Restricted Finite Time Dominance

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Motivation

IN HONOUR OF LOUIS EECKHOUDT

Motivation

- Appraisal of sustainable development policies.
- Intertemporal choices.
- Individual comparisons of lifetime outcomes (health profiles).
- Evaluation of long term investments.
- Cost Benefit Analysis.

Framework

Finite, discrete time $t \in \{0, 1, \ldots, T\}$.

- Mutually exclusive intertemporal alternatives, $a, b \in \mathbb{R}^{T+1}$

$a = (a_0, a_1, ..., a_T) \in \mathbb{R}^{T+1}$, $b = (b_0, b_1, ..., b_T) \in \mathbb{R}^{T+1}$;
stream of net outcomes
$x = (x_0, x_1, ..., x_T) = (a - b) = (a_0 - b_0, ..., a_T - b_T) \in \mathbb{R}^{T+1}$

t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4

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- Temporal preferences $v = (v_0, v_1, \ldots, v_t, \ldots, v_T) \geq \mathbf{0}$ with $v_0 = 1$

$$NPV_v(x) := \sum_{t=0}^{T} v_t \cdot x_t$$

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- V_0 positively valued, with $v_0 = 1$.
- $v \in V_1 \subset V_0$, time impatience.
- $v \in V_2 \subset V_1$, are also strictly convex (decreasing time impatience).
- Restrictions to the set of discounting functions.
  Let $v_t^0 = v_t$; for any $k = 1, 2$, let $v_t^k := v_{t+1}^{k-1} - v_t^{k-1}$

$$\begin{aligned} V_k &: = \{v : v \in V_{k-1}, \text{ and } (-1)^k v_t^k \geq 0\} \\ & \quad \text{that is} \\ V_1 &: = \{v : v \in V_0, \text{ and } \Delta_t = v_t - v_{t+1} \geq 0\}, \\ V_2 &: = \{v : v \in V_1, \text{ and } \Delta_t - \Delta_{t+1} \geq 0\}. \end{aligned}$$

References

- Time dominance. Bøhren and Hansen (J. Fin. 1980) and Ekern (J. Fin. 1981).
- Ranking investment projects/ streams of utilities in terms of Net Present Value NPV, irrespective of the choice of the discount rate. Foster and Mitra (ET, 2002).
- Multidimensional stochastic dominance conditions with applications to the intertemporal evaluations. Karcher et al. (1995) and Trannoy (2003).
- Stochastic dominance relations on unidimensional grids. Fishburn and Lavalle (Math. O. R.1995), Chakravarty and Zoli (JET, 2012).
- Inverse stochastic dominance of degree r ($r - ISD$) in terms of weak r-majorization of the vectors of mean order statistics. De La Cal and Carcamo (J. App. Prob. 2010).
- Almost Stochastic Dominance. Leshno and Levy (Man. Sc. 2002)

Time Dominance

$NPV_v(x) = NPV_v(a) - NPV_v(b)$

Definition

$NPV_v(a) \geq NPV_v(b)$ for all $v \in V_n$ is denoted as $a \succcurlyeq_n b$.

Repeated summations of the net distribution x. $X_t^0 = x_t$.For $n = 1, 2$.

$$X_t^n = \sum_{s=0}^{t} X_s^{n-1}$$

Definition

Project a dominates b by the n^{th} **order time dominance**, denoted by $a \geqslant_n b$, if and only if for the net project $x = a - b$

$$\begin{aligned} X_T^k &\geq 0 \text{ for all } k = 1, 2, ..., n-1 \\ X_t^n &\geq 0 \text{ for all } t \in \{0, 1, ..., T\}. \end{aligned}$$

Example

Let $T = 8$ and $n \in \{1, 2\}$.

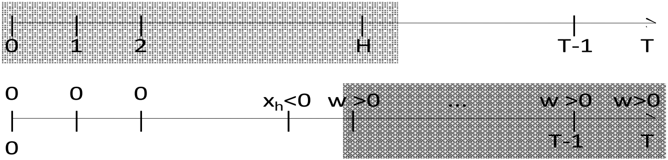
t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4
	↓	↓↗	↓↗	↓↗	↓↗	↓↗	↓↗	↓↗	↓↗
X_t^1	-1	-3	0	3	1	5	8	6	**2**
X_t^2	-1	-4	-4	-1	0	5	13	19	

Theorem (Ekern, J. Fin. 1981)

$a \succcurlyeq_n b$ if and only if $a \geqslant_n b$.

Non Dictatorship of the Present (NDP)

Dictatorship of the present: no compensation between future net outcomes and current outcome if $x_0 < 0$.



Definition (NDP)

Let $H \in \{1, ..., T\}$. For $x_h < 0$, where $h < H$, $x_t = 0$ for all $t \in \{0, 1, ..., h-1\}$ and for any $v \in V_H \subseteq V_2$ there exists $\omega > 0$ with $x_t = \omega$, for all $t \in \{h+1, \dots, T\}$ s.t. $NPV_v(x) > 0$.

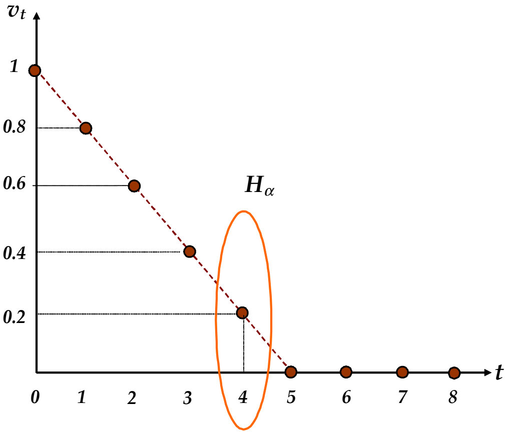
Remark $NPV_v(x)$ for $v \in V_H \subseteq V_2$ satisfies NDP if and only if

- $v_H > 0$ and bounded from below. Alternatively:
- $\exists\ \alpha \in [0, 1)$ s.t. $\Delta_t = v_t - v_{t+1} \leq \alpha$, $\forall t$, with $\sum_{t=0}^{H-1} \alpha = \alpha H < 1$.

Link alpha - H

Example

$\alpha = 0.2;\ H_\alpha = \frac{1}{\alpha} - 1 = 4$



Alfa-Restricted 1st order Time Dominance

Definition

Let $n \in \{1, 2\}$, and $\alpha \in [0, 1)$, then

$$V_n^\alpha := \{v \in V_n \text{ s.t. } \Delta_t = v_t - v_{t+1} \leq \alpha \text{ for all } t\}.$$

α : maximum magnitude of the variation of discounting functions.

Definition

Let $n \in \{1, 2\}$, $NPV_v(a) \geq NPV_v(b)$ for all $v \in V_n^\alpha$ is denoted as $a \succcurlyeq_n^\alpha b$.

Theorem ($\alpha - TD_1$)

For $\alpha \in (0, 1)$ then $a \succcurlyeq_1^\alpha b$ if and only if $G_{X^{1*}}\left(\frac{1}{\alpha} - 1\right) \geq 0$.
If $\alpha = 0$ then then $a \succcurlyeq_1^0 b$ if and only if $X_T^1 \geq 0$.

Alfa - TD1

Example

t	0	1	2	3	4	5	6	7	8
x_t	-1	-2	3	3	-2	4	3	-2	-4
X_t^1	-1	-3	0	3	1	5	8	6	**2**
$X_{[t]}^1$	-3	-1	0	1	2	3	5	6	8
$X_{[t]}^{1*}$	-3	-1	0	1	2	2	2	2	2
$G_{X^{1*}}(t)$	-3	-4	-4	-3	-1	1	3	5	7

Note that $G_{X^{1*}}(4.5) = 0$ thus $NPV_v(x) \geq 0$ if and only if $H_\alpha = \frac{1}{\alpha} - 1 \geq 4.5$ that is $\frac{1}{5.5} = 0.18 \geq \alpha$.

Cut off point $\alpha^* = 0.18$

Alfa-Restricted 2nd order TD

Theorem (α-TD2)

For $\alpha \in (0,1)$ then $a \succsim_2^{\alpha} b$ if and only if:
(A) $X_T^1 \geq 0$, and
(B) $X_t^2 + X_T^1 \cdot [\frac{1}{\alpha} - 1 - t]_+ \geq 0$.
If $\alpha = 0$ then then $a \succsim_2^{0} b$ if and only if $X_T^1 \geq 0$.

where $[x]_+ := \max\{0, x\}$. Let $H_\alpha = \frac{1}{\alpha} - 1$
Condition (B) can be rewritten as:
(B1) $X_t^2 \geq 0$ for all $t \geq H_\alpha$
(B2) $X_t^2 + X_T^1 [H_\alpha - t] \geq 0$ for all $t < H_\alpha$

Example

t=0: $x_0 + X_T^1 [\frac{1}{\alpha} - 1] \geq 0$
t=1: $(2x_0 + x_1) + X_T^1 [\frac{1}{\alpha} - 2] \geq 0$
t=2: $(3x_0 + 2x_1 + x_2) + X_T^1 [\frac{1}{\alpha} - 3] \geq 0$

Example

Example

t	0	1	2	3	**4**	5	6	7	8
x_t	-1	-2	3	3	**-2**	4	3	-2	-4
X_t^1	-1	-3	0	3	**1**	5	8	6	2
X_t^2	-1	-4	-4	-1	**0**	5	13	19	21

note that for $\alpha^o = 0.2$ then $a \succcurlyeq_2^{\alpha^o} b$, in fact:
- time threshold $H_{\alpha^0} = \frac{1}{\alpha^o} - 1 = 4$, with $X_t^2 \geq 0$ for $t \geq 4$
for t=2: $X_2^2 + X_T^1[H_{\alpha^0} - 2] = (-4) + 2 \cdot [4 - 2] = 0$
for t=3: $X_3^2 + X_T^1[H_{\alpha^0} - 3] = (-1) + 2 \cdot [4 - 3] = 1 \geq 0$
for t=0 and t=1 the analogous conditions are also satisfied.

Remarks/Corollaries

Corollary

If $X_T^1 = 0$ then $a \succcurlyeq_n^\alpha b \Longleftrightarrow a \succcurlyeq_n b \Longleftrightarrow a \geqslant_n b$ for $n \in \{1, 2\}$.

That is, α-TD is more decisive w.r.t. standard *TD* only if $X_T^1 > 0$.

Corollary

Let $n \in \{1, 2\}$. If $a \neq b$, if there is no TDn, i.e., neither $a \geqslant_n b$ nor $b \geqslant_n a$, then

(i) if $X_T^1 > 0$ then $\exists\alpha$ s.t. $a \succcurlyeq_n^\alpha b$,
(ii) if $X_T^1 < 0$ then $\exists\alpha$ s.t. $b \succcurlyeq_n^\alpha a$.

No disagreement problems for the $\alpha - TDn$ criterion: it is not possible that there exists an α for which $a \succcurlyeq_n^\alpha b$, and another α' for which $b \succcurlyeq_n^{\alpha'} a$.

Remarks/Corollaries/Links with....

Corollary

Let $\alpha' < \alpha$ *then* $a \succcurlyeq_n^{\alpha} b \Longrightarrow a \succcurlyeq_n^{\alpha'} b$ *for* $n \in \{1, 2\}$.

Corollary

Suppose $a \succcurlyeq_1^{\alpha'} b$ *then* $\exists\ \alpha \geq \alpha'$ *such that* $a \succcurlyeq_2^{\alpha} b$.

Links with...

- Literature on internal rates of return (*IRR*), but ... unique upper bound for α s.t.$a \succcurlyeq_n^{\alpha} b$ versus possible multiple IRR.
- *Almost Stochastic Dominance* (ASD). The $\alpha - TD$ criterion considers a restriction on the *absolute magnitude* of the "marginal" change of the discount function between two adjacent periods. The set of restrictions considered are logically distinct and results are conceptually different. They can provide novel results if re-mapped in the ASD space.