The Utility Premium and The Risk Premium as Jensen's Gaps: A Unified Approach

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Risk and Choice: A Conference in Honor of Louis Eeckhoudt
Toulouse School of Economics

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Motivations

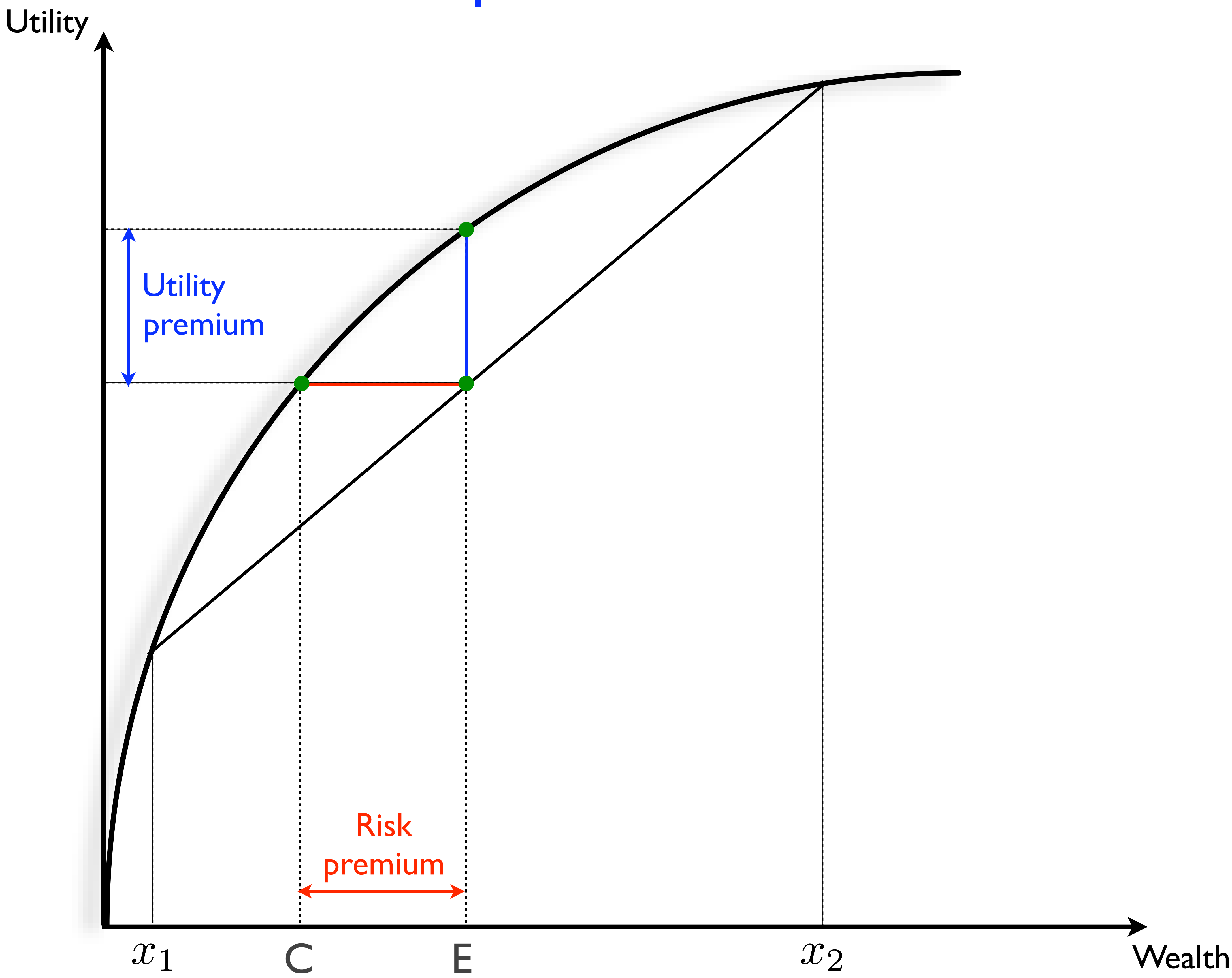
- Louis (and his close friends)'s *coming out* : « Pratt's paper has become too famous », one should focus on the utility premium.

- The current economic crisis has raised serious challenges to the ways choices under uncertainty are modelled.
  - Expected Utility, Non-Expected Utility, ... etc,
  - Distributional assumptions: Tail events, black swan, ... etc.

- Even without the current crisis, far too many so-called puzzles remain, which still beg for explanations:
  - Asset allocation between risk-free and risky assets,
  - The magnitude of the equity risk premium.

- Talking a lot with R. Nock made us discover very promising interactions with computer science & information theory

Graphical Intuition



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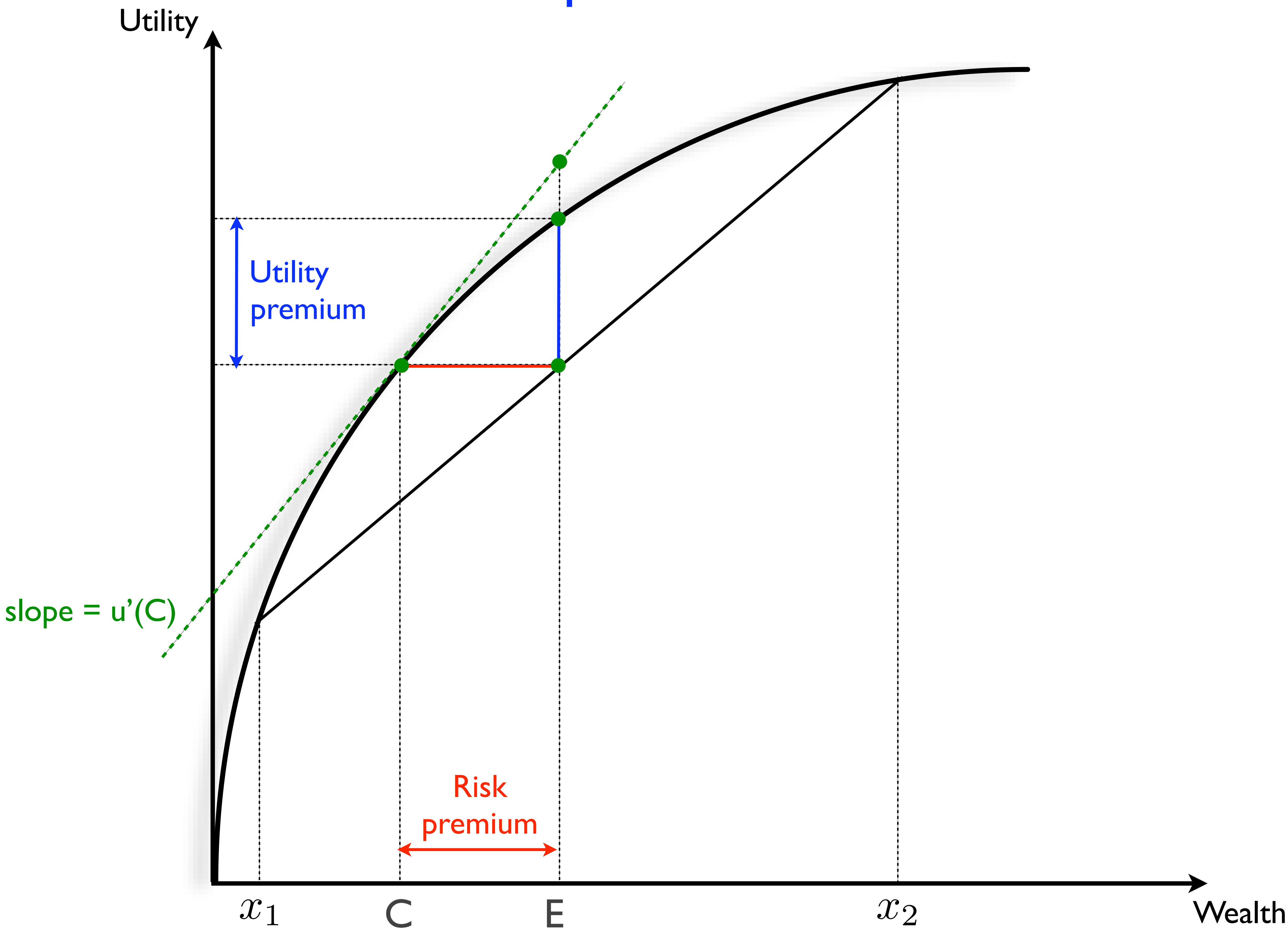

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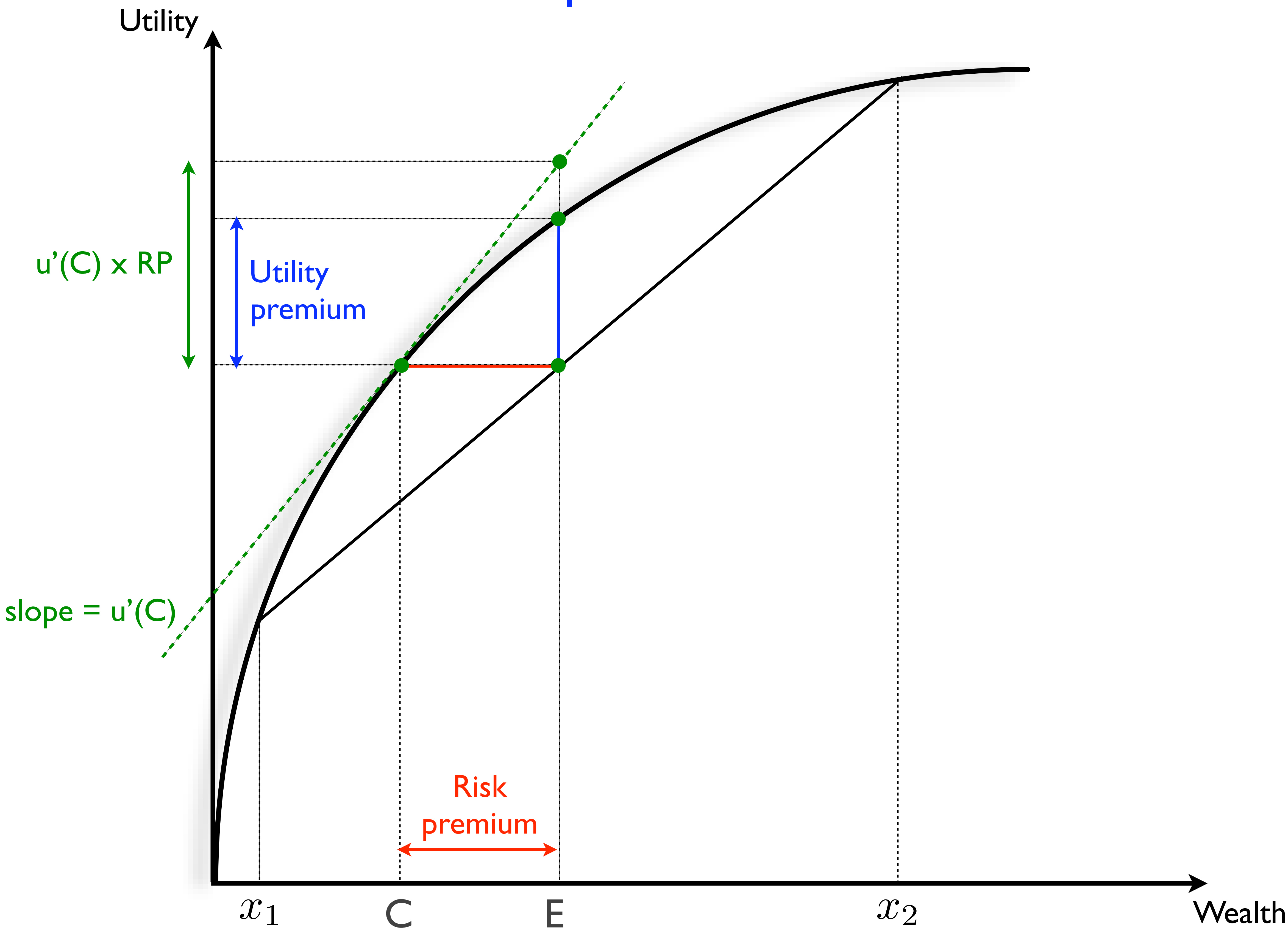

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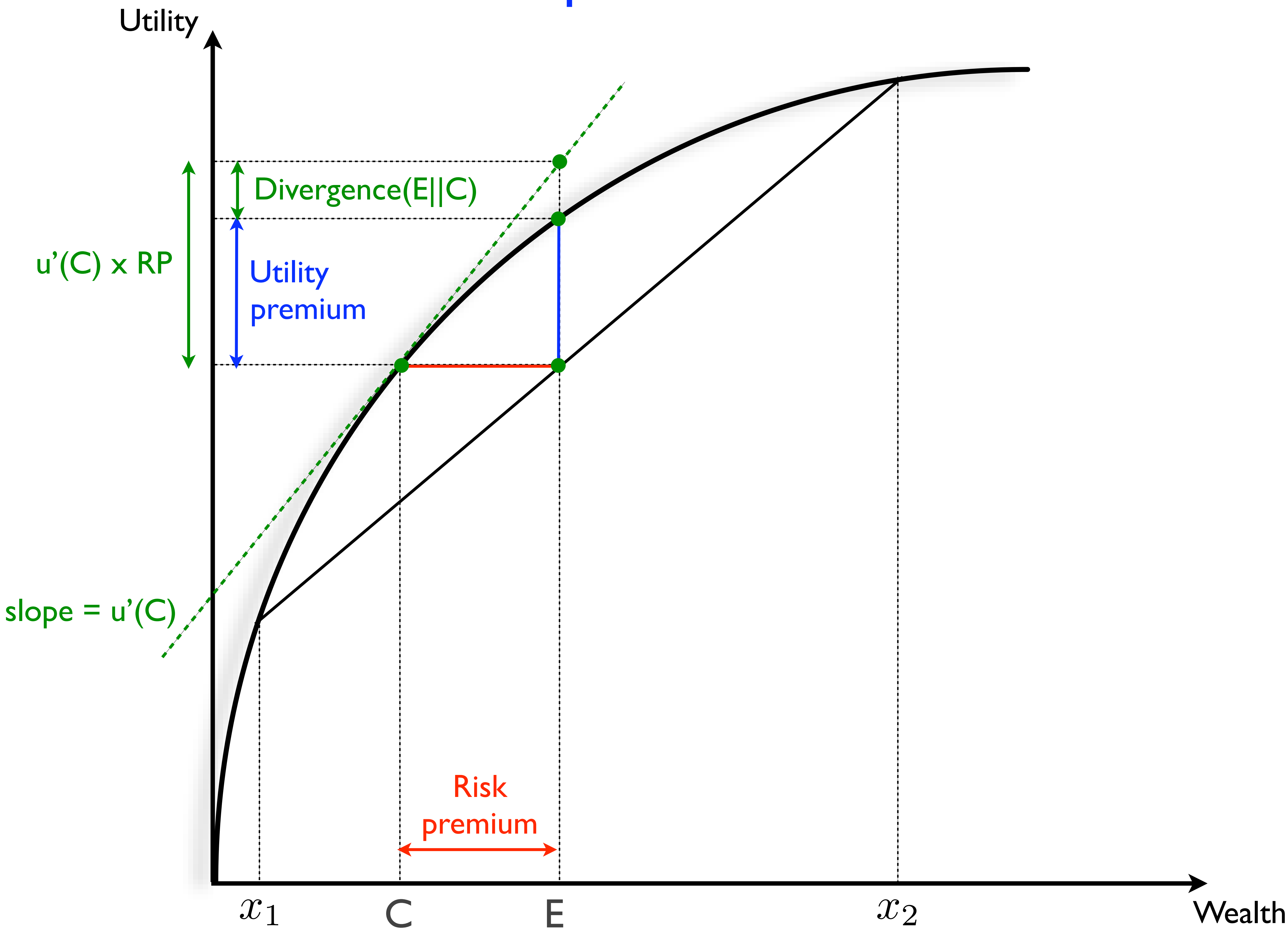

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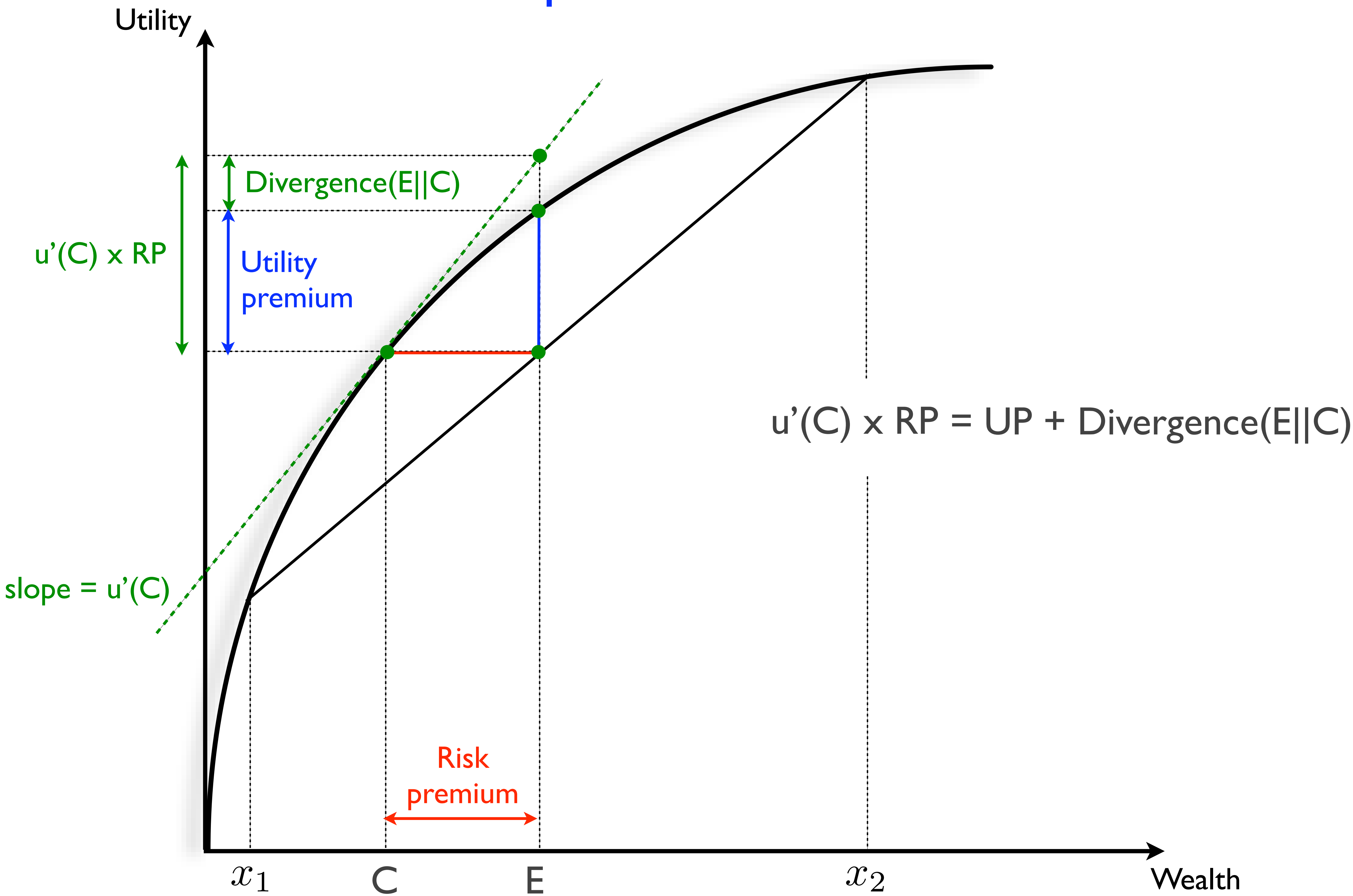

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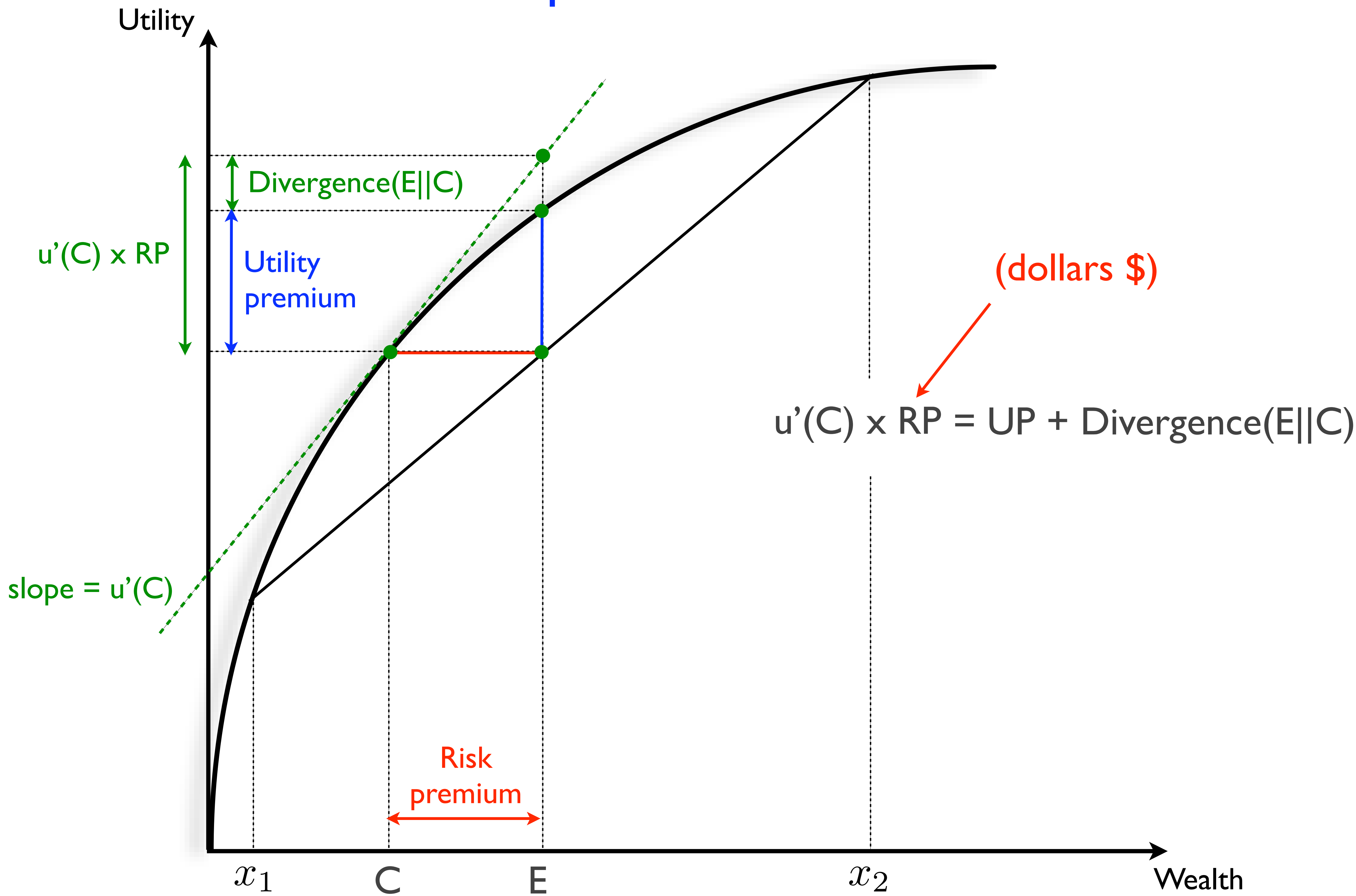

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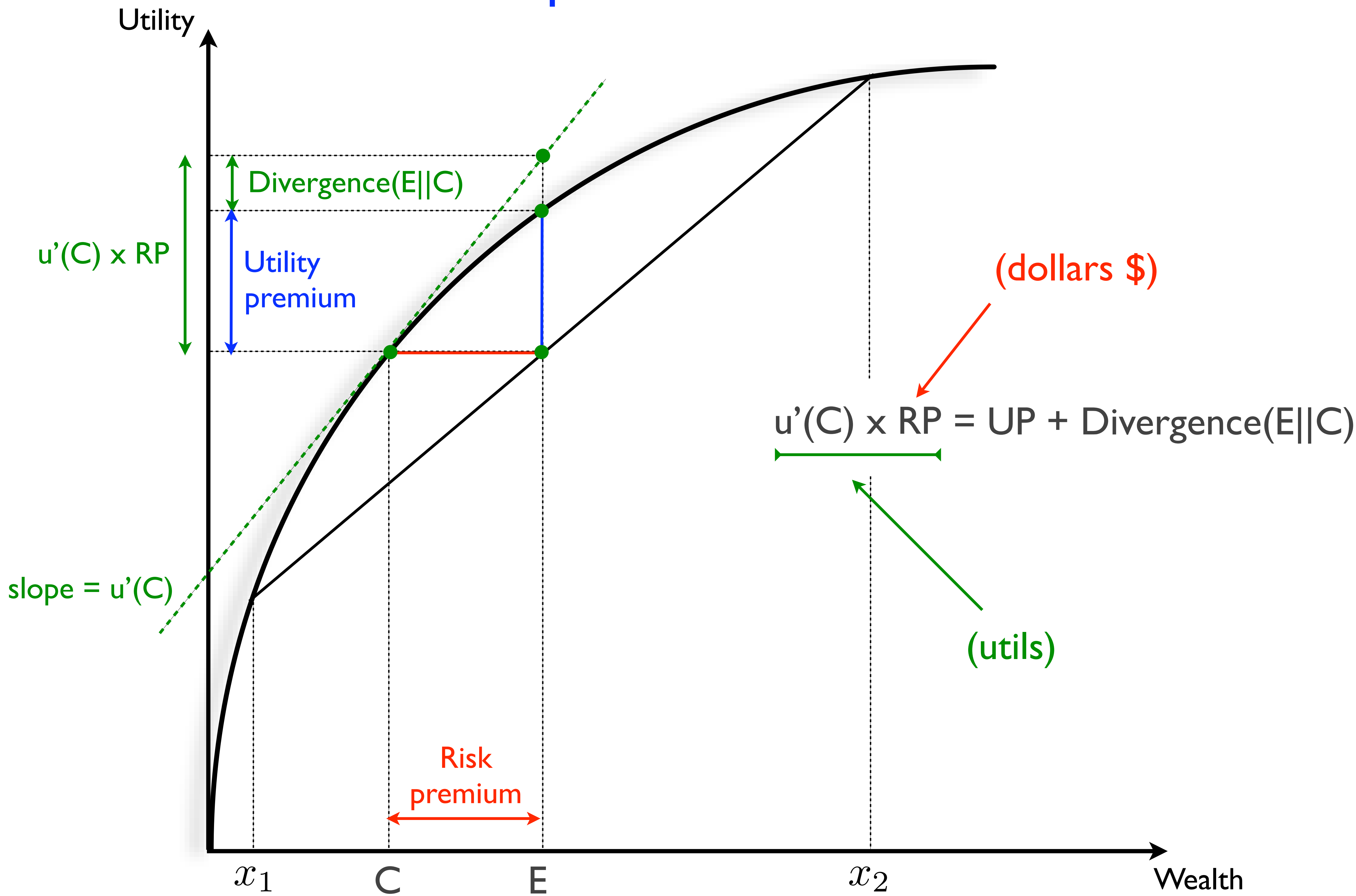


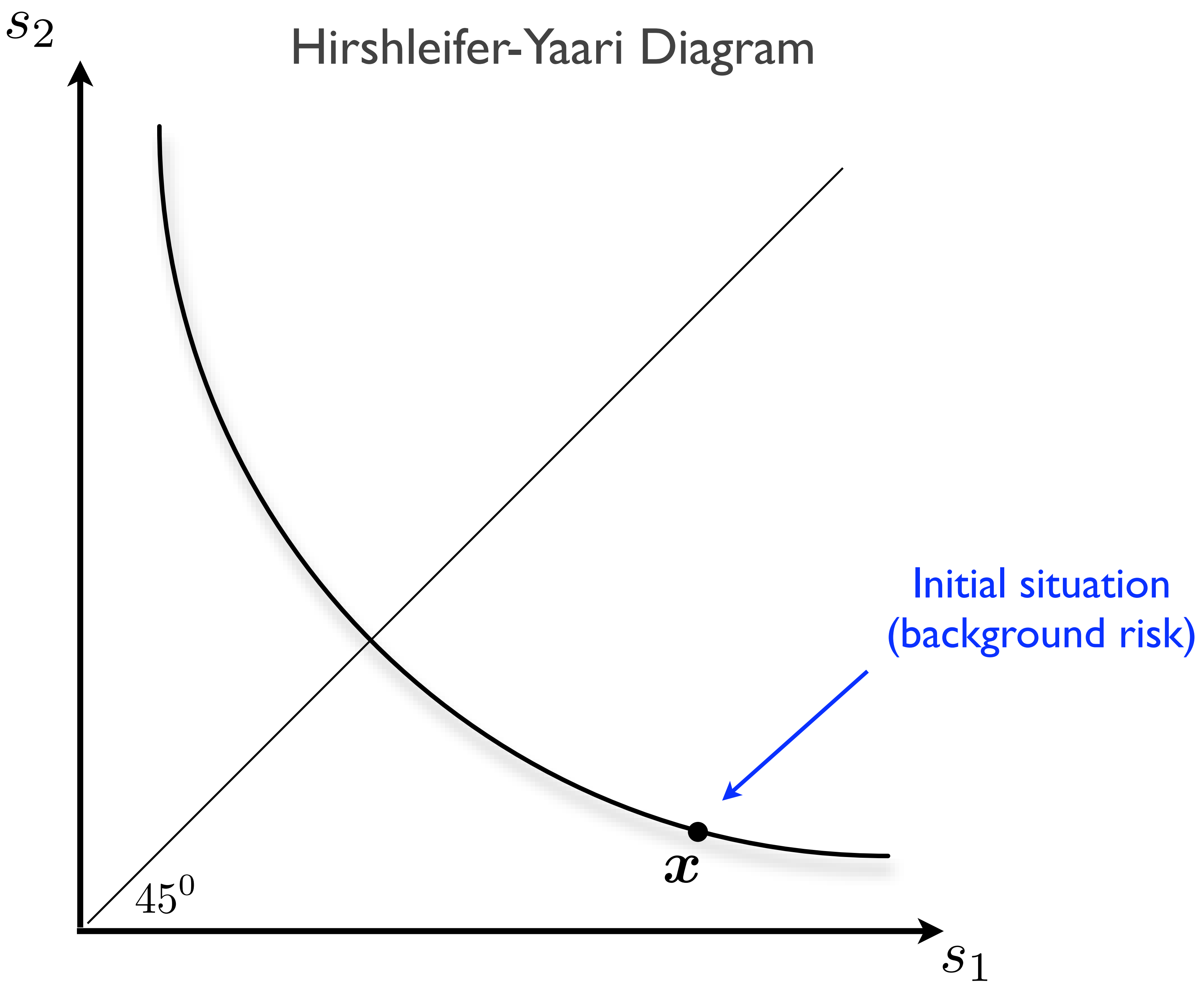
The Framework

- Notation:
  - A finite set of states of the world $\mathscr{S} := \{s_1, s_2, \ldots, s_n\}$
  - A outcome $x_i \in \mathscr{X}$ is associated to each state $s_i \in \mathscr{S}$
  - A risky situation $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathscr{X}^n$
  - A riskless situation $\boldsymbol{c} = (c, \ldots, c) \in \mathscr{X}^n$

- Smooth Preferences:
  - Complete, transitive, continuous, monotone
    $\exists \mathcal{V} : \mathscr{X}^n \longrightarrow \mathbb{R}$ so that $\boldsymbol{x} \succeq \boldsymbol{y} \Longleftrightarrow \mathcal{V}(\boldsymbol{x}) \geq \mathcal{V}(\boldsymbol{y})\,,\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathscr{X}^n$
  - We assume that $\mathcal{V}$ is differentiable

- Expected Utility Model: $\mathcal{V}_{\text{EU}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i\, u(x_i)\,,\ \forall\ \boldsymbol{x} \in \mathscr{X}^n$

The Framework



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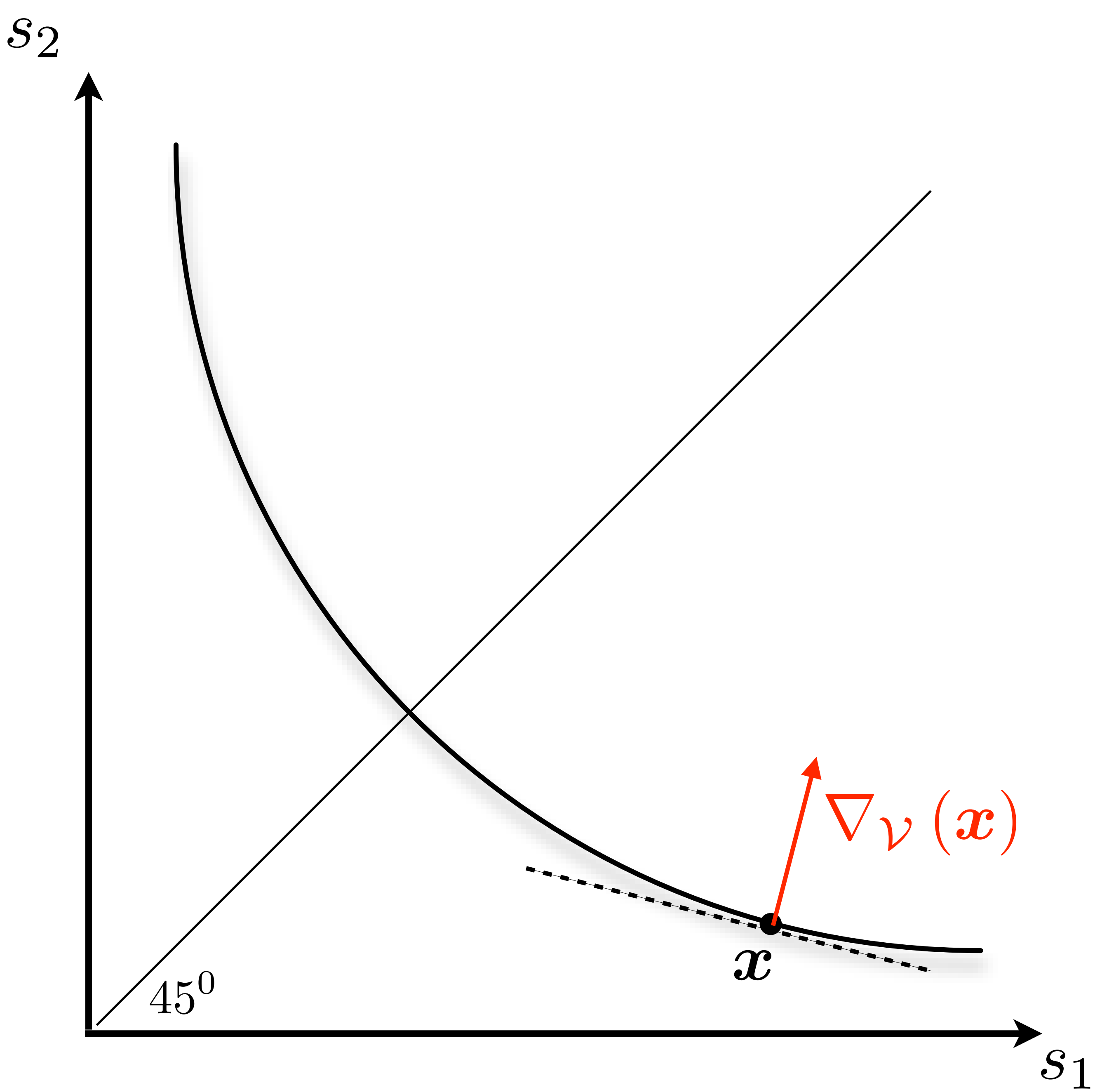

- The gradient of $\mathcal{V}$ at $\boldsymbol{x}$:

$$\nabla_{\mathcal{V}}(\boldsymbol{x}) = \left(\frac{\partial \mathcal{V}(\boldsymbol{x})}{\partial x_1}, \frac{\partial \mathcal{V}(\boldsymbol{x})}{\partial x_2}, \ldots, \frac{\partial \mathcal{V}(\boldsymbol{x})}{\partial x_n}\right)$$

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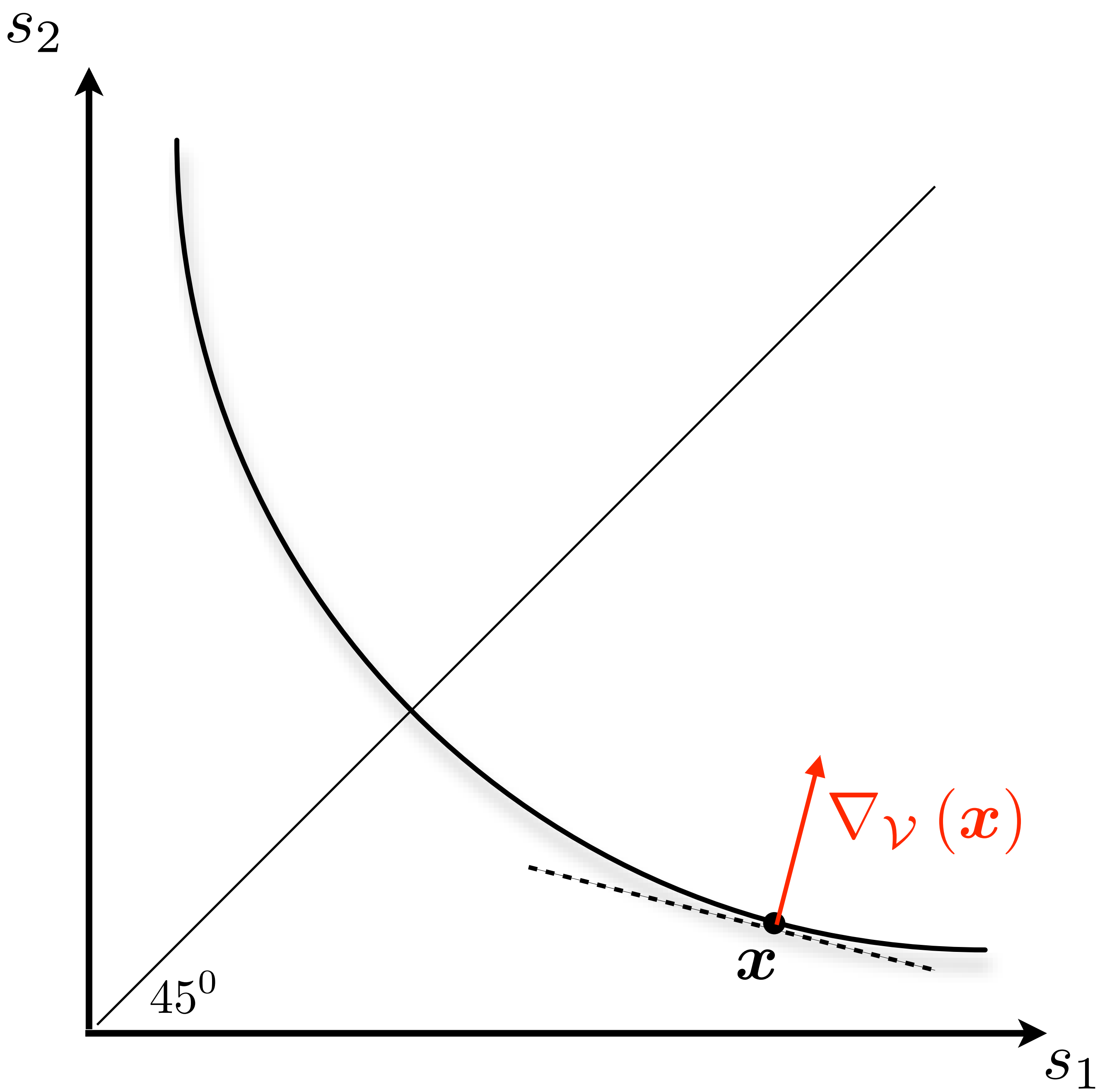

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- The risk-neutral probabilities :

$$\boldsymbol{\pi}(\boldsymbol{x}) = \frac{\nabla_{\mathcal{V}}(\boldsymbol{x})}{\nabla_{\mathcal{V}}(\boldsymbol{x}) \, . \, \mathbf{1}}$$

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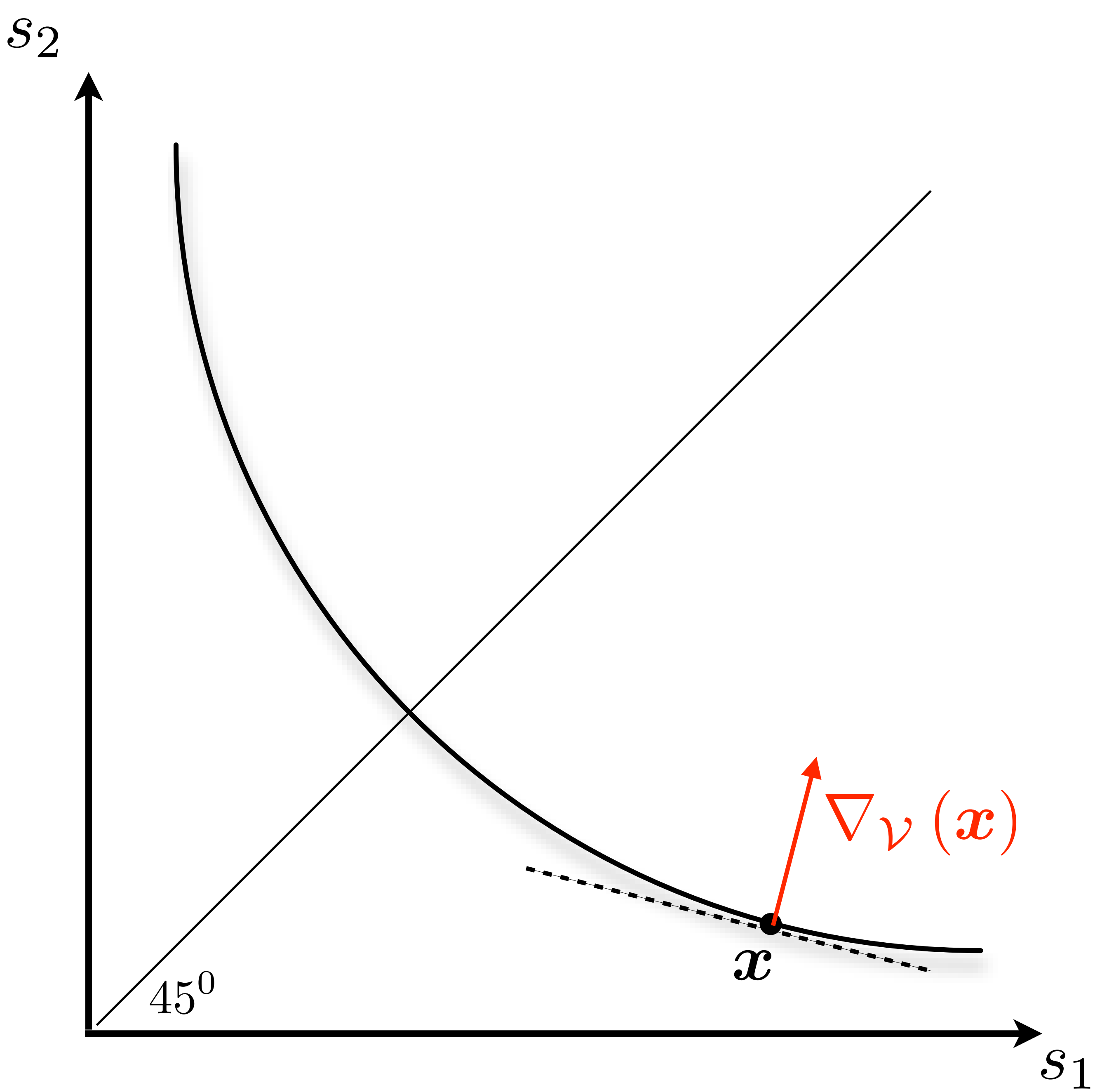

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- The risk-neutral probabilities :

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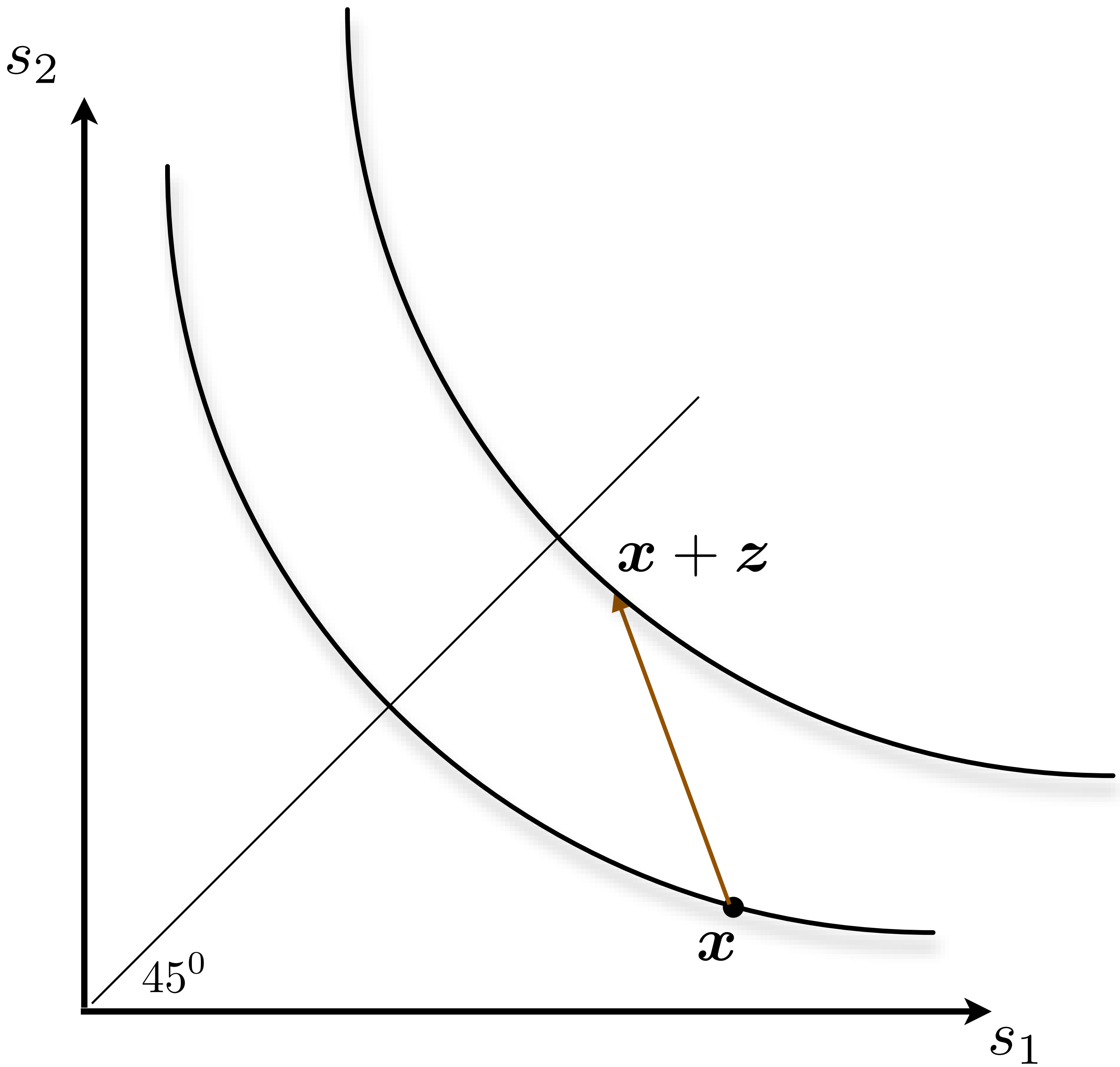
- If EU and riskless initial situation :

Risk-neutral prob. = True prob.

The Framework



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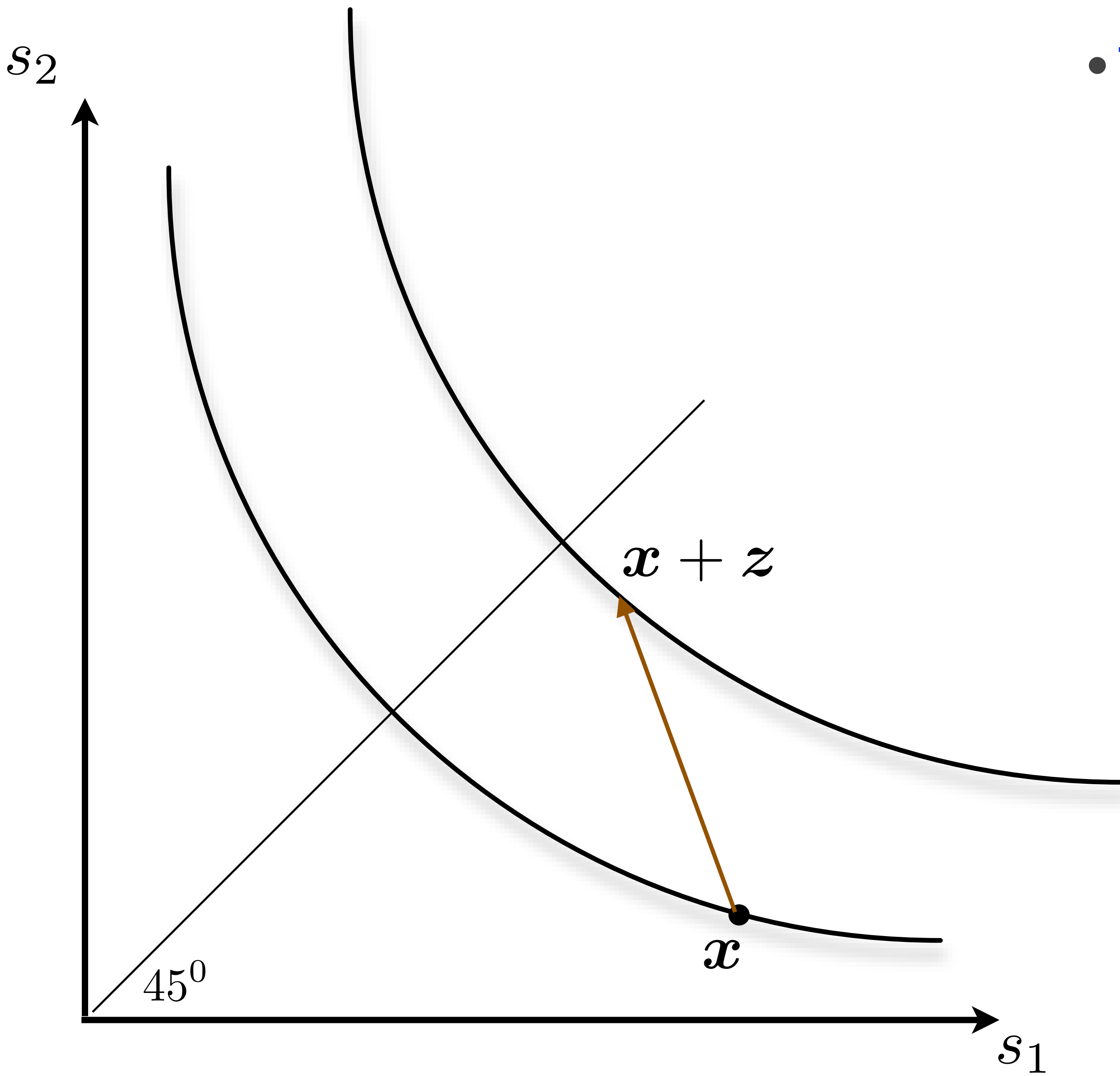
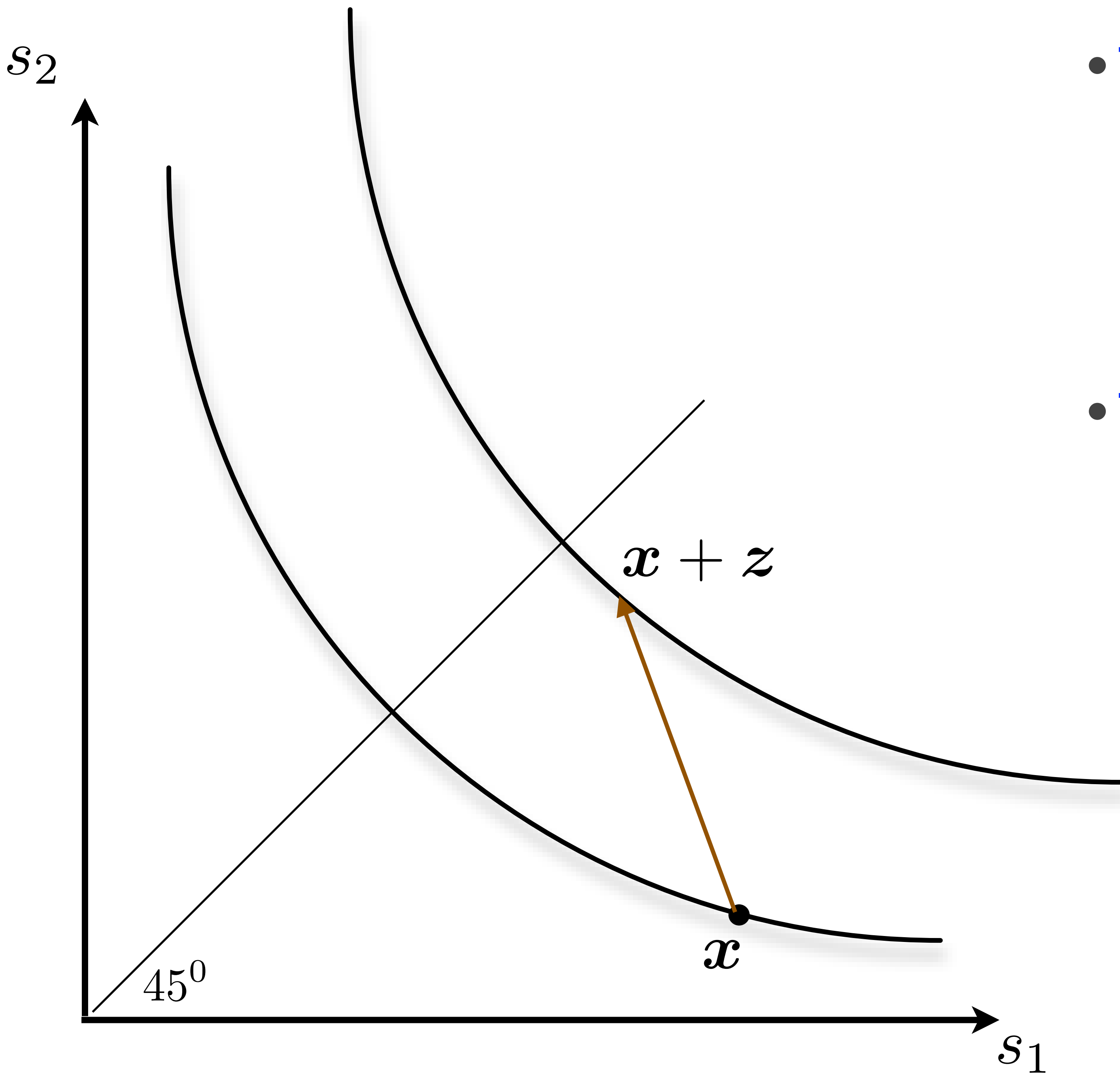


- The Utility Premium :

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = \mathcal{V}(\boldsymbol{x}+\boldsymbol{z}) - \mathcal{V}(\boldsymbol{x})$$

The Framework



- The Utility Premium :

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = \mathcal{V}(\boldsymbol{x}+\boldsymbol{z}) - \mathcal{V}(\boldsymbol{x})$$

- The Risk Premium :

$$\mathcal{V}(\boldsymbol{x}+\boldsymbol{z}) = \mathcal{V}\left(\boldsymbol{x}+\mathcal{C}(\boldsymbol{z};\boldsymbol{x})\,.\,\mathbf{1}\right)$$

where $\mathcal{C}(\boldsymbol{z};\boldsymbol{x}) = \mathrm{E}_{\pi(\boldsymbol{x})}(\boldsymbol{z}) - \mathcal{R}(\boldsymbol{z};\boldsymbol{x})$

Utility Premium for Small Risks

Expected Utility Model

Proposition 2. *Consider a twice differentiable function $u : \mathscr{X} \longrightarrow \mathbb{R}$, an initial riskless situation $\mathbf{c} \in \mathscr{X}^n$ and a small risk $\boldsymbol{z} \in \mathscr{X}^n$. We obtain the following approximation for the utility premium:*

$$\mathcal{J}_{\text{EU}}(\boldsymbol{z}; \mathbf{c}) \approx u'(c) \left[\mathrm{E}(\boldsymbol{z}) - \frac{1}{2} a(c) \, \mathrm{E}(\boldsymbol{z}^2) \right] .$$

where $a(c) = -u''(c)/u'(c)$ be the Arrow-Pratt coefficient of absolute risk aversion, at $c \in \mathscr{X}$.

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Smooth Preferences

Proposition 7. *Consider a function $\mathcal{V} \in \mathscr{D}$ twice differentiable, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$ and a small risk $\boldsymbol{z} \in \mathscr{X}^n$. We obtain the following approximation for the utility premium:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) \approx (\nabla_{\mathcal{V}}(\boldsymbol{x})\,.\,\mathbf{1})\left[\mathrm{E}_{\pi(\boldsymbol{x})}(\boldsymbol{z}) - \frac{1}{2}\,\boldsymbol{z}\,.\,\Sigma_{\pi(\boldsymbol{x})}\,.\,\boldsymbol{z}\right],$$

where $\Sigma_{\pi(\boldsymbol{x})} = -\nabla^2_{\mathcal{V}}(\boldsymbol{x})/\left(\nabla_{\mathcal{V}}(\boldsymbol{x})\,.\,\mathbf{1}\right)$, which depends on the Hessian matrix $\nabla^2_{\mathcal{V}}$ of $\mathcal{V}$.

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Utility Premium & Divergence Measure

Definition 2. *Consider a function* $\phi \in \mathscr{D}$ *and two situations* $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{X}^n$. *A Bregman divergence* $D_\phi : \mathscr{X}^n \times \mathscr{X}^n \longrightarrow \mathbb{R}$ *is defined by:*

$$D_\phi(\boldsymbol{x} \| \boldsymbol{y}) = \phi(\boldsymbol{x}) - \phi(\boldsymbol{y}) - (\boldsymbol{x} - \boldsymbol{y}) \,.\, \nabla_\phi(\boldsymbol{y}) \,.$$

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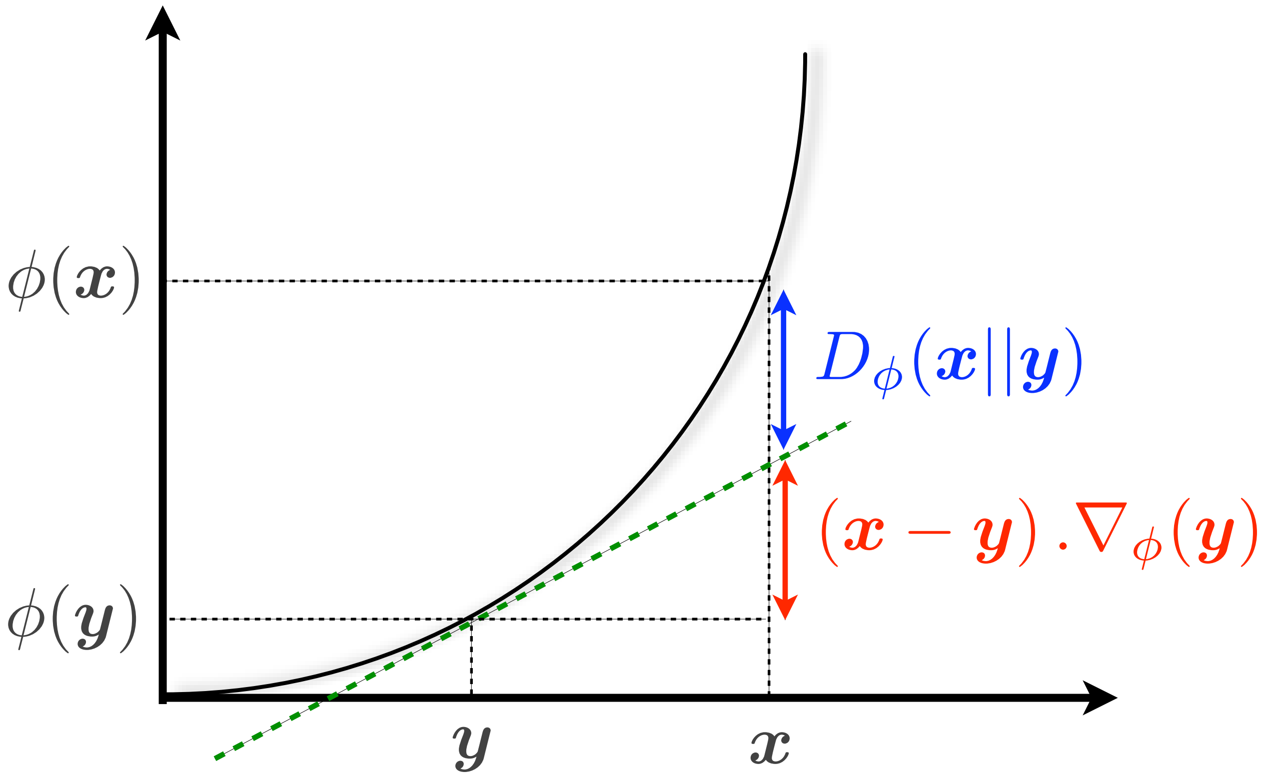
Geometric Interpretation



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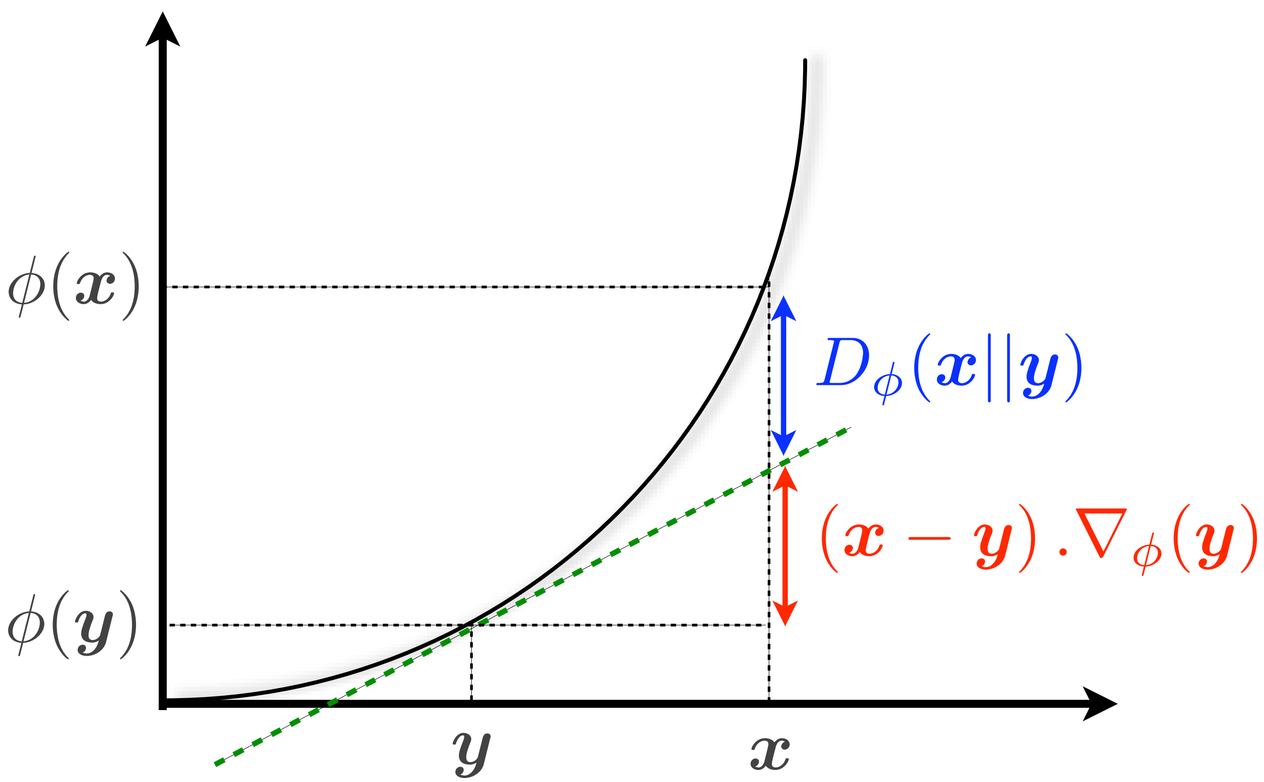

(Differential) Calculus Interpretation

$$\phi(\boldsymbol{x}) = \phi(\boldsymbol{y}) + (\boldsymbol{x} - \boldsymbol{y}) \,.\nabla_\phi(\boldsymbol{y}) + D_\phi(\boldsymbol{x}||\boldsymbol{y})$$

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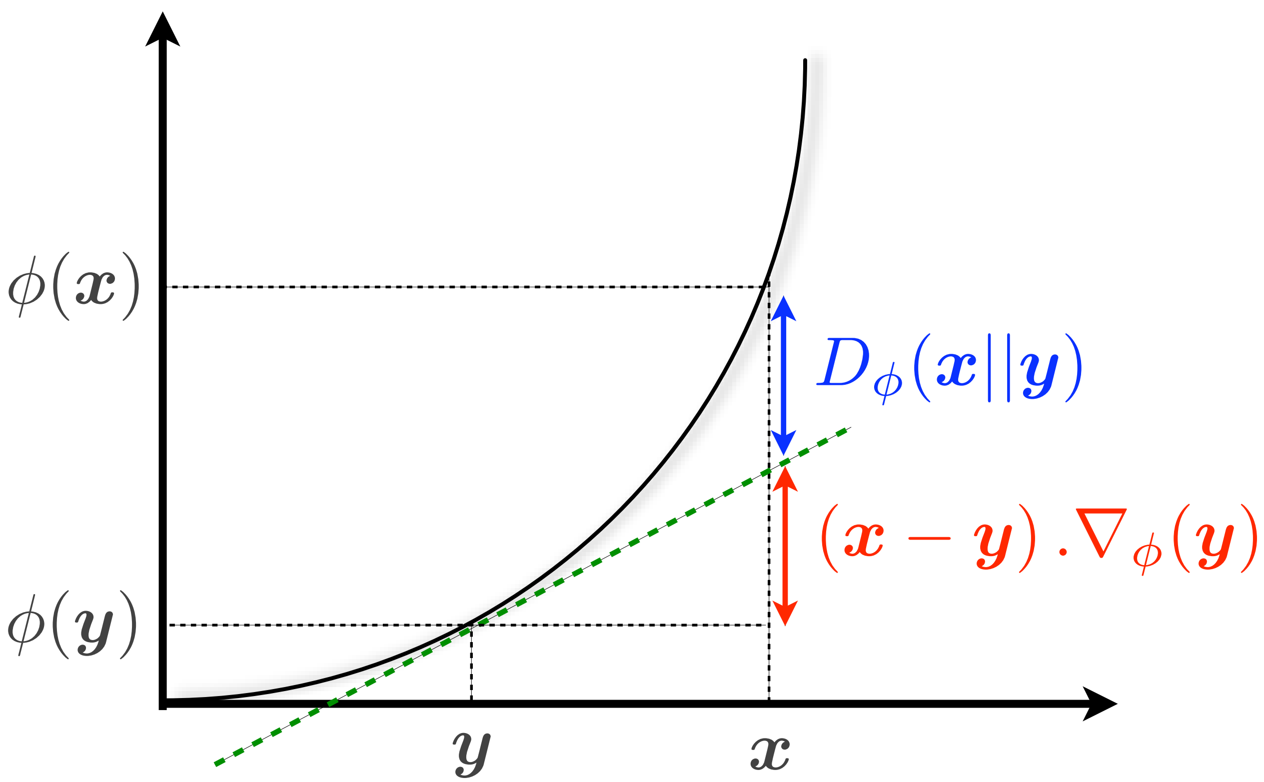

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The rest of a Taylor's series expansion

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Proposition 8. *Consider a function $\mathcal{V} \in \mathscr{D}$, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$ and a risk $\boldsymbol{z} \in \mathscr{X}^n$. The utility premium can be rewritten as:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = (\nabla_\mathcal{V}(\boldsymbol{x}) \,.\, \mathbf{1}) \left[\mathrm{E}_{\pi(\boldsymbol{x})}(\boldsymbol{z}) - D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x}) \right] \,.$$

where $\phi_{\boldsymbol{x}}(\boldsymbol{y}) = -\mathcal{V}(\boldsymbol{y}) / \left(\nabla_\mathcal{V}(\boldsymbol{x}) \,.\, \mathbf{1}\right)$. It is worthwhile observing that, for small risks $\boldsymbol{z} \in \mathscr{X}^n$, $D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x})$ corresponds to the risk premium.

Utility Premium & Divergence Measure

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where $\phi_{\boldsymbol{x}}(\boldsymbol{y}) = -\mathcal{V}(\boldsymbol{y}) / (\nabla_{\mathcal{V}}(\boldsymbol{x}) \,.\, \mathbf{1})$. *It is worthwhile observing that, for small risks* $\boldsymbol{z} \in \mathscr{X}^n$, $D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x})$ *corresponds to the risk premium.*

Illustration: EU with Zero-Mean Risk

$$\max_{\boldsymbol{z}\in\mathscr{X}^n} \mathcal{J}_{\mathrm{EU}}(\boldsymbol{z};\boldsymbol{x}) \quad \Longleftrightarrow \quad \min_{\boldsymbol{z}\in\mathscr{X}^n} D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x}+\boldsymbol{z}\|\boldsymbol{x})$$

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Utility function	**Related Divergence**
Quadratic	Squared Euclidean distance
Logarithmic	Itakura-Saito divergence
Power (CRRA)	Bregman-Csiszar divergence
Exponential (CARA)	(see Magdalou & Nock, JET 2011)

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If riskless initial situation :

Variance (→ Squared Euclidean distance)

2nd Theil measure (→ Itakura-Saito divergence)

Applications

- Generalizing Markowitz:
  - Mean-Divergence / *non-Normal distributions* (Briys & al., 2010)
- Generalizing Tobin's asset allocation:
  - Optimization with *cumulant generating functions* (Martin, 2012)
- ... and applications in other economic disciplines:
  - Measure of *goodness-of-fit* (Cowell, Davidson & Flachaire, 2011),
  - Measure of *distance between fair an unfair income distributions* (Magdalou & Nock, 2011) ... etc.

Thanks for your attention ...

Many thanks to Mr Louis !

More Results

Proposition 1. *Consider a differentiable function* $u : \mathscr{X} \longrightarrow \mathbb{R}$, *an initial riskless situation* $\mathbf{c} \in \mathscr{X}^n$ *and a small risk* $\boldsymbol{z} \in \mathscr{X}^n$. *We obtain the following approximation:*

$$\mathcal{J}_{\mathrm{EU}}(\boldsymbol{z};\mathbf{c}) \approx u'(c)\left[\mathrm{E}(\boldsymbol{z}) - \mathcal{R}_{\mathrm{EU}}(\boldsymbol{z};\mathbf{c})\right] .$$

Proposition 2. *Consider a twice differentiable function* $u : \mathscr{X} \longrightarrow \mathbb{R}$, *an initial riskless situation* $\mathbf{c} \in \mathscr{X}^n$ *and a small risk* $\boldsymbol{z} \in \mathscr{X}^n$. *We obtain the following approximation for the utility premium:*

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where $a(c) = -u''(c)/u'(c)$ *be the Arrow-Pratt coefficient of absolute risk aversion, at* $c \in \mathscr{X}$.

More Results

Proposition 3. *Consider a differentiable function $u : \mathscr{X} \longrightarrow \mathbb{R}$, an initial riskless situation $\boldsymbol{c} \in \mathscr{X}^n$ and a risk $\boldsymbol{z} \in \mathscr{X}^n$. The utility premium can be rewritten as:*

$$\mathcal{J}_{\mathrm{EU}}(\boldsymbol{z};\boldsymbol{c}) = u'(c)\,\left[\mathrm{E}(\boldsymbol{z}) - D_{\phi_c}(\boldsymbol{c}+\boldsymbol{z}\|\boldsymbol{c})\right]\,,$$

where $\phi_c(\boldsymbol{y}) = -\mathcal{V}_{\mathrm{EU}}(\boldsymbol{y})/u'(c)$. It is worthwhile observing that, for small risks $\boldsymbol{z} \in \mathscr{X}^n$, the second term in (3) within bracket is the risk premium.

Proposition 4. *Consider a function $\mathcal{V} \in \mathscr{D}$, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$ and a risk $\boldsymbol{z} \in \mathscr{X}^n$. The utility premium can be rewritten as:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = \int_0^1 \left[\boldsymbol{z}\,.\,\nabla_{\mathcal{V}}(\boldsymbol{x}+\lambda\boldsymbol{z})\right] d\lambda\,.$$

More Results

Proposition 5. *Consider a function* $\mathcal{V} \in \mathscr{D}$*, an initial situation* $\boldsymbol{x} \in \mathscr{X}^n$ *and a risk* $\boldsymbol{z} \in \mathscr{X}^n$*. The utility premium can be rewritten as:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = \int_0^1 \alpha(\boldsymbol{x}+\lambda\boldsymbol{z})P(\boldsymbol{z};\boldsymbol{x}+\lambda\boldsymbol{z})d\lambda\,,$$

where $\alpha(\boldsymbol{y}) = (\nabla_{\mathcal{V}}(\boldsymbol{y})\,.\,\mathbf{1}) \geq 0$ *by vertue of monotonicity of* $\mathcal{V}$*.*

Proposition 6. *Consider a function* $\mathcal{V} \in \mathscr{D}$ *twice differentiable, an initial situation* $\boldsymbol{x} \in \mathscr{X}^n$ *and a small risk* $\boldsymbol{z} \in \mathscr{X}^n$*. We obtain the following approximation:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) \approx (\nabla_{\mathcal{V}}(\boldsymbol{x})\,.\,\mathbf{1})\left[\mathrm{E}_{\pi(\boldsymbol{x})}(\boldsymbol{z}) - \mathcal{R}(\boldsymbol{z};\boldsymbol{x})\right]\,.$$

More Results

Proposition 7. *Consider a function $\mathcal{V} \in \mathscr{D}$ twice differentiable, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$ and a small risk $\boldsymbol{z} \in \mathscr{X}^n$. We obtain the following approximation for the utility premium:*

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where $\Sigma_{\pi(\boldsymbol{x})} = -\nabla^2_{\mathcal{V}}(\boldsymbol{x}) / \left(\nabla_{\mathcal{V}}(\boldsymbol{x}) \,.\, \mathbf{1}\right)$, *which depends on the Hessian matrix* $\nabla^2_{\mathcal{V}}$ *of* $\mathcal{V}$.

Proposition 8. *Consider a function $\mathcal{V} \in \mathscr{D}$, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$ and a risk $\boldsymbol{z} \in \mathscr{X}^n$. The utility premium can be rewritten as:*

$$\mathcal{J}(\boldsymbol{z};\boldsymbol{x}) = (\nabla_{\mathcal{V}}(\boldsymbol{x}) \,.\, \mathbf{1}) \left[\mathrm{E}_{\pi(\boldsymbol{x})}(\boldsymbol{z}) - D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x}) \right] .$$

where $\phi_{\boldsymbol{x}}(\boldsymbol{y}) = -\mathcal{V}(\boldsymbol{y}) / \left(\nabla_{\mathcal{V}}(\boldsymbol{x}) \,.\, \mathbf{1}\right)$. *It is worthwhile observing that, for small risks $\boldsymbol{z} \in \mathscr{X}^n$, $D_{\phi_{\boldsymbol{x}}}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x})$ corresponds to the risk premium.*

More Results

Proposition 9. *Consider a function $\mathcal{V} \in \mathscr{D}$ twice differentiable, an initial situation $\boldsymbol{x} \in \mathscr{X}^n$, a risk $\boldsymbol{z} \in \mathscr{X}^n$ and a function ϕ_x as defined in Proposition 8. We have:*

$$D_{\phi_x}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x}) = \int_0^1 (1 - \lambda) \left[\boldsymbol{z} \,.\, \Sigma_{\pi(\boldsymbol{x})}(\lambda) \,.\, \boldsymbol{z} \right] d\lambda \,.$$

where $\Sigma_{\pi(\boldsymbol{x})}(\lambda) = -\nabla^2_{\mathcal{V}}(\boldsymbol{x}+\lambda\boldsymbol{z}) / \left(\nabla_{\mathcal{V}}(\boldsymbol{x}) \,.\, \mathbf{1}\right)$. For small risks $\boldsymbol{z} \in \mathscr{X}^n$, we have $\Sigma_{\pi(\boldsymbol{x})}(\lambda) \approx \Sigma_{\pi(\boldsymbol{x})}$, as defined in Proposition 7, and $D_{\phi_x}(\boldsymbol{x} + \boldsymbol{z} \| \boldsymbol{x}) \approx \frac{1}{2} \boldsymbol{z} \,.\, \Sigma_{\pi(\boldsymbol{x})} \,.\, \boldsymbol{z}$.