On Prudence, Temperance, and Monoperiodic Portfolio Optimization

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Arrow-Pratt development up to order four

Recall the definition of the risk premium π :

$$u(E[V]-\pi) = E[u(V)]$$

The Arrow-Pratt development up to order four is :

$$\pi \simeq \left[-\frac{u''(E[V])}{u'(E[V])}\right]\frac{\mathrm{var}\,V}{2} - \left[\frac{u'''(E[V])}{u'(E[V])}\right]\frac{m_3(V)}{6} + \left[-\frac{u''''(E[V])}{u'(E[V])}\right]\frac{m_4(V)}{24}$$

where $m_3 = E[(V-E[V])^3]$ and $m_4 = E[(V-E[V])^4]$

Arrow-Pratt development up to order four

Reexpress the risk premium as :

$$\pi \simeq \frac{1}{2} \lambda \,\mathrm{var}\, V - \frac{1}{6} \psi\, m_3(V) + \frac{1}{24} \varphi\, m_4(V)$$

What is the link between the coefficient ψ of appetite to m_3,
and the coefficient φ of aversion to m_4,
to the coefficients of prudence and temperance ?

Interpretation of coefficients

The coefficient of appetite to m_3 satisfies :

$$\psi = \lambda\ \zeta$$

where ζ is the coefficient of prudence :

$$\zeta = -\frac{u'''(E[V])}{u''(E[V])}$$

Interpretation of coefficients

The coefficient of aversion to m_4 satisfies :

$$\varphi = \chi \; \zeta \; \lambda$$

where χ is the coefficient of temperance :

$$\chi = -\frac{u''''(E[V])}{u'''(E[V])}$$

Interpretation of coefficients

It is interesting to reexpress the risk premium as follows :

$$\pi \simeq \frac{1}{2}\lambda\left(\mathrm{var}\,V - \frac{1}{3}\zeta\left(m_3(V) - \frac{1}{4}\chi\, m_4(V)\right)\right)$$

or, more simply at order 3 :

$$\pi \simeq \frac{1}{2}\lambda\left(\mathrm{var}\,V - \frac{1}{3}\zeta\, m_3(V)\right)$$

to let appear the impact of prudence and temperance as perturbation factors.

Prudence and temperance for HARA functions

Consider a HARA utility function :

$$u(x) = \frac{1-a}{a}\left(\frac{b}{1-a}\, x + c\right)^{a}$$

The coefficients of risk aversion, appetite to asymmetry, and aversion to leptokurticity are :

$$\lambda(x) = \frac{1}{\frac{1}{1-a}x + \frac{c}{b}}$$

$$\psi(x) = \frac{(1-a)(2-a)}{\left(x + \frac{c(1-a)}{b}\right)^{2}} \qquad \varphi(x) = \frac{(1-a)(2-a)(3-a)}{\left(x + \frac{c(1-a)}{b}\right)^{3}}$$

Prudence and temperance for CRRA functions

Consider a CRRA utility function :

$$u(x) = \frac{x^a}{a}$$

The coefficients of risk aversion, appetite to asymmetry, and aversion to leptokurticity are :

$$\lambda(x) = \frac{1-a}{x}$$

$$\psi(x) = \frac{(1-a)(2-a)}{x^2} \qquad \varphi(x) = \frac{(1-a)(2-a)(3-a)}{x^3}$$

Prudence and temperance for CRRA functions

The coefficient of prudence can be expressed as :

$$\zeta(x) = \frac{2-a}{x}$$

whereas the coefficient of temperance admits the expression :

$$\chi(x) = \frac{3-a}{x}$$

The portfolio problem

The goal is to compute optimal portfolio compositions,
satisfying :

$$\max E[u(V)] = \max u(V^*)$$

or, equivalently :

$$\max \left(E[V] - \pi\right)$$

Replacing with the expression of π,
we obtain the general problem :

$$\max \left(E[V] - \frac{\lambda}{2} \mathrm{var}(V) + \frac{\psi}{6} m_3(V) - \frac{\varphi}{24} m_4(V) \right)$$

Optimizing with one risky asset

Denote by x the proportion of funds invested in the risky asset, by R the return of the risky asset, and by r the risk-free rate. The final portfolio wealth is worth :

$$V = (V_0 - xV_0)(1 + r) + xV_0(1 + R) = V_0(1 + r) + xV_0(R - r)$$

where the total return of the portfolio is :

$$R_p = \frac{V - V_0}{V_0} = \frac{V_0 r + xV_0(R - r)}{V_0} = r + x(R - r).$$

Optimizing with one risky asset

We first replace the moments of V by those of R_p in the portfolio problem :

$$E[V] = V_0 + V_0 \; E[R_p] \qquad \text{var}(V) = V_0^2 \, \text{var}(R_p)$$

$$m_3(V) = V_0^3 \; m_3(R_p) \qquad m_4(V) = V_0^4 \; m_4(R_p)$$

Optimizing with one risky asset

Then, we replace the moments of R_p by those of R in the portfolio problem :

$$E\left[R_p\right] = r + x(E[R] - r) \qquad \text{var}\left(R_p\right) = x^2 \, \text{var}(R)$$

$$m_3\left(R_p\right) = x^3 \; m_3\left(R\right) \qquad m_4\left(R_p\right) = x^4 \; m_4\left(R\right)$$

Optimizing with one risky asset

The portfolio problem can then be rewritten as :

$$\max_x \left(r + x(E[R] - r) - \frac{x^2\tilde{\lambda}}{2}\text{var}(R) + \frac{x^3\tilde{\psi}}{6}m_3(R) - \frac{x^4\tilde{\varphi}}{24}m_4(R) \right)$$

where $\tilde{\lambda} = \lambda\ V_0$, $\tilde{\psi} = \psi\ V_0^2$, and $\tilde{\varphi} = \varphi V_0^3$ are the coefficients of relative risk aversion, relative appetite to asymmetry and relative aversion to leptokurticity.

The corresponding first order condition is :

$$E[R] - r - x\ \tilde{\lambda}\,\text{var}(R) + x^2\ \frac{\tilde{\psi}}{2}m_3(R) - x^3\ \frac{\tilde{\varphi}}{6}m_4(R) = 0$$

A closed-form formula

The (unique) real solution of the above third-degree equation is :

$$x = \frac{\tilde{\psi} m_3(R)}{\tilde{\varphi}\, m_4(R)} + \frac{2}{\tilde{\varphi}\, m_4(R)} \left(\frac{1}{2} \left(\Lambda_1(R) + \sqrt{\Lambda_1(R)^2 - 4\Lambda_2(R)^3} \right) \right)^{\frac{1}{3}}$$

$$+ \frac{2}{\tilde{\varphi}\, m_4(R)} \left(\frac{1}{2} \left(\Lambda_1(R) - \sqrt{\Lambda_1(R)^2 - 4\Lambda_2(R)^3} \right) \right)^{\frac{1}{3}}$$

where :

$$\Lambda_1(R) = \frac{\tilde{\psi}^3}{4} m_3(R)^3 - \frac{3\tilde{\varphi}\tilde{\psi}\tilde{\lambda}}{4} m_4(R) m_3(R)\, \mathrm{var}(R) + \frac{3\tilde{\varphi}^2}{4} m_4(R)^2 \left(E[R] - r\right)$$

and where :

$$\Lambda_2(R) = \frac{\tilde{\psi}^2}{4} m_3(R)^2 - \frac{\tilde{\varphi}\tilde{\lambda}}{2} m_4(R)\, \mathrm{var}(R)$$

Illustration

We conduct a calibration over the period 1950-2012.
The risk-free rate is deduced from US treasury bills :

$$r = 4.5\ \%$$

The distribution of the annual log-returns
of the SP500 index is characterized by :

$$E[R] = 6.75\ \% \quad \text{Var}(R) = 0.02753$$

$$m_3(R) = -0.00399 \quad m_4(R) = 0.0029636$$

The relative risk aversion parameter is set to 2.5 ;
a CRRA utility function is assumed.

Illustration

The Merton (mean-variance) optimal weight that should be invested in the risky asset is :

$$x^* = 32.75\ \%$$

Correcting for m_3 and prudence yields :

$$x^* = 30.35\ \%$$

A further correction for m_4 and temperance gives :

$$x^* = 29.71\ \%$$

Optimizing with several risky assets

In dimension d, the terminal value of the portfolio is given by :

$$V = V_0(1-\langle \mathbf{x}, \mathbf{1}\rangle)(1+r)+V_0\langle \mathbf{x}, \mathbf{1}+\mathbf{R}\rangle = V_0(1+r)+V_0\langle \mathbf{x}, \mathbf{R}-r\mathbf{1}\rangle$$

The portfolio return R_p becomes :

$$R_p = \frac{V - V_0}{V_0} = \frac{V_0 r + V_0\langle \mathbf{x}, \mathbf{R} - r\mathbf{1}\rangle}{V_0} = r + \langle \mathbf{x}, \mathbf{R} - r\mathbf{1}\rangle$$

Optimizing with several risky assets

The formulas linking the moments of V to the ones of R_p are unchanged. However, one should use the following relationships between the moments of R_p and those of R :

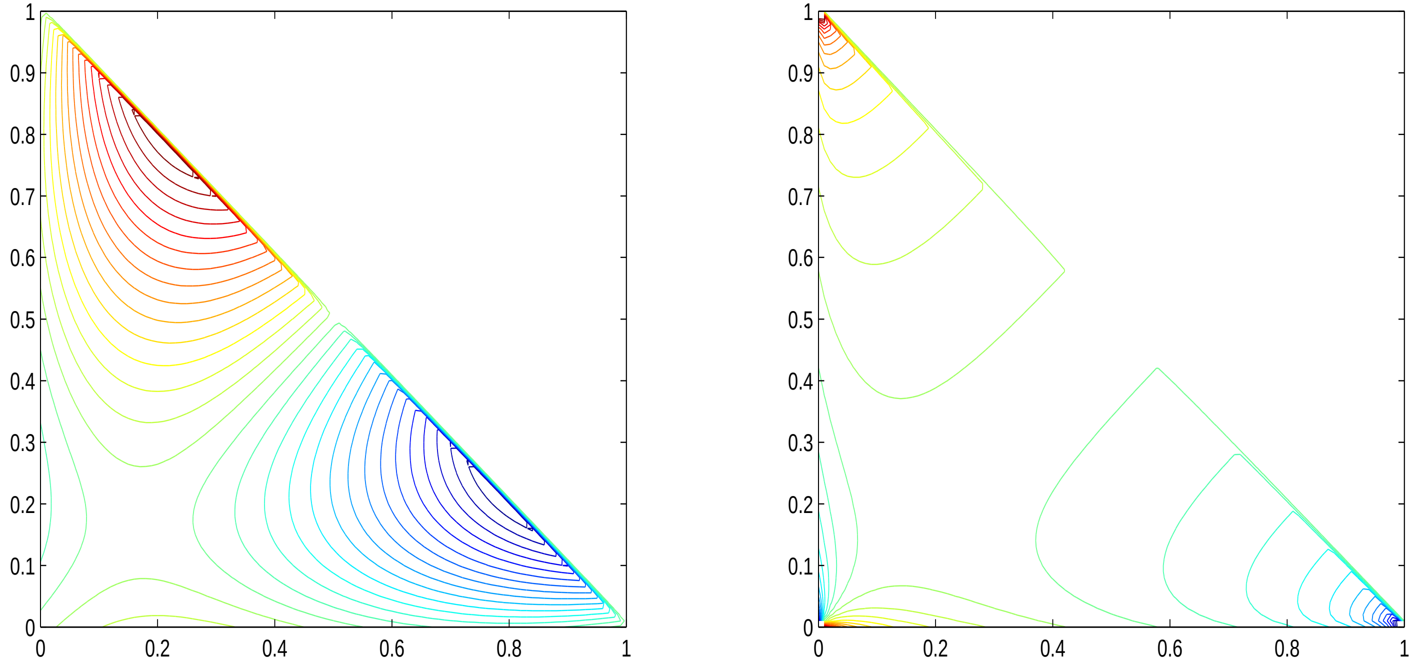
$$E\left[R_p\right] = r + \langle \mathbf{x}, E[\mathbf{R}] - r\mathbf{1} \rangle = r + \sum_{i=1}^{d} x_i (E[R_i] - r)$$

and :

$$\begin{aligned}
\mathrm{var}\left(R_p\right) &= \mathbf{x}' \; M_2(\mathbf{R}) \; \mathbf{x}, \\
m_3\left(R_p\right) &= \mathbf{x}' \; M_3(\mathbf{R}) \; \mathbf{x} \otimes \mathbf{x}, \\
m_4\left(R_p\right) &= \mathbf{x}' \; M_4(\mathbf{R}) \; \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x},
\end{aligned}$$

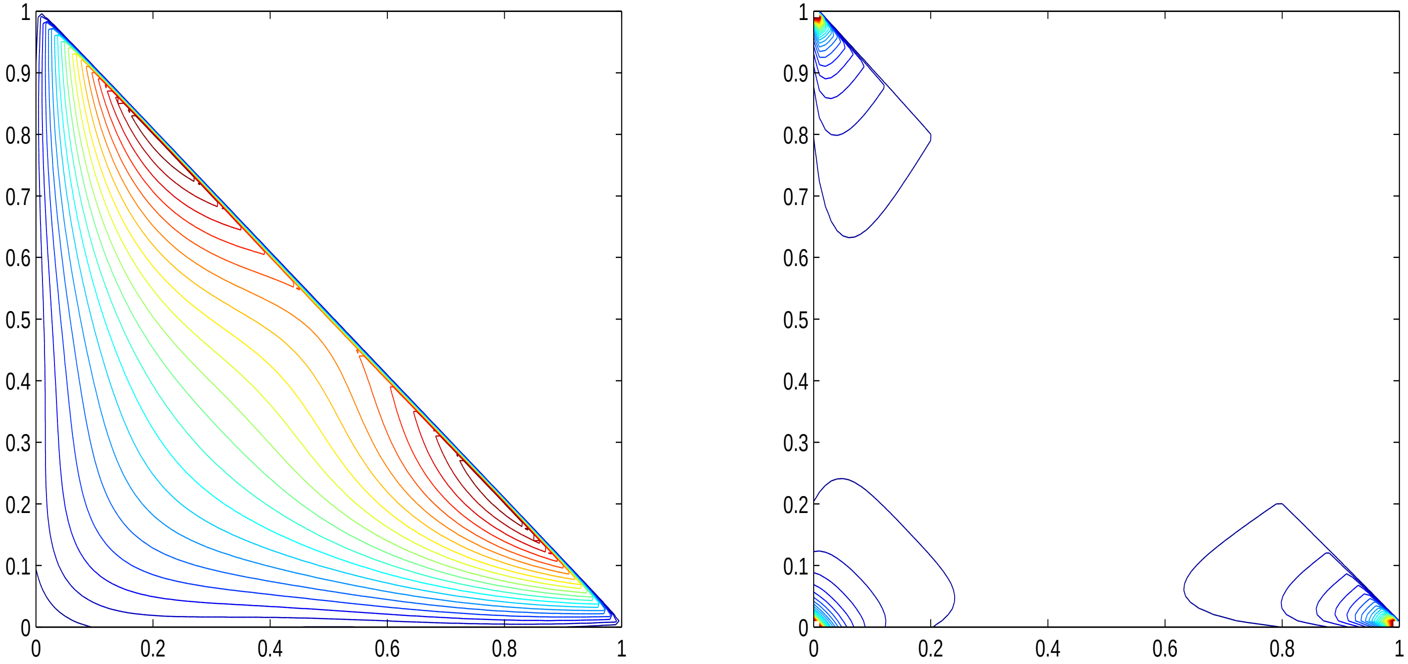
where $\otimes$ is the tensorial product.

Perspectives and conclusion



Iso-m_3 and iso-skewness lines in the Machina triangle.

Perpectives and conclusion



Iso-m_4 and iso-kurtosis lines in the Machina triangle.