Some Similarities Between Greater Risk Aversion and Greater Downside Risk Aversion

by Richard Watt (University of Canterbury)

“I was born on the back-seat of a greyhound bus, rolling down highway 41”; The Allman Brothers (Ramblin’ Man).

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“I was born on the front-seat of an EGRIE bus, chatting with Louis Eeckhoudt”; my interest in downside risk aversion.

Motiviation

Traditional downside risk aversion:

$$\frac{1}{2}Eu(w+z+\widetilde{x})+\frac{1}{2}u(w)>\frac{1}{2}u(w+z)+\frac{1}{2}Eu(w+\widetilde{x})$$

where $z>0$, $E\widetilde{x}=0$, and random variable $\widetilde{x}$ has positive variance.

This is a simple 2-state analysis with $w_1=w+z>w_2=w$.

It is well known that a sufficient and necessary condition for such a preference is that marginal utility is convex, that is, u''', which together with the assumptions of positive marginal utility and risk aversion (concavity of utility), implies that the decision maker is "prudent".

But we are still trying to come to a consensus as to a measure an intensity of downside risk aversion.

Some of the "compensation" literature:

(a) Crainich and Eeckhoudt (2009), m such that

$$\frac{1}{2}Eu(w+z+\widetilde{x})+\frac{1}{2}u(w)=\frac{1}{2}u(w+z+m)+\frac{1}{2}Eu(w+\widetilde{x})$$

(b) Jindapon (2010), q such that

$$\frac{1}{2}Eu(w+z+\widetilde{x})+\frac{1}{2}u(w)=\left(\frac{1}{2}+q\right)u(w+z)+\left(\frac{1}{2}-q\right)Eu(w+\widetilde{x})$$

To go any further, one typically needs to make an assumption of small risk, and use a Taylor's approximation.

A different perspective

I characterize downside risk aversion in terms of the size of the set of lotteries for which the risk $\widetilde{x}$ would be located on the downside (the smaller is this set, the more downside risk averse is the decision maker). The analysis is valid for risks $\widetilde{x}$ of any size (small and large).

If the probability of state 1 was equal to 1, then the zero-mean risk would always be optimally located in state 2, as that way it can be avoided completely which under risk aversion would of course be optimal.

So if a utility function is downside risk averse, and risk averse, by continuity there must exist a probability p^*, defined on $\frac{1}{2} < p^* < 1$, such that

$$p^* Eu(w + z + \widetilde{x}) + (1 - p^*)u(w) = p^* u(w + z) + (1 - p^*)Eu(w + \widetilde{x})$$

p^* splits the set $p = [0,1]$ into two mutually independent parts. For lotteries with $p < p^*$, the decision maker prefers to locate the risk $\widetilde{x}$ on the upside of the primary lottery, and for lotteries with $p > p^*$ locating the risk $\widetilde{x}$ on the downside is preferred.

Therefore, the greater is p^* the smaller is the set of lotteries for which locating $\widetilde{x}$ on the downside is preferred.

It seems natural to state that the greater is p^*, the more downside risk averse is our decision maker.

Such a definition of greater downside risk aversion mimics to a certain degree the traditional sorting of utility functions according to their absolute risk aversion – the more risk averse is a utility function, the smaller is the set of lotteries that would be voluntarily exchanged for a given risk-free wealth allocation.

p^* can be written as

$$p^* = \frac{u(w) - Eu(w + \widetilde{x})}{[u(w) - Eu(w + \widetilde{x})] + [u(w + z) - Eu(w + z + \widetilde{x})]}$$

or, defining $f(w, \widetilde{x}) \equiv u(w) - Eu(w + \widetilde{x})$,

$$p^* = \frac{f(w, \widetilde{x})}{f(w, \widetilde{x}) + f(w + z, \widetilde{x})} = \frac{f(w)}{f(w) + f(w + z)}$$

Although $f(w)$ is not immune to affine transformations, p^* is.

Proposition 1 *Downside risk aversion increases with the size of* $z = w_1 - w_2$.

So the "riskier" is the primary lottery, the greater is downside risk aversion.

Utility transformations

Consider utility transformations of the type $v(w) = G(u(w))$. When can we conclude that v is more downside risk averse than u?

Proposition 2 *Any alteration in $u(w)$ that causes a relative change in $f(w)$ that is greater than (resp. less than, equal to) the relative change in $f(w+z)$ will serve to increase (resp. decrease, not alter) the value of p^*.*

So utility function $v(w) = G(u(w))$ will be more downside risk averse than utility function $u(w)$, in the sense that $p_v^* > p_u^*$, if

$$\frac{f_v(w) - f_u(w)}{f_u(w)} > \frac{f_v(w+z) - f_u(w+z)}{f_u(w+z)} \quad \Rightarrow \quad \frac{f_v(w)}{f_v(w+z)} > \frac{f_u(w)}{f_u(w+z)}$$

This can be summed up as:

Proposition 3 *The intensity of downside risk aversion can be measured by the ratio of the utility premium in state of nature 2 to the utility premium in state of nature 1,* $\frac{f(w)}{f(w+z)}$.

Proposition 4 *A sufficient condition for utility function* v *to be more downside risk averse than utility function* u *is, for all* w

$$-\frac{f_v'(w)}{f_v(w)} > -\frac{f_u'(w)}{f_u(w)}$$

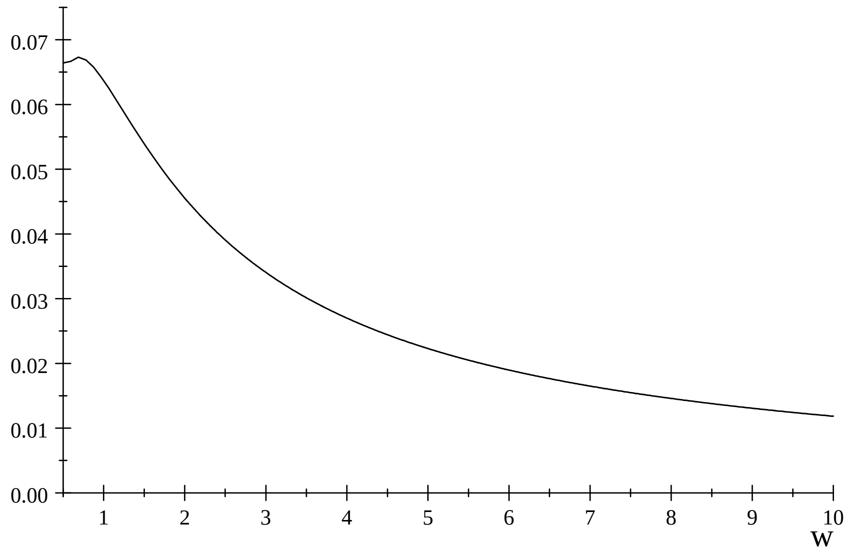
The result relates to Proposition 2 in Jindapon (2010), in which $-\frac{f'(w)}{f(w)}$ is referred to as the measure of "pain elasticity".

Jindapon finds that, for the case of large risks, utility function v is more downside risk averse (in the sense of having a larger probability premium) than is utility function u if v is both more risk averse and has more pain elasticity than does u.

Here, using the characterization of downside risk aversion given by the size of the set of lotteries for which locating the secondary risk on the downside is preferred, we only need the condition on pain elasticity, and not the condition on risk aversion.

Proposition 5 *Utility function* $v(w) = G(u(w))$ *is "almost surely" more downside risk averse than is* $u(w)$ *if* $G' > 0, G'' < 0$ *and* $G''' > 0$.

For example, consider the transform $G(u) = \ln(u^2)$. Notice that $G' = \frac{2}{u} > 0$, $G'' = -\frac{2}{u^2} < 0$, and $G''' = \frac{4}{u^3} > 0$, and take the utility function $u(w) = \sqrt{w}$. We get $v(w) = \ln((\sqrt{w})^2) = \ln(w)$. The graph of the difference, $p_v^* - p_u^*$, is:



Changes in risk-free wealth

We can also consider how p^* changes with wealth, that is, the question of whether downside risk aversion is increasing or decreasing in wealth.

In our set-up, a change in risk-free wealth is simply a change in w.

Deriving p^* with respect to w and simplifying yields

$$\frac{\partial p^*}{\partial w} = \frac{f'(w)f(w+z) - f(w)f'(w+z)}{[f(w)+f(w+z)]^2}$$

This is negative, i.e. downside risk aversion as defined by p^* decreases with w if

$$-\frac{f'(w)}{f(w)} > -\frac{f'(w+z)}{f(w+z)}$$

Of course, since $z > 0$, a sufficient condition for decreasing downside risk aversion is

$$\frac{\partial}{\partial w}\left(-\frac{f'(w)}{f(w)}\right) < 0$$

Carrying out the derivative and simplifying (recalling that $f' < 0$) gives the following condition:

$$-\frac{f''(w)}{f'(w)} > -\frac{f'(w)}{f(w)}$$

This says that in order for $u(w)$ to display decreasing downside risk aversion, it is sufficient that $-u'(w)$ be more downside risk averse than $u(w)$.

This result can be easily seen to mirror the well known result that for $u(w)$ to display decreasing absolute risk aversion, the function $-u'(w)$ should be more risk averse than the function $u(w)$ itself.

Relationship between the slope of downside risk aversion and the slope of risk aversion

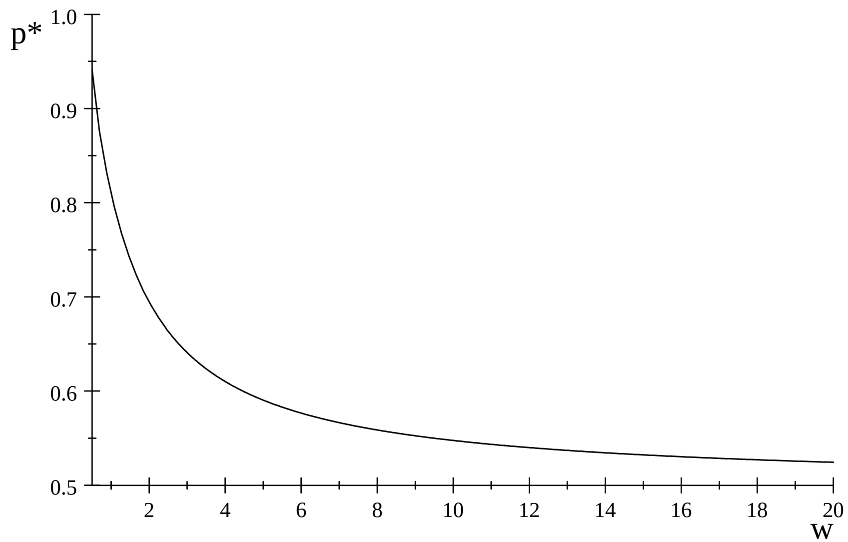
Proposition 6 *If $u(w)$ displays constant absolute risk aversion, then $u(w)$ also displays constant downside risk aversion, $\frac{\partial p^*}{\partial w} = 0$.*

There may be a similar relationship between DARA and decreasing downside risk aversion.

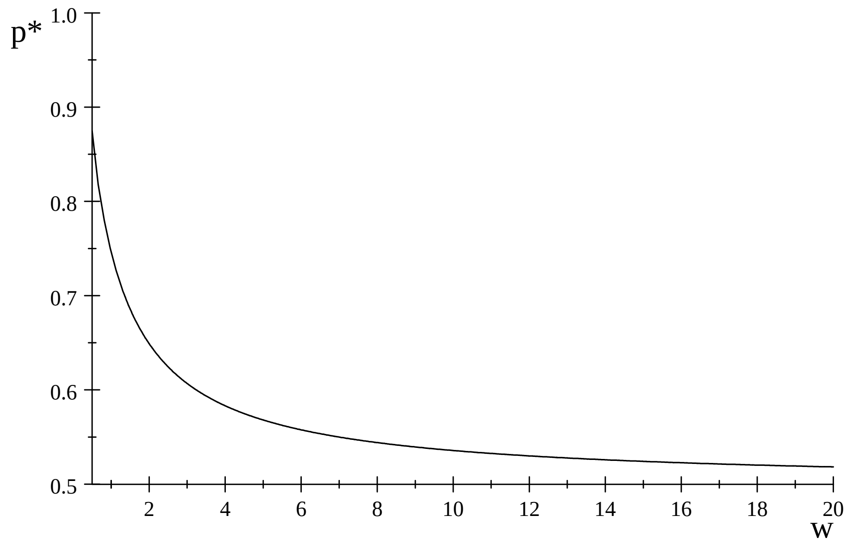
In fact, I hypothesise that $\frac{\partial p^*}{\partial w} \leq 0$ is reasonably robust.

Some examples

(1) Take $u(w) = \ln(w)$, which is DARA. With this example (and taking $z = 1$, and $\tilde{x}$ uniform between -0.5 and 0.5), the graph of p^* is:



(2) With $u(w) = \sqrt{w}$, the graph of p^* is:



So both of these DARA utility functions give decreasing downside risk aversion.

(3) What about IARA? It we use a quadratic; $u(w) = aw^2 + bw$, which does display IARA, we have $u'''(w) = 0$, i.e. marginal utilty is linear, so the function is downside risk neutral. This leads to the utility premium being constant, and $p^* = \frac{1}{2}$ for all w.

Take then the function

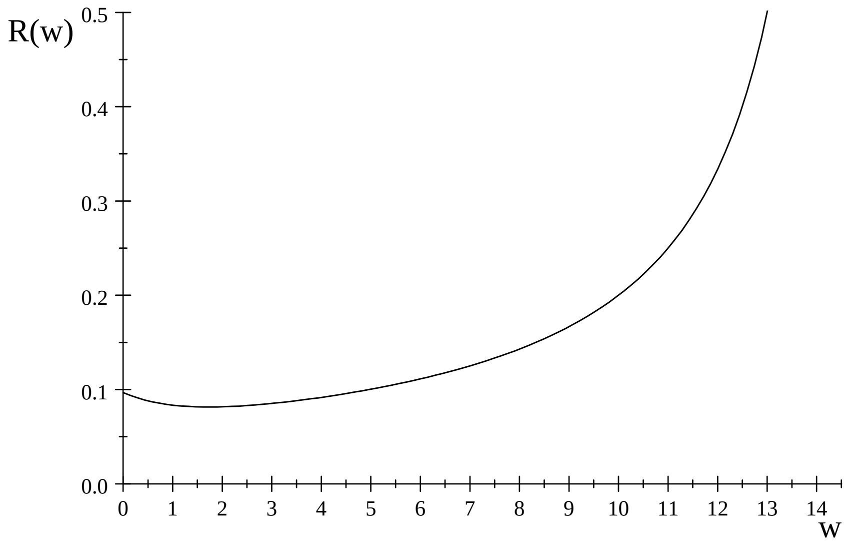
$$u(w) = \lambda\left(\alpha - e^{-\beta w}\right) + (1 - \lambda)\left(aw^2 + bw\right)$$

Since this is a mixture of CARA and IARA, it does have a positive third derivative (i.e. it is downside risk averse). It can also be made to have IARA with an appropriate choice of parameters. Take $\lambda = 0.5$, $\beta = 1$, $a = -1$, and $b = 30$. Using these parameters, marginal utility is positive for $w < 15$ (to 7 significant digits) so we shall restrict w to be on this range.

The absolute risk aversion function is

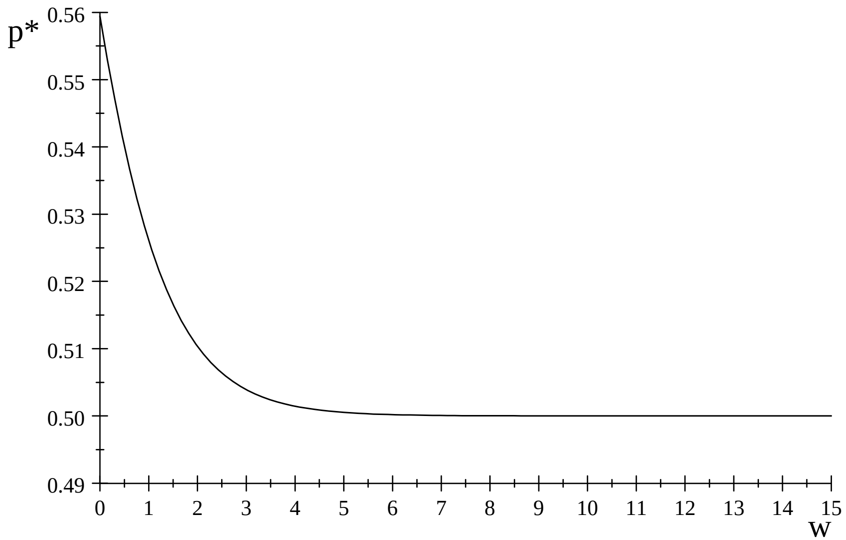
$$R(w) = \frac{e^{-w} + 2}{e^{-w} + (30 - 2w)}$$

The graph of $R(w)$ is



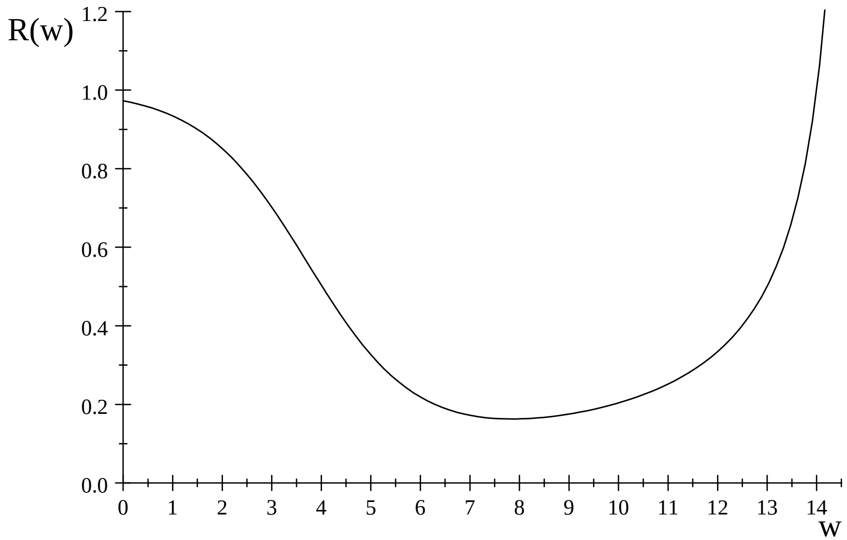
So with this function, absolute risk aversion is decreasing up to $w \approx 1.73$, and increasing after that.

But he graph of p^* (assuming, as above, $z = 1$) is everywhere decreasing:

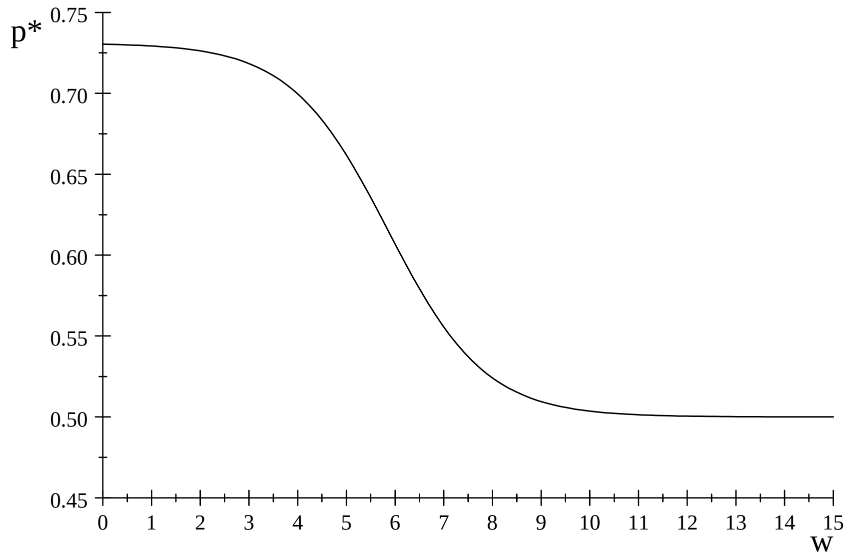


Different things can be found with different choices of λ. For example, take this same example with $\lambda = 0.999$. Marginal utility is strictly positive for all w less than 15.0002.

The graph of risk aversion is now:



And the graph of p^* (assuming, as above, $z = 1$) is



Still everywhere downward sloping, but now not always convex.

Conclusions

1. I propose a different characterization of DSRA, based on the size of preferred sets.

2. The intensity of DSRA can be measured using the ratio of the utility premium over states.

3. The "riskier" is the primary lottery, the greater is DSRA.

4. $u(w)$ displays decreasing DSRA if the function $-u'(w)$ is more DSRA than the function $u(w)$ itself (similar to DARA and the relationship between the RA of $u(w)$ and $-u'(w)$).

5. An increasing, concave and prudent transform of $u(w)$ *probably* increases DSRA.

6. If $u(w)$ is CARA, then it also has constant DSRA.

7. Decreasing DSRA seems to be a reasonably robust characteristic for other kinds of risk aversion.