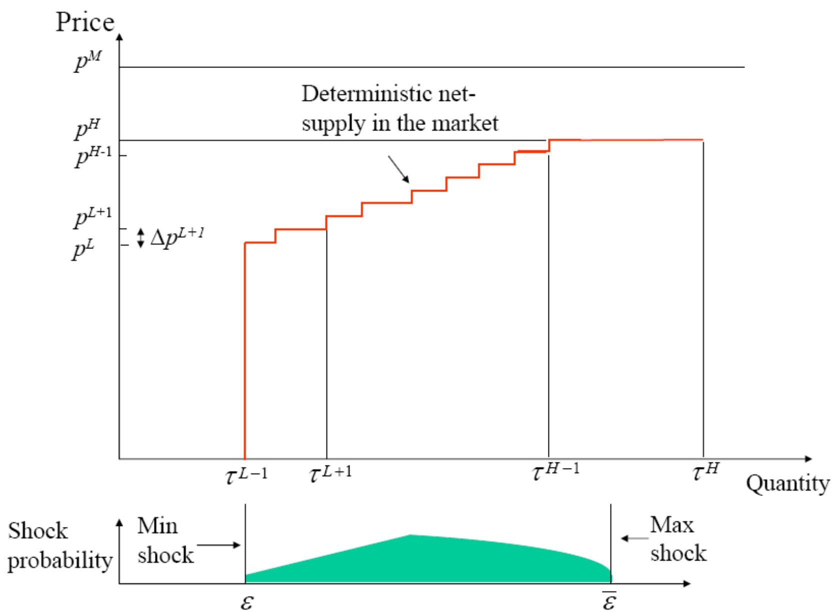

Figure 1: Stepped supply, demand shocks and key price levels.

knowledge.

Electricity consumers are non-strategic. Their demand is stepped and the minimum demand at each price is $d^j + \varepsilon$, where ε is an additive demand shock. Decremental demand is $\Delta d^j = d^j - d^{j-1} \leq 0$, with $\Delta d^j \geq \Delta d^{j+1}$, corresponding to a continuously differentiable concave deterministic demand curve, $d(p)$, in the continuous case. The latter is such that $\lim_{\Delta p^j \to 0} \frac{\Delta d^j}{\Delta p^j} = d'\left(p^j\right)$ and $\lim_{\Delta p^j \to 0} d^j = d\left(p^j\right)$. Note that Δp^j is a local tick-size and that other tick-sizes Δp^k are fixed when these limits are calculated, so that p^j is fixed and $p^{j-1} \to p^j$. The additive demand shock has a continuous probability density, $g(\varepsilon)$, which is strictly positive on its support $[\underline{\varepsilon}, \overline{\varepsilon}]$.

Let $\tau^j = s^j - d^j$ be the deterministic part of total net supply (excluding the stochastic shock) at price p^j, and define the increase in net supply from a positive increment in price as $\Delta\tau^j = \tau^j - \tau^{j-1}$. Similarly, the residual deterministic net supply is $\tau^j_{-i} = s^j_{-i} - d^j$ and its increase is $\Delta\tau^j_{-i} = \tau^j_{-i} - \tau^{j-1}_{-i}$.

The Market Clearing Price (MCP) is the lowest price at which the deterministic net-supply equals the stochastic demand shock. Thus the equilibrium price as a function of the demand shock is left continuous, and the MCP equals p^j if $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$. Given chosen step functions, the market clearing price can be calculated for each demand shock in the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. The lowest and highest prices that are realized are denoted by p^L and p^H, respectively, where $1 \leq L < H \leq M$. Both depend on the available number of price levels, M, as well as the boundary conditions, and these various price levels and the demand shocks are shown in Fig. 1. The lowest and highest realized prices in the corresponding continuous system are a and b respectively.

Given all players' chosen strategies, we can write the clearing price, $p(\varepsilon)$, and producer i's accepted output, $s_i(\varepsilon)$, as functions of the demand shock, so that

$$E(\pi_i) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} [p(\varepsilon)\, s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon. \tag{1}$$

With pro-rata on-the-margin rationing, all supply offers below the MCP, p^j, are accepted, while offers at p^j are rationed pro-rata. Thus for $\varepsilon \in (\tau^{j-1}, \tau^j]$, $\varepsilon - \tau^{j-1}$ is excess demand at p^{j-1}, so the accepted supply of a generator i is given by:

$$s_i(\varepsilon) = s_i^{j-1} + \frac{\Delta s_i^j (\varepsilon - \tau^{j-1})}{\Delta \tau^j} = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{\Delta \tau^j}, \tag{2}$$

(making use of the fact that $\tau^j = \tau_{-i}^j + s_i^j$ and $\Delta\tau^j = \Delta\tau_{-i}^j + \Delta s_i^j$).

2.1 First-order conditions

The contribution to the expected profit of generator i from realizations $\varepsilon \in (\tau^{j-1}, \tau^j]$ is:

$$\begin{aligned} E_i^j = & \int_{\tau^{j-1}}^{\tau^j} \left[p^j s_i(\varepsilon) - C_i(s_i(\varepsilon))\right] g(\varepsilon)\, d\varepsilon = \\ & \int_{s_i^{j-1}+\tau_{-i}^{j-1}}^{s_i^j+\tau_{-i}^j} \left[p^j \left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon-\tau^{j-1})}{\Delta\tau^j}\right) - C_i\left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon-\tau^{j-1})}{\Delta\tau^j}\right)\right] g(\varepsilon)\, d\varepsilon, \end{aligned} \tag{3}$$

where again $\tau^j = \tau_{-i}^j + s_i^j$. Generator i's total expected profit is

$$E(\pi_i(\mathbf{s})) = \sum_{j=1}^{M} E_i^j \left(s_i^j, s_i^{j-1}\right). \tag{4}$$

The Nash equilibrium is found by deriving the best response of each firm given its competitors' chosen stepped supply functions. The first order conditions are found by differentiating the expected profit in (4). Proposition 1 characterizes these first order conditions over the range of possible intersections of aggregate supply with demand (i.e. over the range on which it has positive probability). All proofs are given in the appendix.

Proposition 1 *With stepped supply function offers, $\Gamma_i^j(\mathbf{s}) = \partial E(\pi_i(\mathbf{s})) / \partial s_i^j$ is always well-defined, and the first-order condition for the supply of firm i at a price level j, such that $\underline{\varepsilon} \leq \tau^j \leq \overline{\varepsilon}$, is given by:*

$$\begin{aligned} 0 = \Gamma_i^j(\mathbf{s}) = \frac{\partial E(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p^{j+1} s_i^j g(\tau^j) + \int_{\tau^{j-1}}^{\tau^j} \left[p^j - C_i'(s_i(\varepsilon))\right] \frac{\Delta\tau_{-i}^j(\varepsilon-\tau^{j-1})}{(\Delta\tau^j)^2} g(\varepsilon)\, d\varepsilon \\ + \int_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'(s_i(\varepsilon))\right] \frac{\Delta\tau_{-i}^{j+1}(\tau^{j+1}-\varepsilon)}{(\Delta\tau^{j+1})^2} g(\varepsilon)\, d\varepsilon, \end{aligned} \tag{5}$$

where $s_i(\varepsilon)$ is given by (2) if $\varepsilon \in [\tau^{j-1}, \tau^j]$.

The first point to note, *pace* Dasgupta and Maskin's (1986) result for games with discontinuous profits, is that expected profits $E(\pi_i(\mathbf{s}))$ are differentiable. Thus expected profit is continuous in the strategy variables, and convergence should be less problematic. The first-order condition can be intuitively interpreted as follows. When calculating $\Gamma_i^j(\mathbf{s}) = \partial E(\pi_i(\mathbf{s}))/\partial s_i^j$, supply is increased at p^j, while holding the supply at all other price levels constant. This implies that the offer price of one (infinitesimally small) unit of power is decreased from p^{j+1} to p^j. This decreases the MCP for the event when the unit is price-setting, i.e. when $\varepsilon = \tau^j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition. On the other hand, because of the rationing mechanism, decreasing the price of one unit (weakly) increases the accepted supply for demand outcomes $\varepsilon \in \left(\tau^{j-1}, \tau^{j+1}\right]$. This brings a positive contribution to the expected profit, which corresponds to the two integrals in the first-order condition. The first integral represents $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$ when the MCP is p^j, and the other integral represents $\varepsilon \in \left(\tau^j, \tau^{j+1}\right]$ when the MCP is p^{j+1}.

The first-order condition in Proposition 1 is not directly applicable to parts of the offer curve that are always or never accepted in equilibrium. The appendix shows that, because of pro-rata rationing, a producer's profit is maximized if offers that are never accepted are offered with a perfectly elastic supply (until the capacity constraint binds) at p^H, so that $s_i^H = \overline{s}_i$, and offers that are always accepted are offered below p^L. In particular, we assume that

$$s_i^j = s_i^{L-1} \text{ if } j < L, \tag{6}$$

because this offer curve discourages NE deviations that undercut the price level p^L, and is accordingly most supportive of an NE. In summary, equilibrium supply is constant for $p < p^L$, satisfies (5) for $p \in [p^L,\ p^H)$ and jumps to $\overline{s}_i$ at p^H.

Note that the difference equation in (5) is of the second-order. Thus solutions, should they exist, would be indexed by two boundary conditions that could appear in a variety of forms, e.g., initial and final (boundary) values or, as here, two boundary values at the upper end of the interval. As argued above, one of the boundary conditions is pinned down by the capacity constraint $s_i^H = \overline{s}_i$. This leaves each firm with one remaining free parameter, s_i^{H-1}, that will be tied down with a second boundary condition, $s_i^{H-1} = \widehat{k}_i$, for some constant $\widehat{k}_i$. This latter condition corresponds to the single boundary condition needed for the continuous case, presented shortly. Definition 1 gives the notation for a set of discrete solutions, meaning a list of simultaneous solutions, one for each player i and price level p^j.

Definition 1 *By* $\left\{\left\{\widehat{s}_i^j\right\}_{j=L}^{j=H}\right\}_{i=1}^N$ *or* $\left\{\widehat{s}_i^j\right\}_{L,1}^{H,N}$ *we denote a set of solutions to the system of difference equations (5) given two boundary conditions* $\widehat{s}_i^H = \overline{s}_i$ *and* $\widehat{s}_i^{H-1} = \widehat{k}_i$*, for some constant* $\widehat{k}_i$*. We call this a discrete stationary solution and say this set is a segment of a discrete SFE if* *the set of strategies* $\left\{\widehat{s}_i^j\right\}_{L,1}^{H,N}$ *formed by taking* $s_i^j = \widehat{s}_i^{L-1}$ *if* $j < L$, $s_i^j = \widehat{s}_i^j$ *if* $L \leq j \leq H$, *and* $s_i^j = \overline{s}_i$ *if* $j > H$ *is an SFE for the discrete game.*

Section 2.2 studies convergence of first-order solutions of the discrete system to first-order solutions of the continuous system. The system of first-order conditions in the continuous case is given by Klemperer and Meyer (1989):

$$-s_i(p) + [p - C_i'(s_i(p))](s_{-i}'(p) - d'(p)) = 0. \tag{7}$$

This system has one degree of freedom, and hence an infinite number of potential solutions. As shown by Baldick and Hogan (2001), the system of differential equations can be written in the standard form of an ordinary differential equation (ODE):

$$s_i'(p) = \frac{d'(p)}{N-1} - \frac{s_i(p)}{p - C_i'(s_i(p))} + \frac{1}{N-1}\sum_k \frac{s_k(p)}{p - C_k'(s_k(p))}. \tag{8}$$

We can therefore index the continuum of continuous SFE by boundary conditions $s_i(b) = \breve{k}_i$. In Section 2.2, we will link the discrete and continuous boundary conditions by requiring $\lim_{\Delta p \to 0} \widehat{k}_i = \breve{k}_i$, where we note that $\widehat{k}_i$ depends on Δp or, equivalently, on M.

The shape of the offer curves in the never-price-setting region of the continuous system is the same as for the discrete system; bids that are always accepted are perfectly inelastic and bids that are never accepted are perfectly elastic. This shape discourages competitors from deviating from a potential NE, and is accordingly most supportive of an NE:

$$s_i(p) = s_i(a) \text{ if } p < a, \text{ and } s_i(p) = \overline{s}_i \text{ if } p > b. \tag{9}$$

The next definition provides the notation for solutions to the continuous system.

Definition 2 *By* $\left\{\breve{s}_i(p)\right\}_{i=1}^N$ *we denote a set of continuous solutions to the system of the differential equations* (8) *on the interval* $[a, b]$ *with boundary conditions* $\breve{s}_i(b) = \breve{k}_i$. *We call this a continuous stationary solution. We say* $\left\{\breve{s}_i(p)\right\}_{i=1}^N$ *is a segment of a continuous SFE if the set of strategies* $\{s_i(p)\}_{i=1}^N$ *formed by taking* $s_i(p) = \breve{s}_i(a)$ *if* $p < a$, $s_i(p) = \breve{s}_i(p)$ *if* $p \in [a, b]$, *and* $s_i(p) = \overline{s}_i$ *if* $p > b$, *is an SFE.*

2.2 Convergence of stationary solutions

This section states (and the appendix proves) that for a market for which differentiable solutions to equations in (7) exist, there also exists a discrete stationary solution that converges to the continuous solution as $\Delta p \to 0$, which is non-obvious given the results in von der Fehr and

Harbord (1993). Note that the existence of smooth solutions has been established for a broad class of cases, see e.g. Klemperer and Meyer (1989), so their existence are not disputed. The steps in the convergence proof are related to the steps in the proof of Dahlquist's equivalence theorem[1] for discrete approximations of ODEs (LeVeque, 2007). It is not standard to approximate ODEs by systems that are both non-linear and implicit (since solving an approximating system then requires an iterative procedure at each step of the integration). Nevertheless our convergence proof has to deal with systems of difference equations in Proposition 1 that are implicit and non-linear; we extend the framework of LeVeque (2007) for this purpose. To facilitate the application of approximation theory for ODEs, we restrict attention to the cases defined by:

Assumption 1. *Bounded, increasing and differentiable stationary solutions* $\left\{\breve{s}_i(p)\right\}_{i=1}^{N}$ *of* (8) *exist on the interval* $[a, b]$.

In order to prove that all producers have positive mark-ups, we first show that all producers have positive outputs. Let n be the producer with the highest marginal cost at zero output, $C_n'(0)$. The Klemperer and Meyer equation (7) implies that its output can only be non-positive if the market clears at a price at or below its marginal cost. From the same equation we see that competitors, whose marginal costs at zero output are no higher than $C_n'(0)$, will offer positive outputs with positive mark-ups. Now, if the lowest demand curve crosses their marginal cost curve at a price above $C_n'(0)$ then producer n's output must be positive at this outcome. This is illustrated in Fig. 2, and motivates the following assumption on costs and shocks.

Assumption 2. *Let* n *be the producer with the highest marginal cost at zero output. The lowest shock is such that* $C_{-n}'\left(\underline{\varepsilon} + d\left(C_n'(0)\right)\right) > C_n'(0)$, *where* $C_{-n}(s_{-n})$ *is the minimum (efficient) production cost of competitors producing* s_{-n} *units.*

In case there are several such producers, the assumption is satisfied for all producers that have the same highest marginal cost at zero output. Given this assumption and our assumptions on the shock density, it is straightforward to prove the following:

Lemma 1 *If Assumptions 1 and 2 are satisfied, then the mark-up,* $p - C_i'\left(\breve{s}_i(p)\right)$, *for continuous stationary solutions is bounded below by a positive constant that is independent of* i *and* $p \in [a, b]$.

Now we present the proof strategy and the convergence result; technicalities are relegated to the appendix. Our task is to relate continuous solutions to solutions of the discrete system (5). The first step in proving convergence of stationary solutions is to verify that the discrete system of stationary conditions in Proposition 1 is consistent with the stationary conditions for

[1] The more general Lax-Richtmyer equivalence theorem applies to partial differential equations.

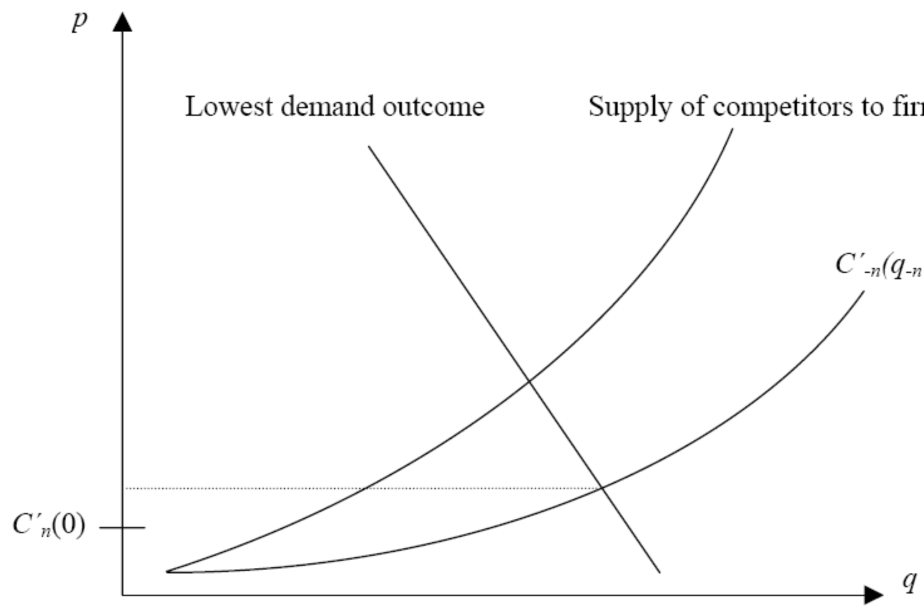

Figure 2: To ensure that producer n, who has the highest marginal cost at zero output, always has a positive output, we assume that the competitors' marginal cost curve crosses the lowest demand curve at a price strictly above $C_n'(0)$.

continuous SFE written as the ODE (8). Lemma 5 of the appendix shows this to be the case. That the discrete system is a consistent approximation of the continuous one implies the former set of equations converges to the latter as the number of price steps M goes to infinity. Thus as $M \to \infty$, the second-order difference equation in (5) converges to a differential equation of the first-order, which corresponds to the Klemperer and Meyer equation (7). If we let the error be the difference between the continuous and discrete solution, this ensures that the error grows at a slower rate per step as the number of steps becomes larger. However, this does not ensure that a discrete stationary solution will exist nor, if it does, that it will converge to the continuous solution, because at the same time the number of steps between p^L and p^H increases. Thus even if the discrete system is consistent with the continuous one, the error could explode when the number of steps becomes large – the unstable case. Hence the second step in the convergence analysis is to establish existence and stability.

Lemma 4 in the appendix shows that a solution at the price level $H-2$ (the first step) is ensured, if the boundary conditions $\left\{s_i^{H-1} = \widehat{k}_i\right\}_{i=1}^N$ and $\left\{s_i^H = \bar{s}_i\right\}_{i=1}^N$ satisfy:

$$\begin{aligned} 0 &< \Delta p s_i^{H-1} g\left(\tau^{H-1}\right) - \int_{\tau^{H-1}}^{\bar{\varepsilon}} \left[p^H - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \Delta\tau_{-i}^H \frac{\left(\tau^H - \varepsilon\right)}{\left(\Delta\tau^H\right)^2} g\left(\varepsilon\right) d\varepsilon \\ &< \min\left[\frac{3\left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right]^2 g_{\min}^2}{2C''_{\max} g_{\max}}, \frac{3\left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right]^2 g_{\min}^2}{2\left(C''_{\max} g_{\max} + \left[p^{H-1} - C_i'\left(0\right)\right] g'_{\max}\right)}\right], \end{aligned} \tag{10}$$

where Δp is the uniform tick-size, $C''_{\max}$ is the highest slope of the marginal cost curves, $g_{\max}$ and $g_{\min}$ are the largest and smallest densities in the support of $g\left(\varepsilon\right)$, and $g'_{\max}$ is the largest slope of the density function in the support of $g\left(\varepsilon\right)$.[2] Recall that we have assumed $g_{\min}$ to be strictly positive. With strictly positive mark-ups, the term on the right is always bounded from below by some positive constant, independent of Δp. Hence, we realize that the condition is always satisfied for some large but finite M (and small Δp) if we choose $\widehat{k}_i$ sufficiently close to the continuous solution. For example, the integral is zero when $\left\{\widehat{k}_i\right\}_{i=1}^N$ are chosen such that $\tau^{H-1} = \bar{\varepsilon}$.

Provided the inequality above is satisfied, Proposition 2 states that the discrete stationary solution exists and is stable for $M \geq M_o$, which is a sufficiently large (but finite) integer number. Moreover, the proposition shows that the solution converges to the continuous stationary solution as $M \to \infty$. Recall that p^L and p^H are the lowest and highest realized prices, and that the indices L and H vary with M (and the boundary conditions).

Proposition 2 *Make Assumption 2 and let* $\left\{\breve{s}_i\left(p\right)\right\}_{i=1}^N$ *be a continuous stationary solution on the interval* $[a,b]$ *that satisfies Assumption 1. Consider the discrete stationary system of difference equations (in Proposition 1) with* $\Delta p^j = \Delta p$ *and boundary conditions* $\widehat{s}_i^{H-1} = \widehat{k}_i$ *and* $\widehat{s}_i^H = \bar{s}_i$ *satisfying (10) for each* i. *If as* $M \to \infty$ *we have* $p^H \to b$ *and* $\widehat{k}_i$ *converges to* $\breve{k}_i = \breve{s}_i\left(b\right)$, *then for* $M \geq M_o$ *there exists a unique discrete solution* $\left\{\widehat{s}_i^j\right\}_{i=1}^N$. *As the number of steps grows* ($M \to \infty$), $\left\{\widehat{s}_i^j\right\}_{i=1}^N$ *converges to* $\left\{\breve{s}_i\left(p\right)\right\}_{i=1}^N$ *in the interval* $[a,b]$.

The meaning of convergence in this result is that if j is chosen to depend on M such that $p^j \to p \in [a,b]$ as $M \to \infty$, then $\widehat{s}_i^j \to \widehat{s}_i(p)$ as $M \to \infty$ for each i.

As an illustration of the discrepancy between consistency and convergence, the following can be noted. To prove consistency in our model it would have been enough to make Assumption 1, which would allow for zero mark-ups when supply is zero. However, the error grows at an infinite rate when the mark-up is zero at zero supply, so the continuous and discrete stationary solutions do not necessarily converge at this point. That is why we make the additional Assumption 2,

[2] We set $g'_{\max} = 0$, in case $g' < 0$ for all ε.

which ensures that output is always positive for all producers. This is related to the instability near zero supply that has been observed when continuous SFE are calculated by means of standard numerical integration methods (Baldick and Hogan, 2002; Holmberg, 2008).

2.3 Sufficient conditions

Here we show that a *non-decreasing* solution of the continuous stationary conditions, presented above, must be an SFE if Assumption 3 below is satisfied. That is, the non-decreasing condition acts rather like a second-order condition in ensuring sufficiency. Lemma 2 gives a corresponding result for the discrete system. These results are of independent interest. For example, Proposition 3, on the sufficiency in the continuous case, extends Klemperer and Meyer's (1989) sufficiency result for symmetric producers. We also generalize Klemperer and Meyer (1989) by considering capacity constraints. But this comes at the price of additional complexity, because competitors' capacity constraints introduce kinks and non-concavities in the pay-off function of a producer. These kinks will start to influence the range of possible equilibria when a producer is pivotal at price $p = b$, i.e. competitors' total production capacity is not sufficient to meet market demand at this price (Green and Newbery, 1992; Baldick and Hogan, 2002; Genc and Reynolds, 2004; Holmberg, 2007; Anderson and Hu, 2008). Pivotal producers will find it profitable to deviate from the stationary solutions with the lowest mark-ups by withholding output to make competitors' capacities bind. For example, the Bertrand equilibria in the model by von der Fehr and Harbord (1993) can be ruled out as soon as one firm is pivotal. To rule out boundary conditions with too low mark-ups for a pivotal producer, i.e. b is too low, we make

Assumption 3.

$$b\breve{k}_i - C_i\left(\breve{k}_i\right) \geq p_d\left[\bar{\varepsilon} + d\left(p_d\right) - \bar{s}_{-i}\right] - C_i\left[\bar{\varepsilon} + d\left(p_d\right) - \bar{s}_{-i}\right] \ \forall p_d \in \left(b, p^M\right].$$

Note that the left-hand side of the inequality is the profit at the boundary condition $p = b$. The right-hand side is the profit when supply is with-held until competitors' capacities bind, so that the price can be increased, $p_d \in \left(b, p^M\right]$. If the assumption is not satisfied then there will always be some shock density $f(\varepsilon)$ (with sufficient probability mass near $\bar{\varepsilon}$), such that $\left\{\breve{s}_i\left(p\right)\right\}_{i=1}^{N}$ with $\breve{s}_i\left(b\right) = \breve{k}_i$ is not a segment of an SFE. Also note that Assumption 3 is always satisfied if b is sufficiently close to the price cap p^M or if producers are non-pivotal. See Genc and Reynolds (2004) for a more detailed analysis of pivotal producer's influence on the existence of SFE. Given Assumption 3 it is straightforward to verify that the stationary solution will have sufficient mark-ups to deter deviations with $p_d > b$ by any pivotal producer for any shock outcome. (See Lemma 6 in the Appendix.). Proposition 3 below rules out deviations resulting in prices $p \leq b$. This is relevant for both pivotal and non-pivotal producers, to prove that non-decreasing, continuous stationary solutions satisfying Assumption 3 are Nash equilibria.[3]

Proposition 3 *Let Assumption 3 hold. If each* $\breve{s}_i(p)$ *is non-decreasing on* $[a,b]$ *then* $\left\{\breve{s}_i(p)\right\}_{i=1}^{N}$ *is a segment of a continuous SFE.*

The next lemma gives a corresponding sufficiency condition for discrete equilibria. The lemma relies on the requirement that the discrete offers are monotonic and that no producer is sufficiently pivotal to find it profitable to increase the price above p^H by withholding production, so that $s_i^H < \bar{\varepsilon} + d(p^H) - \bar{s}_{-i}$ (because $\widehat{s}_{-i}^H = \bar{s}_{-i}$). In Section 2.4 we will prove that these two conditions follow from the assumed properties of the continuous SFE and the proof that the discrete stationary solution converges to the continuous stationary solution. Recall that $\Delta d^j \geq \Delta d^{j+1}$, corresponding to concave demand.

Lemma 2 *Consider a set* $\left\{\widehat{s}_i^j\right\}_{L,1}^{H,N}$ *of discrete stationary solutions to the system of difference equations in (5) under the usual boundary conditions* $\widehat{s}_i^H = \bar{s}_i$ *and* $\widehat{s}_i^{H-1} = \widehat{k}_i$*. Let* $\Delta p^j = \Delta p$ *and suppose that* $p^j - C_i'(s_i^j) > 0$ *for all price levels* $L \leq j \leq H-1$ *and each* $i = 1, \ldots, N$*. If the strategy* $\left\{\widehat{s}_i^j\right\}_{L}^{H}$ *is non-decreasing for each generator* i*, then* $\left\{\widehat{s}_i^j\right\}_{L,1}^{H,N}$ *is a segment of a discrete SFE for* $M \geq M_1$*, unless there are profitable deviations such that* $s_i^H < \bar{\varepsilon} + d(p^H) - \bar{s}_{-i}$*. In the case with uniformly distributed demand we have* $M_1 = 3$*.*

M_1 is a sufficiently large (but finite) integer number. $M_1 = 3$ is the smallest number of price levels in our model.

2.4 Convergence of discrete and continuous SFE

This section states (and the appendix proves) the central result of the paper: that for a market for which a continuous SFE exists, a discrete SFE also exists and converges to the continuous SFE as $\Delta p \to 0$. Section 2.2 proved convergence of stationary solutions, using techniques normally applied to ODEs. We now depart from the theory of ODEs in order to prove convergence of the equilibria themselves. Fortunately this turns out to follow relatively easily from convergence of the stationary solutions. We use the observation that a stationary solution of the continuous system is actually a Nash equilibrium strategy if it is increasing in price: see Proposition 3. The convergence of the discrete stationary solution to the continuous one, proved in Proposition

[3] In the proof we use the assumption that demand is concave. This is to avoid undercutting incentives at the price a, but if a would equal producers' marginal cost at zero output (i.e. Assumption 2 is not satisfied) as in Klemperer and Meyer (1989) then concave demand is not needed to prove our general equilibrium result.

2, ensures that the discrete stationary solution inherits three important properties from the continuous stationary solution: 1) the discrete stationary solution is increasing, 2) mark-ups are strictly positive, and 3) the stationary solution has sufficiently high mark-ups that any pivotal producer will not wish to deviate to a price $p_d > p^H \to b$. According to Lemma 2, these properties are enough to ensure that the discrete stationary solution is a Nash equilibrium, and the proof of Theorem 1 is complete.

Theorem 1 *Let Assumptions 1-3 hold, then:*

a) $\left\{ \breve{s}_i(p) \right\}_{i=1}^{N}$ *is a segment of a continuous SFE.*

b) In addition, suppose $\Delta p^j = \Delta p$ *and that boundary conditions* $\widehat{s}_i^{H-1} = \widehat{k}_i$ *and* $\widehat{s}_i^H = \overline{s}_i$ *satisfy (10) for each* i. *If as* $M \to \infty$ *we have* $p^H \to b$ *and* $\widehat{k}_i$ *converges to* $\breve{k}_i = \breve{s}_i(b)$, *then for* $M \geq M_2$ *there exists a unique discrete solution* $\left\{ \widehat{s}_i^j \right\}_{i=1}^{N}$ *that is a segment of a discrete SFE. As the number of steps grows* ($M \to \infty$), $\left\{ \widehat{s}_i^j \right\}_{i=1}^{N}$ *converges to* $\left\{ \breve{s}_i(p) \right\}$ *in the interval* $[a, b]$.

M_2 is a sufficiently large (but finite) integer number. Recall that p^L and p^H are the lowest and highest realized prices, and that the indices L and H vary with M (and the boundary conditions). The meaning of convergence in this result is that if j is chosen to depend on M such that $p^j \to p \in [a, b]$ as $M \to \infty$, then $\widehat{s}_i^j \to \widehat{s}_i(p)$ as $M \to \infty$ for each i. Note that the convergence result is valid for general convex cost functions, asymmetric producers and general probability distributions of the demand shock. From Proposition 1 we know that the latter influences the discrete difference equation for a finite number of steps, but apparently this dependence disappears in the limit when the discrete solution converges to the continuous one, which does not depend on the shock density.

One implication of Theorem 1 is that with a sufficient number of finite steps, existence of discrete SFE is ensured if a corresponding continuous SFE exists. As an example, Klemperer and Meyer (1989) establish the existence of continuous SF equilibria if firms are symmetric, ε has strictly positive density everywhere on its support $[\underline{\varepsilon}, \overline{\varepsilon}]$, the cost function is C_2 and convex, and the demand function $D(p, \varepsilon)$ is C_2, concave and with a negative first derivative. Thus with a sufficient number of finite steps, discrete SFE will exist under those circumstances as well.

In the study of SFEs there is little work that relates discrete games to their continuous counterparts by convergence analysis. Anderson and Hu (2008) discretise a continuous SFE system in order to get a numerically convenient discrete system with straightforward convergence to the continuous solution. This is a valuable numerical scheme for approximating continuous SFE. By contrast, we start with a class of self-contained discrete games with relevance to actual electricity markets and demonstrate both existence and convergence of SFE for the discrete system to those of the continuous system. This is a hitherto missing bridge from continuous SFE theory to discrete SFE practice.

Appendix Proposition 4 reverses the implication of Theorem 1 to show that if a discrete stationary solution is non-decreasing and converges to a set of smooth functions (one per player) with positive mark-ups, then the limiting set of functions is a continuous SFE. That is, the family of increasing smooth SFE with positive mark-ups is asymptotically in one-to-one correspondence with the family of corresponding well-behaved discrete SFE. This is in itself a useful contribution to existence results for continuous SFEs.

3 EXAMPLE

Consider a market with two symmetric firms that have infinite production capacity. Each producer has linear increasing marginal costs $C_i' = s_i$. Demand at each price level is by assumption given by $d\left(p^j, \varepsilon\right) = \varepsilon - 0.5p^j$. The demand shock, ε, is assumed to be uniformly distributed on the interval $[1.5, 3.5]$, i.e. $g(\varepsilon) = 0.5$ in this range.

In the continuous case, there is a continuum of symmetric stationary solutions to the differential equation in (7). The chosen solution depends on the end-condition. Klemperer and Meyer (1989) and Green and Newbery (1992) show that in the continuous case, the symmetric solution slopes upwards between the marginal cost curve and the Cournot schedule, while it slopes downwards (or backwards) outside this wedge. The Cournot schedule is the set of Cournot solutions that would result for all possible realizations of the demand shock, and the continuous SFE is vertical at this line (with price on the y-axis). In the other extreme, when price equals marginal cost the solution becomes horizontal. Infinite production capacities ensure that Assumption 3 is satisfied and in this case a continuous symmetric solution constitutes an SFE if and only if the solution is within the wedge for all realized prices. Fig. 3 plots the most and least competitive continuous SFE. All solutions of the differential equations (7) or (8) in-between the most and least competitive continuous cases are also continuous SFE.[4]

For the marginal cost and demand curves assumed in this example, the difference equation in Proposition 1 can be simplified to:

$$-\Delta p s_i^j + \frac{1}{2}\left(p^j - \frac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\Delta\tau_{-i}^j + \frac{1}{2}\left(p^{j+1} - \frac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\Delta\tau_{-i}^{j+1} = 0. \tag{11}$$

In a symmetric duopoly equilibrium with $\Delta d = -0.5\Delta p$, $\Delta\tau_{-i}^j = s_i^j - s_i^{j-1} + 0.5\Delta p$. Thus the first-order condition can be written:

$$\begin{aligned}&-\Delta p s_i^j + \tfrac{1}{2}\left(p^j - \tfrac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\left(s_i^j - s_i^{j-1} + 0.5\Delta p\right) + \\ &\tfrac{1}{2}\left(p^{j+1} - \tfrac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\left(s_i^{j+1} - s_i^j + 0.5\Delta p\right) = 0.\end{aligned}$$

[4] The dotted continuous SFs are very close to the stepped SF and for the most competitive case are essentially indistinguishable.

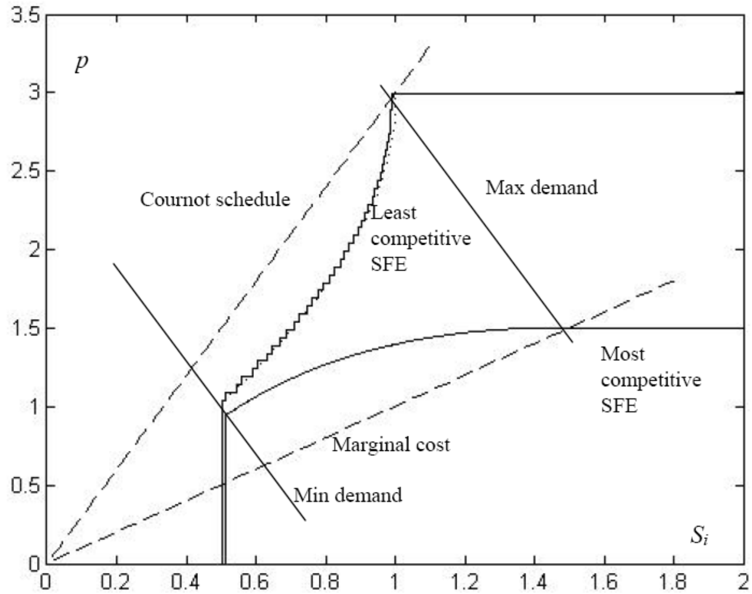

Figure 3: The most and least competitive continuous SFE (dotted) and their discrete approximations (solid). The discrete approximations have a tick-size of $\Delta p = 0.05$ (non-competitive case) and $\Delta p = 0.001$ (competitive case).

In Fig. 3 the discrete stationary solutions are plotted. As producers are non-pivotal and demand is uniformly distributed, it follows from Lemma 2 that these solutions are discrete SFE, and so are all discrete non-decreasing stationary solutions in-between them. Our experience is that we need a much smaller tick-size in the most competitive case compared to the least competitive case in order to get a monotonic solution. We believe that it is related to that convergence is poorer when mark-ups are small due to the singularity at zero mark-ups.

4 CONCLUDING REMARKS

Green and Newbery (1992), and Newbery (1998) assume that the allowed number of steps in the supply function bids of electricity auctions is so large that equilibrium bids can be approximated by continuous SFE. This is a very attractive assumption, because it implies that a pure-strategy equilibrium can be calculated analytically for simple cases and numerically for general cost functions and asymmetric producers. The pure-strategy equilibrium that has inherently stable prices also justifies empirical models of strategic bidding in electricity auctions, such as Wolak (2003) who is able to deduce contract positions, marginal costs and the price-cost mark-up from observed bids.

von der Fehr and Harbord (1993), however, argue that as long as the number of steps is finite, then continuous SFE are not a valid representation of bidding in electricity auctions. Under the extreme assumption that prices can be chosen from a continuous distribution so that the price tick size is negligible, von der Fehr and Harbord (1993) show that uniform price electricity auctions have an inherent price instability. If demand variation is sufficiently large, so that no producer is pivotal at minimum demand and at least one firm is pivotal at maximum demand, then there are no pure strategy NE, only mixed strategy NE. The intuition behind the non-existence of pure strategy NE is that producers slightly undercut each other's step bids until mark-ups are zero. Whenever producers are pivotal they have profitable deviations from such an outcome.

We claim that the von der Fehr and Harbord result is not driven by the stepped form of the supply functions, but rather by their discreteness assumption. We consider the other extreme in which the price tick size is significant and the quantity multiple is negligible. We show that in this case step equilibria converge to continuous supply function equilibria. The intuition for the existence of pure strategy equilibria is that with a significant price tick size, it is not necessarily profitable to undercut perfectly elastic segments in competitors' bids.

Our results imply that the concern that electricity auctions have an inherent price instability and that they cannot be modelled by continuous SFE is not necessarily correct. We also claim that this potential problem can be avoided if tick sizes are such that the number of price levels is small compared to the number of possible quantity levels, which is the case in many electricity markets. To avoid price instability, we also recommend that restrictions in the number of steps should be as lax as possible, even if some restrictions are probably administratively necessary. Restricting the number of steps increases each producer's incremental supply offered at each step, encouraging price randomization.

Our recommendation to have small quantity multiples contrasts with that of Kremer and Nyborg (2004b) who recommend a large minimum quantity increment relative to the price tick size to encourage competitive bidding. Their recommendation is correct for markets in which bidders are non-pivotal for all demand realizations, because in such markets pure strategy equilibria with very low mark-ups are possible. For example, von der Fehr and Harbord's (1993) model has a Bertrand equilibrium in this case. However, when one or several producers are pivotal for some demand realization, encouraging producers to undercut competitors' bids can lead to non-existence of pure strategy Nash equilibria and not necessarily lower average mark-ups (von der Fehr and Harbord, 1993). Even if mark-ups would be lower also in this case, the market participants would bear the cost of uncertainty caused by the inherent price instability.

Because of a singularity at zero mark-up, equilibrium bid-curves tend to be numerically unstable and easily non-monotonic near such points (Baldick and Hogan, 2002; Holmberg, 2008). We have the same experience with our stepped offer curves. The policy implication is that smaller tick-sizes, and even smaller quantity multiples, are needed in competitive markets with small mark-ups in order to get stable prices.

General convergence results for finite-dimensional games by Dasgupta and Maskin (1986) and Simon (1987) are not necessarily applicable to our problem, which is infinitely-dimensional in the limit. But their results suggest that the risk of non-convergence and price instability in electricity auctions would be lower if payoffs were continuous, for example by allowing piece-wise linear offers as in Nord Pool (Nordic countries) and Powernext (France). This conjecture is supported by Anderson and Hu (2008) who show that equilibria in such auctions converge to continuous SFE provided that the piece-wise linear offer curves are constructed to avoid the influence of kinks in residual demand. Moreover, as illustrated by Parisio and Bosco (2003), pure-strategy equilibria in von der Fehr and Harbord's (1993) model exist if production costs are private information to some extent, as then uncertainty about competitors' offers would make expected profits continuous. In spite of this additional uncertainty, we believe that pure-strategy equilibria in such a market can be approximated by a continuous SFE if demand uncertainty dominates uncertainty about competitor's production costs. We leave this as an interesting topic for future research.

Still an electricity market may fail to have a pure-strategy NE if quantity increments are large and producers are pivotal. In this case we conjecture that if firms can choose a sufficiently large number of steps (and most firms have a large number of individual generating sets), then the range over which each price is randomized may shrink as the number of possible price choices increases. In future research, it may be possible to demonstrate convergence of step SFEs to the continuous SFEs even when the possible price steps are smaller than the quantity steps. If so, the price instability at any level of demand would be small, and errors in using continuous representations also small.

In case discrete NE are useful as a method of numerically calculating continuous SFE, it should be noticed that the assumed price tick size does not necessarily have to correspond to the tick size of the studied auction. In a numerically efficient solver, it might be of interest to vary the tick size with the price. Our discrete model where quantities are chosen from a continuous set and prices from a discrete set has the nice property that pay-off functions are continuous, which should ensure existence of (mixed-strategy) equilibria in the discrete model. This may turn out to be a useful property in existence proofs of equilibria in the limit game with continuous supply functions.

We show that never-accepted out-of-equilibrium bids of rational producers are perfectly elastic at the highest realized market price in uniform-price procurement auctions with stepped supply functions and pro-rata on-the-margin rationing. This theoretical prediction, which should not depend on the size of tick-sizes and quantity multiples or on whether costs are private information, can be used to empirically test whether producers in electricity auctions believe that some of their offers are accepted with zero-probability, which is assumed in many theoretical models of electricity auctions. Another by-product of our analysis is the result that any set of, not necessarily symmetric, solutions to Klemperer and Meyer's system of differential equations constitute a continuous SFE if supply functions are increasing for all realized prices, demand is concave, and if a pivotal producer does not have a profitable deviation at the highest demand outcome.

Finally, we would not claim that the apparent tension between tractable but unrealistic continuous SFEs and realistic but intractable step SFEs is the only, or even the main, problem in modelling electricity markets. First, there are multiple SFE if some offers are always accepted or never accepted. Then under reasonable conditions, there is a continuum of continuous SFE bounded by (in the short run) a least and most profitable SFE. Second, the position of the SFEs depends on the contract position of all the generators, and determining the choice of contracts and their impact on the spot market is a hard and important problem. The greater the extent of contract cover, the less will be the incentive for spot market manipulation (Newbery, 1995), and as electricity demand is very inelastic and markets typically concentrated, this is an important determinant of market performance. Newbery (1998) argued that these can be related, in that incumbents can choose contract positions to keep both the contract and average spot price at the entry-deterring level, thus simultaneously solving for prices, contract positions, and embedding the short-run SFE within a longer run investment and entry equilibrium. A full long-run model of the electricity market should also be able to investigate whether some market power is required for (or inimical to) adequate investment in reserve capacity to maintain adequate security of supply. With such a model one could also make a proper assessment of how many competing generators are needed to deliver a workably competitive but secure electricity market.

REFERENCES

Anderson, E.J. and A. B. Philpott (2002). 'Using supply functions for offering generation into an electricity market', Operations Research 50 (3), pp. 477-489.

Anderson, E. J. and H. Xu (2004). 'Nash equilibria in electricity markets with discrete prices', Mathematical Methods of Operations Research 60, pp. 215–238.

Anderson, E. J. and X. Hu (2008). 'Finding Supply Function Equilibria with Asymmetric Firms', Operations Research 56 (3), pp. 697-711.

Bagh, A. (2010). 'Variational convergence: Approximation and existence of equilibria in discontinuous games', Journal of Economic Theory 145 (3), pp. 1244-1268.

Baldick, R., and W. Hogan (2002). Capacity constrained supply function equilibrium models for electricity markets: Stability, non-decreasing constraints, and function space iterations, POWER Paper PWP-089, University of California Energy Institute.

Baldick, R., R. Grant, and E. Kahn (2004). 'Theory and Application of Linear Supply Function Equilibrium in Electricity Markets', Journal of Regulatory Economics 25 (2), pp. 143-67.

Dasgupta P. and E. Maskin (1986). 'The existence of equilibrium in discontinuous economic games, I: Theory.', Review of Economic Studies 53, pp. 1-27.

Fabra, N., N-H. M. von der Fehr and D. Harbord (2006). 'Designing Electricity Auctions', RAND Journal of Economics 37 (1), pp. 23-46.

von der Fehr, N-H. M. and D. Harbord (1993). 'Spot Market Competition in the UK Electricity Industry', Economic Journal 103 (418), pp. 531-46.

Gans, J.S. and F.A. Wolak (2007). 'A Comparison of Ex Ante versus Ex Post Vertical Market Power: Evidence from the Electricity Supply Industry', available at ftp://zia.stanford.edu/pub/papers/vertical_mkt_power.pdf

Gatti, R.J. (2005). 'A note on the existence of Nash equilibrium in games with discontinuous payoffs', Cambridge Economics Working Paper No. 0510.

Genc, T. and S. Reynolds (2004), 'Supply Function Equilibria with Pivotal Electricity Suppliers', Eller College Working Paper No.1001-04, University of Arizona.

Green, R.J. and D.M. Newbery (1992). 'Competition in the British Electricity Spot Market', Journal of Political Economy 100 (5), pp. 929-53.

Green, R.J. (1996), 'Increasing competition in the British Electricity Spot Market', Journal of Industrial Economics 44 (2), pp. 205-216.

Holmberg, P. (2008). 'Numerical calculation of asymmetric supply function equilibrium with capacity constraints', European Journal of Operational Research 199 (1), pp. 285-295.

Holmberg, P. (2007). 'Supply Function Equilibrium with Asymmetric Capacities and Constant Marginal Costs', Energy Journal 28 (2), pp. 55-82.

Hortacsu, A. and S. Puller (2008), 'Understanding Strategic Bidding in Multi-Unit Auctions: A Case Study of the Texas Electricity Spot Market', Rand Journal of Economics 39 (1), pp. 86-114.

Kastl, J. (2008) 'On the properties of equilibria in private value divisible good auctions with constrained bidding', mimeo Stanford, available at http://www.stanford.edu/~jkastl/share.pdf

Klemperer, P. D. and M.A. Meyer, (1989). 'Supply Function Equilibria in Oligopoly under

Uncertainty', Econometrica, 57 (6), pp. 1243-1277.

Kremer, I and K.G. Nyborg (2004a). 'Divisible Good Auctions: The Role of Allocation Rules', RAND Journal of Economics 35, pp. 147–159.

Kremer, I. and K.G. Nyborg (2004b). 'Underpricing and Market Power in Uniform Price Auctions', The Review of Financial Studies 17 (3), pp. 849-877.

LeVeque, R. (2007). Finite Difference Methods for Ordinary and Partial Differential Equations, Philadelphia: Society for Industrial and Applied Mathematics.

Madlener, R. and M. Kaufmann (2002). Power exchange spot market trading in Europe: theoretical considerations and empirical evidence, OSCOGEN Report D 5.1b, Centre for Energy Policy and Economics, Switzerland.

Newbery, D.M. (1995). 'Power Markets and Market Power', Energy Journal 16 (3), 41-66.

Newbery, D. M. (1998a). 'Competition, contracts, and entry in the electricity spot market', RAND Journal of Economics 29 (4), pp. 726-749.

Niu, H., R. Baldick, and G. Zhu (2005). 'Supply Function Equilibrium Bidding Strategies With Fixed Forward Contracts', IEEE Transactions on power systems 20 (4), pp. 1859-1867.

Parisio, L. and B. Bosco (2003). 'Market power and the power market: Multi-unit bidding and (in)efficiency in electricity auctions', International tax and public finance 10 (4), pp. 377-401.

Reny, P. (1999). 'On the existence of pure and mixed strategy Nash equilibria in discontinuous games', Econometrica 67, pp. 1029–1056.

Rudkevich, A., M. Duckworth and R. Rosen (1998). 'Modelling electricity pricing in a deregulated generation industry: The potential for oligopoly pricing in poolco', The Energy Journal 19 (3), pp. 19-48.

Simon, L.K. (1987). 'Games with discontinuous payoffs.' Review of Economic Studies 54, pp. 569-97.

Sioshansi, R. and S. Oren (2007). 'How Good are Supply Function Equilibrium Models: An Empirical Analysis of the ERCOT Balancing Market', Journal of Regulatory Economics 31 (1), pp. 1-35.

Sweeting, A. (2007). 'Market Power in the England and Wales Wholesale Electricity Market 1995-2000', Economic Journal 117 (520), pp. 654-85.

Wilson, R. (1979). 'Auctions of Shares', Quarterly Journal of Economics 93 (4), pp. 675-689.

Wolak, F. A. (2003). 'Identification and Estimation of Cost Functions Using Observed Bid Data: An Application to Electricity Markets', in Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress, ed. by M. Dewatripont, L. Hansen, and S. Turnovsky, vol. 2, pp. 115–149. Cambridge University Press, Cambridge.

Wolak, F.A. (2004). 'Quantifying the Supply-Side Benefits from Forward Contracting in

Wholesale Electricity Markets', available from http://www.stanford.edu/~wolak.

APPENDIX - PROOFS OF PROPOSITIONS

4.1 The discrete first-order condition

Proof of Proposition 1: To find an equilibrium we need to determine the best response of firm i given its competitors' offers. The best response necessarily satisfies a first-order condition for each price level, found by differentiating (3) with respect to s_i^j and s_i^{j-1}, noting that the limits are functions of s_i^j and s_i^{j-1}, as $\tau^j = \tau_{-i}^j + s_i^j$:

$$\frac{\partial E_i^j}{\partial s_i^j} = \int_{\tau^{j-1}}^{\tau^j} \left(p^j - C_i'(\cdot)\right) \frac{\Delta\tau_{-i}^j \left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g(\varepsilon)\, d\varepsilon + \left[p^j s_i^j - C_i\left(s_i^j\right)\right] g\left(\tau^j\right) \tag{12}$$

and

$$\frac{\partial E_i^j}{\partial s_i^{j-1}} = \int_{\tau^{j-1}}^{\tau^j} \left(p^j - C_i'(\cdot)\right) \Delta\tau_{-i}^j \frac{\left(\tau^j - \varepsilon\right)}{\left(\Delta\tau^j\right)^2} g(\varepsilon)\, d\varepsilon - \left[p^j s_i^{j-1} - C_i\left(s_i^{j-1}\right)\right] g\left(\tau^{j-1}\right).$$

From the last expression it follows that:

$$\frac{\partial E_i^{j+1}}{\partial s_i^j} = \int_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'(\cdot)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g(\varepsilon)\, d\varepsilon - \left[p^{j+1} s_i^j - C_i\left(s_i^j\right)\right] g\left(\tau^j\right). \tag{13}$$

Combining (12) and (13) gives the first-order condition for step supply functions:

$$\begin{aligned} \frac{\partial E(\pi_i(\mathbf{s}))}{\partial s_i^j} &= \frac{\partial E_i^j}{\partial s_i^j} + \frac{\partial E_i^{j+1}}{\partial s_i^j} = -\Delta p^{j+1} s_i^j g\left(\tau^j\right) + \int_{\tau^{j-1}}^{\tau^j} \left[p^j - C_i'\left(s_i(\varepsilon)\right)\right] \frac{\Delta\tau_{-i}^j\left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g(\varepsilon)\, d\varepsilon \\ &+ \int_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'\left(s_i(\varepsilon)\right)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g(\varepsilon)\, d\varepsilon = 0, \end{aligned} \tag{14}$$

where $s_i(\varepsilon)$ is given by (2) if $\varepsilon \in \left[\tau^{j-1}, \tau^j\right]$.

$\partial E\left(\pi_i(\mathbf{s})\right)/\partial s_i^j$ is always well-defined, as from our definitions and assumed restrictions on the bids it follows that $\Delta\tau^j \geq \Delta\tau_{-i}^j \geq 0$ and $\Delta\tau^j \geq \varepsilon - \tau^{j-1} \geq 0$ if $\varepsilon \in \left[\tau^{j-1}, \tau^j\right]$. ■

The first-order condition in Proposition 1 is not directly applicable to parts of the offer curves that are never accepted in equilibrium, i.e. for price levels p^j such that $\tau^j > \overline{\varepsilon}$. Recall that p^H is the highest price level that is realized with a positive probability. By differentiating the expected profit in (4), one can show that

$$0 < \frac{\partial E\left(\pi_i(\mathbf{s})\right)}{\partial s_i^H} = \int_{\tau^{H-1}}^{\overline{\varepsilon}} \left[p^H - C_i'\left(s_i(\varepsilon)\right)\right] \frac{\Delta\tau_{-i}^H\left(\varepsilon - \tau^{H-1}\right)}{\left(\Delta\tau^H\right)^2} g(\varepsilon)\, d\varepsilon,$$

because $g(\varepsilon) = 0$ for $\varepsilon > \overline{\varepsilon}$. Thus to maximize its expected profit a firm should offer all of its remaining capacity at p^H. The intuition for this result is as follows: due to pro-rata on-the-margin rationing, maximizing the supply at p^H maximizes the firm's share of the accepted supply at p^H, and, because of the bounded range of demand shocks, there is no risk that an increased supply at p^H will lead to a lower price for any realized event. Hence $s_i^H = \overline{s}_i$. Our discreteness and uncertainty assumptions should not be critical for this result. Intuitively, we expect never-accepted offers to be perfectly elastic in any uniform price auction with stepped supply functions and pro-rata on the margin rationing.

Now, consider offers that are always infra-marginal. Recall that p^L is the lowest price that is realized with positive probability. Differentiate expected profit in (4):

$$0 < \frac{\partial E(\pi_i(\mathbf{s}))}{\partial s_i^{L-1}} = \int_{\tau^{L-1}}^{\tau^L} \left[p^L - C_i'(s_i(\varepsilon))\right] \Delta\tau_{-i}^L \left(\frac{\Delta\tau^L - \left(\varepsilon - \tau^{L-1}\right)}{\left(\Delta\tau^L\right)^2}\right) g(\varepsilon)\, d\varepsilon \text{ if } \tau^{L-1} < \underline{\varepsilon}, \quad (15)$$

because $g(\varepsilon) = 0$ for $\varepsilon < \underline{\varepsilon}$. Hence $\tau^{L-1} = \underline{\varepsilon}$. This result makes sense intuitively. To increase the accepted supply with pro-rata on-the-margin rationing at the price level p^L, infra-marginal offers that are never price-setting should be offered below p^L rather than at p^L, because bids at p^L are rationed for the lowest shock outcome. Again, we intuitively believe that always-accepted offers are generally offered below p^L in any uniform price auction with stepped supply functions and a pro-rata on the margin rationing mechanism.

Lemma 3 below derives a Taylor expansion and other properties of the discrete first-order condition - very useful when we later show that discrete SFE converge to continuous SFE.

Lemma 3 *We can make the following statements if $p^j - C_i'\left(s_i^j\right) > 0$ for all price levels such that $L \leq j \leq H-1$:*

1. *The difference $s_i^{j+1} - s_i^j$ is of the order $\Delta p^{j+1} \quad \forall j = L \ldots H-2$, and $\overline{\varepsilon} - \tau^{H-1}$ is of order Δp^H.*

2. *The discrete first-order condition in (14) can be approximated by the following Taylor series expansions:*

$$\Gamma_i^j(\mathbf{s}) \equiv \frac{\partial E(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p^{j+1} s_i^j g\left(\tau^j\right) + \frac{\left[p^j - C_i'\left(s_i^j\right)\right]\left(\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}\right) g\left(\tau^j\right)}{2}$$
$$+ O\left(\max\left(\left(\Delta p^j\right)^2, \left(\Delta p^{j+1}\right)^2\right)\right) \text{ if } L \leq j < H-1, \text{ and}$$

$$\Gamma_i^{H-1}(\mathbf{s}) \equiv \frac{\partial E(\pi_i(\mathbf{s}))}{\partial s_i^{H-1}} = -\Delta p^H s_i^{H-1} g\left(\tau^{H-1}\right) + \frac{\left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] g\left(\tau^{H-1}\right) \Delta\tau_{-i}^{H-1}}{2}$$
$$+ \left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] \Delta\tau_{-i}^H g\left(\tau^{H-1}\right) \frac{\left(\overline{\varepsilon} - \tau^{H-1}\right)}{\Delta\tau^H} + O\left(\max\left(\left(\Delta p^H\right)^2, \left(\Delta p^{H-1}\right)^2\right)\right).$$

Proof: The sum

$$\int_{\tau^{j-1}}^{\tau^j} \left[p^j - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \frac{\Delta\tau_{-i}^j\left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g\left(\varepsilon\right) d\varepsilon \tag{16}$$
$$+ \int_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g\left(\varepsilon\right) d\varepsilon > 0$$

must be of the order Δp^{j+1}, otherwise the first-order condition in (5) cannot be satisfied for small Δp^{j+1}. Now, if there is some difference Δs_i^{j+1} that is of the order 1, then differences $\Delta\tau^{j+1}$ and $\Delta\tau_{-m}^{j+1}$ will also be of the order 1 for some producer $m \neq i$. But this would lead to the contradiction that the sum in (16) is of the order 1, as by assumption $p^j - C_i'\left(s_i^j\right) > 0$ for all price levels such that $L \leq j \leq H-1$. Hence, we can conclude that differences $\left\{\Delta s_i^{j+1}\right\}_{i=1}^N$ must be of the order Δp^{j+1} for $L \leq j \leq H-2$. We have $\bar{\varepsilon} \leq \tau^H$, so differences $\left\{\Delta s_i^H\right\}_{i=1}^N$ are not necessarily of the order Δp^H. However, it must always be the case that $\bar{\varepsilon} - \tau^{H-1}$ is of the order Δp^H.

Given this result we now derive the Taylor expansions of the first-order condition.

$$\begin{aligned}
0 = \Gamma_i^j\left(\mathbf{s}\right) &\equiv \tfrac{\partial E\left(\pi_i\left(\mathbf{s}\right)\right)}{\partial s_i^j} = -\Delta p^{j+1} s_i^j g\left(\tau^j\right) + \left[p^j - C_i'\left(s_i^j\right)\right] \tfrac{\Delta\tau_{-i}^j}{\left(\Delta\tau^j\right)^2} g\left(\tau^j\right) \int_{\tau^{j-1}}^{\tau^j} \left(\varepsilon - \tau^{j-1}\right) d\varepsilon \\
&+ \left[p^j - C_i'\left(s_i^j\right)\right] \tfrac{\Delta\tau_{-i}^{j+1}}{\left(\Delta\tau^{j+1}\right)^2} g\left(\tau^j\right) \int_{\tau^j}^{\tau^{j+1}} \left(\tau^{j+1} - \varepsilon\right) d\varepsilon + O\left(\max\left(\left(\Delta p^j\right)^2, \left(\Delta p^{j+1}\right)^2\right)\right) \\
&= -\Delta p^{j+1} s_i^j g\left(\tau^j\right) + \tfrac{\left[p^j - C_i'\left(s_i^j\right)\right] g\left(\tau^j\right)\left(\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}\right)}{2} + O\left(\max\left(\left(\Delta p^j\right)^2, \left(\Delta p^{j+1}\right)^2\right)\right),
\end{aligned} \tag{17}$$

if $L \leq j < H-1$, which gives us the first half of the second statement. But when $j = H-1$ we get:

$$\begin{aligned}
0 = \Gamma_i^{H-1}\left(\mathbf{s}\right) &\equiv \tfrac{\partial E\left(\pi_i\left(\mathbf{s}\right)\right)}{\partial s_i^j} = -\Delta p^H s_i^{H-1} g\left(\tau^{H-1}\right) + \left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] \tfrac{\Delta\tau_{-i}^{H-1}}{\left(\Delta\tau^{H-1}\right)^2} g\left(\tau^{H-1}\right) \int_{\tau^{H-2}}^{\tau^{H-1}} \left(\varepsilon - \tau^{H-2}\right) d\varepsilon \\
&+ \left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] \tfrac{\Delta\tau_{-i}^H}{\left(\Delta\tau^H\right)^2} g\left(\tau^{H-1}\right) \int_{\tau^{H-1}}^{\bar{\varepsilon}} \left(\tau^H - \varepsilon\right) d\varepsilon + O\left(\max\left(\left(\Delta p^{H-1}\right)^2, \left(\Delta p^H\right)^2\right)\right) \\
&= -\Delta p^H s_i^{H-1} g\left(\tau^{H-1}\right) + \tfrac{\left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] g\left(\tau^{H-1}\right) \Delta\tau_{-i}^{H-1}}{2} + \left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right] \Delta\tau_{-i}^H g\left(\tau^{H-1}\right) \tfrac{\left(\bar{\varepsilon} - \tau^{H-1}\right)}{\Delta\tau^H} \\
&+ O\left(\max\left(\left(\Delta p^H\right)^2, \left(\Delta p^{H-1}\right)^2\right)\right),
\end{aligned} \tag{18}$$

which is the other half of the second statement. ■

4.2 Convergence of stationary solutions

The first step is to prove that mark-ups are strictly positive under our assumptions.

Proof of Lemma 1: The market design only allows for non-negative outputs. Hence, mark-ups must be non-negative according to (7). Thus with continuous marginal costs, non-decreasing offer curves and non-negative mark-ups, it follows from Assumption 2 that competitors to firm n must offer strictly less than $\underline{\varepsilon} + d\left(C_n'\left(0\right)\right)$ at the price $C_n'\left(0\right)$, and according to (7) the output of firm n is zero at this price. The demand curve is differentiable and Assumption 1 makes the same assumption for the supply curves. Thus the market must clear at a price strictly larger than $C_n'\left(0\right)$ for the lowest demand outcome. According to (7) this implies that all firms have strictly positive outputs at the lowest price. Supply curves are non-decreasing with respect to price, so the output must be strictly positive for all shock outcomes.

From equation (7), strict positivity of $s_i(p)$ implies that both the mark-up $p - C_i'(s_i(p))$ and the difference $s_{-i}'(p) - d'(p)$ take nonzero values for $p \in [a, b]$. In fact, since $s_{-i}(p)$ is non-decreasing and $d(p)$ non-increasing, $s_{-i}'(p) - d'(p)$ is non-negative, hence positive. Thus the mark-up, which equals $s_i(p)/(s_{-i}'(p) - d'(p))$, must be strictly positive. Moreover, continuity of $C_i(\cdot)$ and $s_i(p)$ yield that the mark-up is bounded below by a positive constant for all p in the compact set $[a, b]$. The smallest of these constants over all i furnishes the result. ∎

Lemma 4 below states that the system of first-order conditions implied by Proposition 1 has a unique solution for the price level p^{j-1} if Δp^j is sufficiently small and if supplies for the two previous steps, p^j and p^{j+1}, are known and satisfy certain properties and if producers never bid below their marginal cost. We will later use Lemma 4 iteratively to ensure that we will be able to find unique solutions to the discrete first-order condition for multiple price levels under given boundary conditions and other specified circumstances. We use the notation that $C_{\max}''$ is the highest slope of the marginal cost curves, $g_{\max}$ and $g_{\min}$ are the largest and smallest densities in the support of $g\left(\varepsilon\right)$, and $g_{\max}'$ is the largest slope of the density function in the support of $g\left(\varepsilon\right)$. We set $g_{\max}' = 0$ if the slope of the density function is always negative. Recall that we have made the assumption that $g_{\min}$ is strictly positive.

Lemma 4 *Assume that* $\left\{s_i^{j+1}\right\}_{i=1}^N$ *and* $\left\{s_i^j\right\}_{i=1}^N$ *are such that there exists* $\delta > 0$, *s.t.* $p^{j+1} - C_i'\left(s_i^{j+1}\right) \geq \delta > 0 \ \forall i = 1 \ldots N$ *and* $p^j - C_i'\left(s_i^j\right) \geq \delta > 0 \ \forall i = 1 \ldots N$, *and that they satisfy*

$$0 < \Delta\widetilde{p}s_i^j g\left(\tau^j\right) - \int_{\tau^j}^{\min\left(\overline{\varepsilon},\tau^{j+1}\right)} \left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \frac{\Delta\tau_{-i}^{j+1}\left(\tau^{j+1} - \varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g\left(\varepsilon\right) d\varepsilon \qquad (19)$$

$$< \min\left[\frac{3\left[p^j - C_i'\left(s_i^j\right)\right]^2 g_{\min}^2}{2C_{\max}'' g_{\max}}, \frac{3\left[p^j - C_i'\left(s_i^j\right)\right]^2 g_{\min}^2}{2\left(C_{\max}'' g_{\max} + \left[p^j - C_i'\left(0\right)\right] g_{\max}'\right)}\right]$$

for local tick-sizes $\Delta p^j = \Delta p^{j+1} = \Delta\widetilde{p}$. Under these circumstances, there exists a unique solution $\left\{s_i^{j-1}\right\}_{i=1}^N$ that together with $\left\{s_i^j\right\}_{i=1}^N$, $\left\{s_i^{j+1}\right\}_{i=1}^N$ and $\Delta p^j = \Delta p^{j+1} = \Delta\widetilde{p}$ satisfy the first-order condition $\left\{\Gamma_i^j\right\}_{i=1}^N$ in Proposition 1.

Proof: We want to determine $\left\{s_i^{j-1}\right\}_{i=1}^N$, i.e. a set of solutions for price level $j-1$. As $\Delta p^{j+1} = \Delta p^j = \Delta\widetilde{p}$, the implicit function Γ, defined by the first-order condition in Proposition 1, can be written:

$$\begin{aligned}\Gamma_i^j\left(\boldsymbol{s}^{j-1}, \boldsymbol{s}^j, \boldsymbol{s}^{j+1}, d^{j-1}, d^j, d^{j+1}, \Delta\widetilde{p}\right) &= -\Delta\widetilde{p}\left(\tau^j - \tau_{-i}^j\right) g\left(\tau^j\right) \\ &+ \int_{\tau^{j-1}}^{\tau^j} \left[p^j - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \frac{\Delta\tau_{-i}^j\left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g\left(\varepsilon\right) d\varepsilon \\ &+ \int_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g\left(\varepsilon\right) d\varepsilon = 0.\end{aligned} \tag{20}$$

Note that τ^{j+1}, τ^j, τ_{-i}^{j+1} and τ_{-i}^j are known, whereas τ^{j-1} and τ_{-i}^{j-1} depend on the unknown vector s^{j-1}.

The first step in the application of the implicit function theorem is to fix a point for which (20) is satisfied for all firms. This is straightforward, because it is easy to show that $s^{j-1} = s^j$ is a solution to (20) when $\Delta\widetilde{p} = 0$, $s^j = s^{j+1}$ and $d^{j-1} = d^j = d^{j+1}$. The next step is to prove that the Jacobian $\left\{\frac{\partial\Gamma_i^j}{\partial s_k^{j-1}}\right\}$ is invertible at this fixed point. It follows from (2) that $\left.\frac{\partial s_i(\varepsilon)}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed } \tau^{j-1}} = \frac{\varepsilon - \tau^j}{\Delta\tau^j}$ and $\left.\frac{\partial s_i(\varepsilon)}{\partial\tau^{j-1}}\right|_{\text{fixed } \tau_{-i}^{j-1}} = \frac{\left(\tau^j - \varepsilon\right)\Delta\tau_{-i}^j}{\left(\Delta\tau^j\right)^2}$ if $\varepsilon \in \left(\tau^{j-1}, \tau^j\right)$. By differentiating (20), it is now straightforward to show that:

$$\begin{aligned}\left.\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed } \tau^{j-1}} &= \int_{\tau^{j-1}}^{\tau^j} C_i''\left(s_i\left(\varepsilon\right)\right) \frac{\Delta\tau_{-i}^j\left(\varepsilon - \tau^{j-1}\right)\left(\tau^j - \varepsilon\right)}{\left(\Delta\tau^j\right)^3} g\left(\varepsilon\right) d\varepsilon \\ &- \int_{\tau^{j-1}}^{\tau^j} \left[p^j - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \frac{\left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g\left(\varepsilon\right) d\varepsilon \\ &\leq \frac{\Delta\tau_{-i}^j C_{\max}'' g_{\max}}{\left(\Delta\tau^j\right)^3} \int_{\tau^{j-1}}^{\tau^j} \left(\varepsilon - \tau^{j-1}\right)\left(\tau^j - \varepsilon\right) d\varepsilon - \frac{\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}}{\left(\Delta\tau^j\right)^2} \int_{\tau^{j-1}}^{\tau^j} \left(\varepsilon - \tau^{j-1}\right) d\varepsilon \\ &= \frac{\Delta\tau_{-i}^j C_{\max}'' g_{\max}}{6} - \frac{\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}}{2}\end{aligned} \tag{21}$$

and that

$$\left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}} = -\int_{\tau^{j-1}}^{\tau^j} C_i''(s_i(\varepsilon))\,\frac{\left(\Delta\tau_{-i}^j\right)^2\left(\varepsilon-\tau^{j-1}\right)\left(\tau^j-\varepsilon\right)}{\left(\Delta\tau^j\right)^4}g(\varepsilon)\,d\varepsilon$$
$$+\int_{\tau^{j-1}}^{\tau^j}\left[p^j - C_i'(s_i(\varepsilon))\right]\Delta\tau_{-i}^j\frac{2\varepsilon-\tau^{j-1}-\tau^j}{\left(\Delta\tau^j\right)^3}g(\varepsilon)\,d\varepsilon \tag{22}$$
$$\leq \frac{\left[p^j-C_i'\left(s_i^{j-1}\right)\right]\Delta\tau_{-i}^j}{\left(\Delta\tau^j\right)^3}\int_{\tau^{j-1}}^{\tau^j}\left(2\varepsilon-\tau^{j-1}-\tau^j\right)\left(g\left(\frac{\tau^{j-1}+\tau^j}{2}\right)+g'_{\max}\left(\varepsilon-\left(\frac{\tau^{j-1}+\tau^j}{2}\right)\right)\right)d\varepsilon$$
$$= \frac{\left[p^j-C_i'\left(s_i^{j-1}\right)\right]\Delta\tau_{-i}^j g'_{\max}}{6},$$

so it follows from (22) and (21) that

$$\left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}} + \left.\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed }\tau^{j-1}} \leq \frac{\Delta\tau_{-i}^j C''_{\max}g_{\max}}{6}$$
$$+\frac{\left[p^j - C_i'\left(s_i^{j-1}\right)\right]\Delta\tau_{-i}^j g'_{\max}}{6} - \frac{\left[p^j - C_i'\left(s_i^j\right)\right]g_{\min}}{2}. \tag{23}$$

For convenience let $\alpha_i = \frac{\partial\Gamma_i^j}{\partial s_i^{j-1}} = \left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}}$ and $\beta_i = \frac{\partial\Gamma_i^j}{\partial s_{k\neq i}^{j-1}} = \left.\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}\right|_{\text{fixed }\tau_{-i}^{j-1}} + \left.\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}\right|_{\text{fixed }\tau^{j-1}}$. The Working Paper (Holmberg, Newbery and Ralph, 2008) proves that the Jacobian $\left\{\frac{\partial\Gamma_i^j}{\partial s_k^{j-1}}\right\}$ is invertible whenever $\beta_i < \alpha_i$ and $\beta_i < 0$. It follows from (22) and (23) that this is true when

$$\Delta\tau_{-i}^j < \min\left[\frac{3\left[p^j - C_i'\left(s_i^j\right)\right]g_{\min}}{C''_{\max}g_{\max}}, \frac{3\left[p^j - C_i'\left(s_i^j\right)\right]g_{\min}}{C''_{\max}g_{\max}+\left[p^j - C_i'\left(s_i^{j-1}\right)\right]g'_{\max}}\right], \tag{24}$$

which obviously is satisfied at the fixed point where $s^{j-1} = s^j = s^{j+1}$, $d^{j-1} = d^j = d^{j+1}$ (so that $\Delta\tau_{-i}^j = 0$) and $\Delta\widetilde{p} = 0$. It is straightforward to verify that the functions $\Gamma_1^j \dots \Gamma_N^j$ are continuously differentiable in $\Delta\widetilde{p}$, s^j, s^{j+1}, d^{j-1}, d^j, and d^{j+1}. Thus we can conclude from the Implicit Function Theorem that there is a unique solution $\mathbf{s}^{j-1}$ to the difference equation in Proposition 1 around the fixed point given by $\mathbf{s}^{j-1} = \mathbf{s}^j = \mathbf{s}^{j+1}$, $d^{j-1} = d^j = d^{j+1}$ and $\Delta\widetilde{p} = 0$. We have assumed that $\left\{s_i^{j+1}\right\}_{i=1}^N$ and $\left\{s_i^j\right\}_{i=1}^N$ are such that

$$\Delta\widetilde{p}s_i^j g\left(\tau^j\right) - \int_{\tau^j}^{\min\left(\overline{\varepsilon},\tau^{j+1}\right)}\left[p^{j+1} - C_i'(s_i(\varepsilon))\right]\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2}g(\varepsilon)\,d\varepsilon > 0,$$

so that $\Delta\tau_{-i}^j > 0$ from (20). It also follows from (20) that

$$\Delta\widetilde{p}s_i^j g\left(\tau^j\right) - \int_{\tau^j}^{\min\left(\overline{\varepsilon},\tau^{j+1}\right)}\left[p^{j+1} - C_i'(s_i(\varepsilon))\right]\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2}g(\varepsilon)\,d\varepsilon$$
$$\geq \frac{\left[p^j - C_i'\left(s_i^j\right)\right]g_{\min}\Delta\tau_{-i}^j}{2}.$$

Hence,

$$\Delta\tau_{-i}^{j} \leq \frac{2\Delta\widetilde{p}s_i^j g\left(\tau^j\right) - 2\int\limits_{\tau^j}^{\min\left(\overline{\varepsilon},\tau^{j+1}\right)} \left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g\left(\varepsilon\right) d\varepsilon}{\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}}.$$

Thus it follows from (24) that the the Jacobian $\left\{\frac{\partial\Gamma_i^j}{\partial s_k^{j-1}}\right\}$ is invertible and unique solutions will exist whenever

$$\begin{aligned} &\frac{2\Delta\widetilde{p}s_i^j g\left(\tau^j\right) - 2\int\limits_{\tau^j}^{\tau^{j+1}} \left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right] \Delta\tau_{-i}^{j+1} \frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2} g\left(\varepsilon\right) d\varepsilon}{\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}} \\ &< \min\left[\frac{3\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}}{C''_{\max} g_{\max}}, \frac{3\left[p^j - C_i'\left(s_i^j\right)\right] g_{\min}}{C''_{\max} g_{\max} + \left[p^j - C_i'\left(s_i^{j-1}\right)\right] g'_{\max}}\right]. \end{aligned}$$

■

Lemma 5 *Under Assumption 2, the difference equation in Proposition 1 for price levels $j = L, \ldots, H-2$ is consistent with the continuous equation in (8) if $\left\{\breve{s}_i\left(p\right)\right\}_{i=1}^N$ satisfies Assumption 1, and $\Delta p^j = \Delta p$.*

Proof: A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small (LeVeque, 2007). The local truncation error is the discrepancy between the continuous slope and its discrete estimate when discrete values s_i^j are replaced with samples of the continuous solution $\breve{s}_i(p^j)$. Under Assumption 1 and 2, Lemma 1 implies that mark-ups of the continuous solution are strictly positive, so we can use the Taylor approximation from Lemma 3 and $\Delta p^j = \Delta p$ to approximate the difference equation in (14):

$$-\Delta p s_i^j g\left(\tau^j\right) + \frac{\left[p^j - C_i'\left(s_i^j\right)\right]\left(\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}\right) g\left(\tau^j\right)}{2} + O\left(\Delta p^2\right) = 0.$$

We have assumed that g is bounded away from zero. Thus

$$-\Delta p s_i^j + \frac{\left[p^j - C_i'\left(s_i^j\right)\right]\left(\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}\right)}{2} + O\left(\Delta p^2\right) = 0. \tag{25}$$

Samples of the continuous solution have positive mark-ups. Hence, (25) can be rewritten as:

$$\frac{-\Delta p s_i^j}{p^j - C_i'\left(s_i^j\right)} + \frac{s_{-i}^{j+1} - s_{-i}^{j-1} - d^{j+1} + d^{j-1}}{2} = O\left(\Delta p^2\right). \tag{26}$$

Summing the corresponding expressions of all firms and then dividing by $N-1$ yields:

$$\frac{s^{j+1}-s^{j-1}}{2}-\frac{N}{N-1}\frac{\Delta d^{j+1}+\Delta d^{j}}{2}-\frac{1}{N-1}\sum_{k}\frac{\Delta p s_k^j}{p^j-C_k'\left(s_k^j\right)}=O\left(\Delta p^2\right). \tag{27}$$

By subtracting (26) from (27) followed by some rearrangements we obtain:

$$\begin{aligned}\frac{s_i^{j+1}-s_i^{j-1}}{2\Delta p}&=\frac{1}{N-1}\frac{\Delta d^{j+1}+\Delta d^{j}}{2\Delta p}-\frac{s_i^j}{p^j-C_i'\left(s_i^j\right)}\\&+\frac{1}{N-1}\sum_{k=1}^{N}\frac{s_k^j}{p^j-C_k'\left(s_k^j\right)}+O\left(\Delta p^2\right).\end{aligned} \tag{28}$$

We know from the definition of the demand in the continuous system that $d'\left(p^j\right)=\lim_{\Delta p\to 0}\frac{\Delta d^j}{\Delta p^j}$. Hence,

$$\lim_{\Delta p\to 0}\frac{s_i^{j+1}-s_i^{j-1}}{2\Delta p}=\frac{d'\left(p^j\right)}{N-1}-\frac{s_i^j}{p^j-C_i'\left(s_i^j\right)}+\frac{1}{N-1}\sum_{k=1}^{N}\frac{s_k^j}{p^j-C_k'\left(s_k^j\right)}. \tag{29}$$

It remains to show that if s_i^j and s_k^j in the right-hand side of (29) are replaced by samples of the continuous solution $\breve{s}_i\left(p^j\right)$ and $\breve{s}_k\left(p^j\right)$ then the right-hand side converges to $\breve{s}_i'\left(p^j\right)$. But this follows from (8). Thus the local truncation error is zero and we can conclude that the discrete system is a consistent approximation of the continuous system if $\Delta p^j=\Delta p$. ■

We use this consistency property when proving convergence below. Recall that L and H are the lowest and highest price indices, j, such that price p^j occurs with positive probability, and varies with M (and the boundary conditions).

Proof of Proposition 2: Lemma 5 states that the discrete difference equation is a consistent approximation of the continuous differential equation. To show that the discrete stationary solution converges to the continuous stationary solution, we need to prove that the discrete stationary solution exists and is stable, i.e. the error does not explode as the number of steps increases without limit. The proof is inspired by LeVeque's (2007) convergence proof for general one-step methods.

Define the vector of global errors at the price p^j, $\boldsymbol{E}^j=\boldsymbol{s}^j-\breve{\boldsymbol{s}}(\boldsymbol{p}^j)$, and the corresponding vector for the local truncation error:

$$\boldsymbol{v}_i^j=\frac{\breve{s}_i(p^{j+1})-\breve{s}_i(p^j)}{\Delta p^{j+1}}.$$

It is useful to introduce a Lipschitz constant λ (LeVeque, 2007). Let it be some constant that satisfies the inequality[5]

$$\lambda > \left\| \frac{p - C_i'(\breve{s}_i(p)) + \breve{s}_i(p) C_i''(\breve{s}_i(p))}{\left[p - C_i'(\breve{s}_i(p)\right]^2} \right\|_\infty + \tag{30}$$

$$\frac{1}{N-1} \sum_k \left\| \frac{p - C_k'(\breve{s}_k(p)) + \breve{s}_k(p) C_k''(\breve{s}_k(p))}{\left[p - C_k'(\breve{s}_k(p)\right]^2} \right\|_\infty \quad \forall p \in (a,b).$$

Such a Lipschitz constant exists since the mark-up, which appears in the denominator of each fraction in (30), is bounded away from zero (because of Lemma 1), the cost function is twice continuously differentiable, and the prices and corresponding strategy values are bounded. For sufficiently small Δp, λ puts a bound on the sensitivity of the vector $\boldsymbol{s}^{j-1}$ to small changes in the solution of the previous step. It is also useful to introduce another constant κ, such that

$$\kappa > \frac{d'(p)}{N-1} + \left\| \frac{\breve{s}_i(p)}{p - C_i'\left(\breve{s}_i(p)\right)} \right\|_\infty + \frac{1}{N-1} \sum_k \left\| \frac{\breve{s}_k(p)}{p - C_k'\left(\breve{s}_k(p)\right)} \right\|_\infty \quad \forall p \in (a,b). \tag{31}$$

The constant κ will bound the difference between the vectors $\boldsymbol{s}^j$ and $\boldsymbol{s}^{j-1}$. Again we know that such a constant will exist, because the continuous solutions are bounded according to Assumption 1 and mark-ups are bounded away from zero on the interval according to Lemma 1. It has been assumed that boundary conditions are chosen such that the inequality in (10) is satisfied, so it follows from Lemma 4 that $\left\{s_i^{H-2}\right\}_{i=1}^N$ can be uniquely determined. For sufficiently small Δp, it also follows from (28) and (30) that the global error satisfies the following inequality:

$$\begin{aligned}\left\|\boldsymbol{E}^{H-2}\right\|_\infty &= \left\|\boldsymbol{s}^{H-2} - \breve{\boldsymbol{s}}\left(p^{H-2}\right)\right\|_\infty \leq \left\|\boldsymbol{E}^{H-1}\right\|_\infty + \lambda \Delta p \left\|\boldsymbol{s}^{H-1} - \breve{\boldsymbol{s}}\left(p^{H-1}\right)\right\|_\infty + \Delta p \left\|\boldsymbol{v}^{H-1}\right\|_\infty \\ &= (1 + \lambda \Delta p) \left\|\boldsymbol{E}^{H-1}\right\|_\infty + \Delta p \left\|\boldsymbol{v}^{H-1}\right\|_\infty .\end{aligned}$$

Thus if Δp is sufficiently small, so that the initial error $\left\|\boldsymbol{E}^{H-1}\right\|_\infty$ and the local truncation error $\left\|\boldsymbol{v}^{H-1}\right\| \Delta p$ are small enough, then $\left\|\boldsymbol{E}^{H-2}\right\|_\infty$ is sufficiently small. It now follows from the assumed properties of the continuous solution that $s_i^{H-1} - s_i^{H-2} \geq 0$ and that $p^{H-2} - C_i'\left(s_i^{H-2}\right) \geq \delta > 0 \; \forall i = 1 \ldots N$. Similar to the proof of claim 2 in Lemma 3, a Taylor expansion of the existence condition in (19) yields:

$$0 < \Delta p s_i^j g\left(\tau^j\right) - \frac{\left[p^{j+1} - C_i'\left(s_i^{j+1}\right)\right] \Delta \tau_{-i}^{j+1} g\left(\tau^j\right)}{2} + O\left(\Delta p^2\right) \tag{32}$$

$$< \min\left[\frac{3\left[p^j - C_i'\left(s_i^j\right)\right]^2 g_{\min}^2}{2 C''_{\max} g_{\max}}, \frac{3\left[p^j - C_i'\left(s_i^j\right)\right]^2 g_{\min}^2}{2\left(C''_{\max} g_{\max} + \left[p^j - C_i'(0)\right] g'_{\max}\right)}\right]$$

[5] Note that $\|\|_\infty$ is the max-norm, i.e. $\left\|E^j\right\|_\infty = \max_{1 \leq i \leq N} \left|E^j\right|$ (LeVeque, 2007).

if $L \leq j < H-1$. If Δp is sufficiently small, so that the discrete solution of the previous step is close to the continuous solution, the term on the right is bounded below by a positive constant, independently of i, j and Δp from Lemma 1. Thus the right inequality is obviously satisfied for M larger than some sufficiently large finite number, so that Δp becomes sufficiently small. This is also true for the left inequality, because it follows from the consistency property and (7) that

$$\lim_{\Delta p \to 0} \frac{\left[p^{j+1} - C_i'\left(s_i^{j+1}\right)\right] \Delta \tau_{-i}^{j+1}}{2\Delta p} = \frac{\left[p^j - C_i'\left(\breve{s}_i(p^j)\right)\right]\left(\breve{s}'_{-i}(p^j) - d'(p^j)\right)}{2} = \frac{\breve{s}_i(p^j)}{2}.$$

The argument above and Lemma 4 can be used to prove that $\left\{s_i^{H-3}\right\}_{i=1}^N$ can be uniquely determined for M larger than some sufficiently large finite number. We know from (28) and (31) that $s_i^{H-1} - s_i^{H-2} \leq \kappa \Delta p$. Thus if Δp is sufficiently small, then the argument for the vector $\boldsymbol{s}^{H-2}$ can be repeated iteratively to prove that the vector $\boldsymbol{s}^k$ $\forall k = L, ..., H-3$ can be uniquely determined and that

$$\left\|\boldsymbol{E}^k\right\|_\infty = \left\|\boldsymbol{s}^k - \breve{\boldsymbol{s}}\left(p^k\right)\right\|_\infty \leq (1+\lambda\Delta p)\left\|\boldsymbol{E}^{k+1}\right\|_\infty + \Delta p \left\|\boldsymbol{\upsilon}^{k+1}\right\|_\infty. \tag{33}$$

Let $\upsilon_{\max}^k = \max\left\{\|\boldsymbol{\upsilon}^n\|_\infty\right\}_{n=k}^{H-1}$. From the inequality in (33), we can show by induction that:

$$\begin{aligned} &\left\|\boldsymbol{E}^k\right\|_\infty \leq (1+\lambda\Delta p)^{H-1-k}\left\|\boldsymbol{E}^{H-1}\right\|_\infty + \Delta p \sum_{m=k+1}^{H-1} \|\boldsymbol{\upsilon}^m\|_\infty (1+\lambda\Delta p)^{m-k-1} < \\ &(1+\lambda\Delta p)^{H-1-k}\left(\left\|\boldsymbol{E}^{H-1}\right\|_\infty + (H-k-1)\Delta p \upsilon_{\max}^{k+1}\right) \\ &\leq (1+\lambda\Delta p)^{H-1-k}\left(\left\|\boldsymbol{E}^{H-1}\right\|_\infty + (H-L)\Delta p \upsilon_{\max}^{L}\right). \end{aligned} \tag{34}$$

Thus we can bound the global error at each price level by choosing a sufficiently small Δp. This and the properties of the continuous solution now imply that there will always be some sufficiently large but finite M_0, such that the condition for a unique solution in (32) is satisfied for $L+1 \leq j \leq H-2$ and $M \geq M_0$. In the limit as $\Delta p \to 0$ then $\left\|\boldsymbol{E}^{H-1}\right\|_\infty \to 0$, $\upsilon_{\max}^{L+1} \to 0$ (because of Lemma 5), $(H-L)\Delta p \to b-a$, and $(1+\lambda\Delta p)^{H-L} \to e^{\lambda(b-a)}$. Thus from (34), $\left\|\boldsymbol{E}^k\right\|_\infty \to 0$ when $\Delta p \to 0$, proving that the discrete stationary solution converges to the continuous one. ■

4.3 Sufficient conditions

So far we have proven convergence of the stationary solutions. Now we want to look at sufficiency conditions that will later be used to prove convergence of the equilibria as well. Lemma 6 shows that if it is not profitable to deviate from the continuous stationary solution by withholding production until the capacity constraints of the competitors bind when the highest shock outcome is realized, then this particular type of deviation will not be profitable for lower shock outcomes either.

Lemma 6 *If Assumption 3 is satisfied then*

$$\begin{aligned} p(\varepsilon)\breve{s}_i(p(\varepsilon)) - C_i\left(\breve{s}_i(p(\varepsilon))\right) &> p_d\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right] - C_i\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right], \\ \forall p_d &\in \left(b, p^M\right] \text{ and } \varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}). \end{aligned}$$

Proof: Let

$$\chi(\varepsilon) = p\breve{s}_i(p) - C_i\left(\breve{s}_i(p)\right) - p_d\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right] + C_i\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right]$$

where $p = p(\varepsilon)$. Now differentiate χ with respect to ε:

$$\chi'(\varepsilon) = \left[p - C_i'\left(\breve{s}_i(p)\right)\right]\breve{s}_i'(p)\,p'(\varepsilon) + \breve{s}_i(p)\,p'(\varepsilon) - \left[p_d - C_i'\left(\varepsilon + d(p_d) - \overline{s}_{-i}\right)\right].$$

We have from (7) that $\breve{s}_i(p) = \left[p - C_i'\left(\breve{s}_i(p)\right)\right]\left(\breve{s}_{-i}'(p) - d'(p)\right)$, so

$$\chi' = \left[p - C_i'\left(\breve{s}_i(p)\right)\right]\left[\breve{s}_i'(p) - d'(p)\right]p'(\varepsilon) - \left[p_d - C_i'\left(\varepsilon + d(p_d) - \overline{s}_{-i}\right)\right].$$

But $\breve{s}(p) - d(p) \equiv \varepsilon$, so $\left[\breve{s}'(p) - d'(p)\right]p'(\varepsilon) = 1$. Thus

$$\chi' = \left[p - C_i'\left(\breve{s}_i(p)\right)\right] - \left[p_d - C_i'\left(\varepsilon + d(p_d) - \overline{s}_{-i}\right)\right] < 0,$$

because $p_d > b \geq p$ and $\breve{s}_i(p) > \varepsilon + d(p_d) - \overline{s}_{-i}$. We know from Assumption 3 that $\chi(\overline{\varepsilon}) \geq 0$, so the above proves that $\chi(\varepsilon) > 0 \; \forall \varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon})$ or equivalently that

$$p(\varepsilon)\breve{s}_i(p(\varepsilon)) - C_i\left(\breve{s}_i(p(\varepsilon))\right) > p_d\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right] - C_i\left[\varepsilon + d(p_d) - \overline{s}_{-i}\right] \;\; \forall \varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon})$$

if Assumption 3 is satisfied. ■

In both the discrete and continuous case, only non-decreasing solutions of the first-order system can constitute valid SFE, because electricity auctions do not accept decreasing offers. Thus a necessary condition for an SFE is that solutions are non-decreasing. Proposition 3 states that a set of increasing solutions to the continuous first-order conditions is a sufficient condition for supply function equilibrium if Assumption 3 is satisfied, no producer is sufficiently pivotal, and the demand curve is concave.

Proof of Proposition 3: Consider an arbitrary firm i. Assume that its competitors follow the strategy implied by the continuous stationary solution. The question is whether it will be a best response of firm i to do the same. The profit of producer i for the outcome ε is given by

$$\pi_i(p, \varepsilon) = \left(\varepsilon + d(p) - \breve{s}_{-i}(p)\right)p - C_i\left(\varepsilon + d(p) - \breve{s}_{-i}(p)\right).$$

Hence

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \left[d'(p) - \breve{s}'_{-i}(p)\right]\left[p - C_i'\left(\varepsilon + d(p) - \breve{s}_{-i}(p)\right)\right] + \varepsilon + d(p) - \breve{s}_{-i}(p). \tag{35}$$

From the first-order condition in (7) it is known that

$$\left[d'(p) - \breve{s}'_{-i}(p)\right]\left[p - C_i'\left(\breve{s}_i(p)\right)\right] + \breve{s}_i(p) = 0 \ \forall p \in [a,b].$$

Subtracting this expression from (35) yields:

$$\begin{aligned} \tfrac{\partial \pi_i(\varepsilon,p)}{\partial p} = &\left[\breve{s}'_{-i}(p) - d'(p)\right]\left[C_i'\Bigg(\underbrace{\varepsilon + d(p) - \breve{s}_{-i}(p)}_{s_i}\Bigg) - C_i'\left(\breve{s}_i(p)\right)\right] + \\ &\Bigg(\underbrace{\varepsilon + d(p) - \breve{s}_{-i}(p)}_{s_i} - \breve{s}_i(p)\Bigg) \ \ \forall p \in [a,b]. \end{aligned} \tag{36}$$

Let $p_d \in [a,b]$ be the clearing price when producer i deviates and sells s_i units at the shock ε rather than $\breve{s}_i$ units. Due to monotonicity of the supply functions we know that $\breve{s}'_{-i}(p) - d'(p) \geq 0$ and that

$$p_d \leq p \Leftrightarrow s_i \geq \breve{s}_i \Leftrightarrow C_i'(s_i) \geq C_i'\left(\breve{s}_i\right).$$

Thus for every $p \in [a,b)$ we can conclude from (36) that

$$\frac{\partial \pi_i(\varepsilon,s_i)}{\partial p} \geq 0 \text{ if } p_d \leq p, \text{ and } \frac{\partial \pi_i(\varepsilon,s_i)}{\partial p} \leq 0 \text{ if } p_d \geq p.$$

Hence, given $\breve{s}_{-i}(p)$ and ε, the profit of firm i is pseudo-concave in the price range $[a,b)$ and the profit maximum is given by the first-order condition if prices would have been restricted to this range. The next step in the proof is to rule out profitable deviations outside this price range. We know from (9) that the price can never be higher than b unless the capacity constraints of the competitors bind, but such deviations are never profitable according to Assumption 3 and Lemma 6. It is possible to push market prices below a, but as will be shown such deviations will be unprofitable as well. The assumptions in (9) imply that all supply functions of the potential equilibrium are perfectly inelastic below a. This assumption and concavity of the demand curve implies that $\frac{d\left\{d'(p) - \breve{s}'_{-i}(p)\right\}}{dp} \leq 0$ if $p \leq a$. Thus we have from (35)

$$\frac{\partial^2 \pi_i(\varepsilon,p)}{\partial p^2} = \left[\underbrace{d''(p) - \breve{s}''_{-i}(p)}_{\leq 0}\right]\left[p - C_i'\left(\underbrace{\varepsilon + d(p) - \breve{s}_{-i}(p)}_{s_i}\right)\right] +$$

$$\left[\underbrace{d'(p) - \breve{s}'_{-i}(p)}_{\leq 0}\right]\left(\underbrace{1 - C_i''\underbrace{\left(\varepsilon + d(p) - \breve{s}_{-i}(p)\right)}_{\geq 0}\underbrace{\left[d'(p) - \breve{s}'_{-i}(p)\right]}_{\leq 0}}_{\geq 0}\right) +$$

$$\left[\underbrace{d'(p) - \breve{s}'_{-i}(p)}_{\leq 0}\right] \leq 0 \ \forall p \in [C_i'(s_i), a].$$

Hence, given that competitors stick to the strategies implied by the continuous stationary solutions $\breve{s}'_{-i}(p)$, the profit function is concave in the range $[C_i'(s_i), a]$. Offering positive supply below marginal cost can never be profit maximizing. Thus we can conclude that $\breve{s}_i(p)$ must be a best response to $\breve{s}_{-i}(p)$. This is true for any firm and we can conclude that the stationary solution is an equilibrium. ■

For a sufficiently large number of price levels, Lemma 2 shows that being non-decreasing is also a sufficient condition for a discrete SFE (so the non-decreasing condition acts rather like a second-order condition in ensuring sufficiency). Note that the result relies on the assumptions that $\Delta d^j \geq \Delta d^{j+1}$, i.e. concave demand, and that mark-ups are sufficiently high to deter possible pivotal producers.

Proof of Lemma 2: Consider a set of discrete stationary solutions $\widehat{\boldsymbol{s}} = \{\widehat{\boldsymbol{s}}_1, \ldots, \widehat{\boldsymbol{s}}_N\}$. The shock distribution is such that p^L and p^H are the lowest and highest realized prices. In what follows it will be shown that an arbitrary chosen firm i has no incentive to unilaterally deviate from the supply schedule $\widehat{\boldsymbol{s}}_i = \left\{\widehat{s}_i^1, \ldots, \widehat{s}_i^M\right\}$ to any $\boldsymbol{s}_i = \left\{s_i^1, \ldots, s_i^M\right\}$ given that Δp is sufficiently small and that competitors stick to $\widehat{\boldsymbol{s}}_{-i} = \left\{\widehat{s}_{-i}^1, \ldots, \widehat{s}_{-i}^M\right\}$. Thus $\widehat{\boldsymbol{s}} = \{\widehat{\boldsymbol{s}}_1, \ldots, \widehat{\boldsymbol{s}}_N\}$ constitutes a Nash equilibrium. Now, assume that competitors stick to $\widehat{\boldsymbol{s}}_{-i} = \left\{\widehat{\boldsymbol{s}}_{-i}^1, \ldots, \widehat{\boldsymbol{s}}_{-i}^M\right\}$ and calculate the total differential of the expected profit of firm i for some $\boldsymbol{s}_i$:

$$dE\left(\pi_i\left(\boldsymbol{s}_i\right)\right) = \sum_{j=1}^{M} \frac{\partial E\left(\pi_i\left(\boldsymbol{s}_i\right)\right)}{\partial s_i^j} ds_i^j.$$

As a first step in the proof of a global maximum we first verify monotonicity properties of $\Gamma_i^j(\mathbf{s}) = \frac{\partial E(\pi_i(\boldsymbol{s}_i))}{\partial s_i^j}$. First we look at the case where demand is uniformly distributed, so that $g(\varepsilon) = g$. By means of Proposition 1 and Leibniz' rule we can show that:

$$\frac{\partial^2 E(\pi_i(\mathbf{s}))}{\partial\left(s_i^j\right)^2} = -\Delta p g + \left[p^j - C_i'\left(s_i^j\right)\right]\frac{\Delta\tau_{-i}^j}{\Delta\tau^j} g - g\int_{\tau^{j-1}}^{\tau^j} C_i''\left(s_i\left(\varepsilon\right)\right)\frac{\partial s_i(\varepsilon)}{\partial s_i^j}\frac{\Delta\tau_{-i}^j\left(\varepsilon-\tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2}d\varepsilon$$
$$-2g\int_{\tau^{j-1}}^{\tau^j}\left[p^j - C_i'\left(s_i\left(\varepsilon\right)\right)\right]\frac{\Delta\tau_{-i}^j\left(\varepsilon-\tau^{j-1}\right)}{\left(\Delta\tau^j\right)^3}d\varepsilon - \left[p^{j+1} - C_i'\left(s_i^j\right)\right]\frac{\Delta\tau_{-i}^{j+1}}{\Delta\tau^{j+1}} g \tag{37}$$
$$-g\int_{\tau^j}^{\tau^{j+1}} C_i''\left(s_i\left(\varepsilon\right)\right)\frac{\partial s_i(\varepsilon)}{\partial s_i^j}\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2}d\varepsilon + 2g\int_{\tau^j}^{\tau^{j+1}}\left[p^{j+1} - C_i'\left(s_i\left(\varepsilon\right)\right)\right]\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^3}d\varepsilon.$$

Using (2) it is straightforward to verify that $\frac{\partial s_i(\varepsilon)}{\partial s_i^j} \geq 0$ if $\varepsilon \in \left[\tau^{j-1}, \tau^{j+1}\right]$. Thus monotonic marginal costs and supply functions imply that:

$$\frac{\partial^2 E(\pi_i(\mathbf{s}))}{\partial\left(s_i^j\right)^2} \leq -\Delta p g + \left[p^j - C_i'\left(s_i^j\right)\right]\frac{\Delta\tau_{-i}^j}{\Delta\tau^j} g - 2g\left[p^j - C_i'\left(s_i^j\right)\right]\int_{\tau^{j-1}}^{\tau^j}\frac{\Delta\tau_{-i}^j\left(\varepsilon-\tau^{j-1}\right)}{\left(\Delta\tau^j\right)^3}d\varepsilon$$
$$-\left[p^{j+1} - C_i'\left(s_i^j\right)\right]\frac{\Delta\tau_{-i}^{j+1}}{\Delta\tau^{j+1}} g + 2g\left[p^{j+1} - C_i'\left(s_i^j\right)\right]\int_{\tau^j}^{\tau^{j+1}}\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^3}d\varepsilon \tag{38}$$
$$= -\Delta p g$$

if $L \leq j < H-1$. Note that $\frac{\partial^2 E(\pi_i(\mathbf{s}))}{\partial\left(s_i^j\right)^2} \leq -\Delta p g$ also holds for $j = H-1$, because the last integral in (38) becomes larger if we replace $\overline{\varepsilon}$ by τ^H.

The case with general demand distributions is more complicated, but we can show the following by means of the Taylor approximations in Lemma 3

$$\frac{\partial\left(\frac{\Gamma_i^j(\mathbf{s})}{g(\tau^j)}\right)}{\partial s_i^j} = -\Delta p - \frac{C_i''\left(s_i^j\right)\left(\Delta\tau_{-i}^j + \Delta\tau_{-i}^{j+1}\right)}{2} + O\left(\Delta p^2\right)$$

if $L \leq j < H-1$, and

$$\frac{\partial\left(\frac{\Gamma_i^{H-1}(\mathbf{s})}{g(\tau^{H-1})}\right)}{\partial s_i^{H-1}} = -\Delta p - \frac{C_i''\left(s_i^{H-1}\right)\Delta\tau_{-i}^{H-1}}{2} - C_i''\left(s_i^{H-1}\right)\Delta\tau_{-i}^H\frac{\left(\overline{\varepsilon}-\tau^{H-1}\right)}{\left(\Delta\tau^H\right)}$$
$$-\left[p^{H-1} - C_i'\left(s_i^{H-1}\right)\right]\Delta\tau_{-i}^H\frac{\left(\tau^H-\overline{\varepsilon}\right)}{\left(\Delta\tau^H\right)^2} + O\left(\Delta p^2\right).$$

Thus we can always find a sufficiently large integer number M_1, so that $\frac{\partial\left(\Gamma_i^j/g\left(\tau^j\right)\right)}{\partial s_i^j} < 0$ if $M \geq M_1$ for all $j = L \ldots H-1$ and $i = 1 \ldots N$. In the case with uniformly distributed demand we have already shown that it is enough with $M_1 = 3$ (the smallest possible number of price levels in our model). As $g\left(\varepsilon\right) > 0$ for $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$ this implies that $\frac{\partial E(\pi_i(\boldsymbol{s}_i))}{\partial s_i^j} > 0$ for $s_i^j < \widehat{s}_i^j$ and $\frac{\partial E(\pi_i(\boldsymbol{s}_i))}{\partial s_i^j} < 0$ for $s_i^j > \widehat{s}_i^j$. Thus no firm has any incentives to unilaterally deviate from the supply schedule $\widehat{\boldsymbol{s}}_i$ for price levels $j = L \ldots H-1$. For deviations such that $\tau^{L-1} > \underline{\varepsilon}$, the argument above is valid for the price level $j = L-1$ as well, so it is never profitable to increase supply by undercutting p^L. By assumption we also know that it is not profitable for pivotal producers to withhold production and push the price above p^H . Thus there are no profitable unilateral deviations, and we can conclude that $\widehat{\boldsymbol{s}}_i = \left\{\widehat{s}_i^1, \ldots, \widehat{s}_i^M\right\}$ must be a Nash equilibrium. ■

4.4 Convergence of equilibria

Given the results of Propositions 2 and 3 and Lemma 2 we are now ready to prove Theorem 1:

Proof of Theorem 1: Part (a) is a restatement of Proposition 3. Our next step is to show part (b). Proposition 2 ensures that the discrete stationary solution exists for $M \geq M_0$, and that it will converge to the continuous one. Thus Assumption 1 together with consistency (proved in Lemma 5) implies that the discrete solution is non-decreasing for M larger than some sufficiently large finite number of price levels. Moreover, convergence of the stationary solutions and Lemma 1 ensure that mark-ups in the discrete stationary solution are strictly positive for M larger than some sufficiently large finite number of price levels. Finally, convergence of competitors' supply curves implies that the difference between a producer's profits in the discrete and continuous system will converge to zero, and this is also true for all possible deviations of the producer. Hence, because of Lemma 6, there are no profitable deviations from the discrete stationary solution such that $s_i^H < \overline{\varepsilon} + d\left(p^H\right) - \overline{s}_{-i}$ if M is larger than some sufficiently large finite number. Hence, in the limit, all conditions in Lemma 2 are satisfied and we can conclude there is some sufficiently large M_2, such that the discrete stationary solution is a segment of a discrete SFE when $M \geq M_2$. ■

Theorem 1 established that for any well-behaved continuous SFE we can find a well-behaved discrete SFE, which is stable and converges to the continuous one. The result below ensures the reverse result. Whenever a well-behaved discrete equilibrium exists in the limit, when the number of steps becomes arbitrarily large, then there always exists a corresponding continuous equilibrium. This establishes that the family of increasing smooth SFE with positive mark-ups is asymptotically in one-to-one correspondence with the family of corresponding well-behaved discrete SFE. To prove the next result we make the assumption below, which ensures that no producer is sufficiently pivotal.

Assumption 4.

$$p^H \widehat{s}_i^{H-1} - C_i\left(\widehat{s}_i^{H-1}\right) \geq p_d\left[\overline{\varepsilon} + d^d - \overline{s}_{-i}\right] - C_i\left[\overline{\varepsilon} + d^d - \overline{s}_{-i}\right] \;\; \forall p_d \in \left(p^H, p^M\right].$$

This assumption is analogous to Assumption 3. The left-hand side of the inequality is the profit at the boundary condition when $p = p^{H-1}$. The right-hand side is the profit for prices $p_d \in \left(p^H, p^M\right]$ when competitors' capacities bind. Note that if Assumption 4 is not satisfied then there will always be some shock density $g(\varepsilon)$ (with sufficient probability mass near $\overline{\varepsilon}$), such that $\left\{ \left\{ \widehat{s}_i^j \right\}_{j=L}^{j=H} \right\}_{i=1}^{N}$ is not a segment of a discrete SFE. But also note that Assumption 4 is always satisfied if p^H is sufficiently close to the price cap p^M or if producers are non-pivotal.

Proposition 4 *Assume for a sufficiently large number of equidistant steps M that there exists a discrete stationary solution $\left\{ \widehat{s}_i^j \right\}_{i=1}^{N}$ with strictly positive mark-ups that is a discrete SFE. If the solution satisfies Assumption 4 and is stable, so that it converges to a set of continuous functions $\{\tilde{s}_i(p)\}_{i=1}^N$ on $[a, b]$, where $a = \lim_{\Delta p \to 0} p^L$ and $b = \lim_{\Delta p \to 0} p^H$, then $\{\tilde{s}_i(p)\}_{i=1}^N$ is a segment of a continuous SFE.*

Proof: Only non-decreasing offers are valid in the auction, so the discrete stationary solution and its limit, $\{\tilde{s}_i(p)\}_{i=1}^N$, must be non-decreasing as well. Convergence of the discrete solution implies that $\left\{ \widehat{s}_i^j \right\}_{i=1}^{N}$ is bounded in the limit. Convergence and properties of the difference equation outlined in Lemma 3 also imply that $\lim_{\Delta p \to 0} \frac{s_i^{j+1} - s_i^{j-1}}{2\Delta p}$ exists, so that $\{\tilde{s}_i(p)\}_{i=1}^N$ is differentiable. Moreover, the limit $\{\tilde{s}_i(p)\}_{i=1}^N$ satisfies the difference equation in Proposition 1, so it follows from the properties of $\{\tilde{s}_i(p)\}_{i=1}^N$ and Lemma 5 that $\{\tilde{s}_i(p)\}_{i=1}^N$ will satisfy the continuous differential equation in (7). In the limit, convergence of competitors' supply curves implies that Assumption 3 is satisfied if Assumption 4 is satisfied. As $\{\tilde{s}_i(p)\}_{i=1}^N$ is a set of non-decreasing functions in the interval $[a, b]$, it now follows from Proposition 3 that $\{\tilde{s}_i(p)\}_{i=1}^N$ is a segment of a continuous SFE. ■