Economic Consequences of Nth-Degree Risk Increases and Nth-Degree Risk Attitudes

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The Problem

$$Max_x E\left[U\left(K - xp, x\tilde{\mu} + \tilde{\alpha}\right)\right].$$

- What is the effect of an increase in additive -endowment- risk $(\tilde{\alpha})$ or multiplicative -asset payoff- risk $(\tilde{\mu})$ in the choice variable x?
    - General problem with R&S changes in risk: Dardanoni (1988)
    - Increases in Nth-degree risk (Ekern, 1980): Precautionary saving (Eeckhoudt and Schlesinger, 2008), precautionary labor supply (Chiu and Eeckhoudt, 2010), portfolio choice (Chiu, Eeckhoudt, Rey, 2011), Environmental uncertainty (Baiardi and Menegatti, 2011).
- Focus of extant literature: Finding *sufficient* conditions for unambiguous comparative statics of risk

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  1. The direction of the response to a change in Nth-degree risk
  2. Properties of the bivariate utility function
  3. Preferences towards a particular class of bivariate lottery pairs.

A Preliminary Result

Lemma

Let q be a given real valued function that is N times continuously differentiable on $\mathbb{R}_+$. The following are equivalent.

1. *For all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, we have $E\left[q(\widetilde{\alpha}_2)\right] \geq E\left[q(\widetilde{\alpha}_1)\right]$.*
2. *For all $x \geq 0$, we have $(-1)^N q^{(N)}(x) \geq 0$.*

- Ekern (1980) shows that *2.* implies *1.*
- We show the reverse implication that *1.* implies *2.*

Endowment Risk

Proposition

For all initial endowments, any increase in Nth-degree risk increases the optimal level of the choice variable if and only if for all (y, z) *we have* $(-1)^{N+1} pU^{(1,N)}(y,z) + (-1)^{N} \mu U^{(0,N+1)}(y,z) \geq 0$.

- Example: For the classical 2-date precautionary saving problem with $U(y,z) = u(y) + v(z)$, we obtain that an increase in Nth-degree risk increases saving if and only if $(-1)^{N} v^{(N+1)} \geq 0$. The "if" part of this result has been established by Eeckhoudt and Schlesinger (2008).

Uncertainty over the asset's payoff

Examples: saving with rate of return risk, labor supply with risky wages, portfolio choice

Proposition

For all initial endowments, any increase in Nth-degree risk over the asset's payoff increases the optimal level of the choice variable if and only if for all (y, z) *we have* $(-1)^N U^{(1,N)}(y,z) \leq 0$,
$(-1)^N \left(zU^{(0,N+1)}(y,z) + NU^{(0,N)}(y,z) \right) \geq 0$ *and*
$(-1)^N U^{(0,N)}(y,z) \geq 0$.

- Note *necessary* condition $(-1)^N U^{(0,N)}(y,z) \geq 0$: No consumer with mixed risk aversion in z (i.e. with $(-1)^N U^{(0,N)}(y,z) \leq 0$) always increases the demand for the asset!!

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- Note *necessary* condition $(-1)^N U^{(0,N)}(y,z) \geq 0$: No consumer with mixed risk aversion in z (i.e. with $(-1)^N U^{(0,N)}(y,z) \leq 0$) always increases the demand for the asset!!
- Measures of partial or relative risk attitudes cannot be applied to characterize *increases* in the choice variable for all initial endowment levels. Usually *assume* $(-1)^N U^{(0,N)}(y,z) \leq 0$.

Uncertainty over the asset's payoff

- It is true, however, that the optimal level of the choice variable *decreases* if we reverse the signs of the conditions above
  - For example, if relative prudence is positive and lower than 2 in the saving problem with separable utility and a mean preserving increase in return risk.

Lottery Choices and Optimal Exposure to Risk

Following the literature on apportioning of risks (e.g.*), we propose concepts of directional Nth-degree risk aversion that are characterized via preferences for harms disaggregation across outcomes of 50-50 bivariate lotteries.

* Eeckhoudt and Schlesinger (2006), Eeckhoudt, Rey, and Schlesinger (2007), Eeckhoudt, Schlesinger, and Tsetlin (2009), and Chiu, Eeckhoudt, and Rey (2011)

A location experiment

Starting from the 50-50 lottery $\left[\left(y, z+\widetilde{\alpha}_1\right); \left(y, z+\widetilde{\alpha}_2\right)\right]$, where $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, the DM must locate the bundles $\left(x_1\rho_y, x_1\rho_z\right)$ and $\left(x_2\rho_y, x_2\rho_z\right)$, with $x_2 > x_1$, across the outcomes of the lottery. To which outcome will she affect each bundle?

Additive risks

Definition

We say that preferences display *Nth-degree risk aversion in the direction of* $\left(\rho_y, \rho_z\right)$ if, for all $(y, z, x_1, x_2) \in \mathbb{R}^4_+$ such that $x_2 > x_1$ and for all pair of random variables $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, we have

$$\begin{aligned} & \left[\left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_2\right); \left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_1\right)\right] \\ \succ \; & \left[\left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_2\right); \left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_1\right)\right]. \quad (1) \end{aligned}$$

- For example, if $\rho_y > 0$ and $\rho_y > 0$, $\left(x_1\rho_y, x_1\rho_z\right)$ is the relatively "bad bundle", which a DM displaying the preferences in (1) prefers to locate with the relatively "good risk" $\widetilde{\alpha}_1$

Additive risks

Corollary

The following properties are equivalent:

1. *For all initial endowments, any increase in Nth-degree risk over the second attribute initial endowment increases the optimal level of the choice variable*
2. *For all* (y, z) *we have* $(-1)^{N+1} pU^{(1,N)}(y, z) + (-1)^{N} \mu U^{(0,N+1)}(y, z) \geq 0.$
3. *The preferences represented by U display Nth-degree risk aversion in the direction of* $\left(\rho_y, \rho_z\right) = (-p, \mu)$.

Additive risks

In the case with $\rho_y = 0$ and $\rho_z > 0$, and for a fixed value of y, Eeckhoudt et al. (2009) established that *2.* implies *3.* The contribution of our proposition is twofold.

- First, our bivariate notion of directional Nth-degree risk aversion is more encompassing.
- Second, the proposition characterizes a unique set of expected utility maximizers that display Nth-degree risk aversion in the direction of $\left(\rho_y, \rho_z\right)$.
- This is important because, without the "only if" part, we would not be able to establish a direct link between lottery choices and optimal exposure to risk, i.e. the equivalency of *3.* and *1.*

Multiplicative risks

Now $x_1\rho_z$ and $x_2\rho_z$ scale the risks $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$. The consumer evaluates the following lotteries:

$$\begin{aligned} L_1 &= \left[\left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_2\right); \left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_1\right)\right] \\ L_2 &= \left[\left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_2\right); \left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_1\right)\right]. \end{aligned} \quad (2)$$

Definition

We say that preferences display *Nth-degree multiplicative-risk attraction (resp aversion) in the direction of* $\left(\rho_y, \rho_z\right)$ if, for all (y, z, x_1, x_2) such that $x_2 > x_1 \geq 0$ and for all pair of random variables $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, we have $L_1 \succ L_2$ (resp. $L_2 \succ L_1$).

Multiplicative risks

Corollary

The following properties are equivalent:

1. *For all initial endowments and all asset's cost and payoff, an increase in Nth-degree risk over the asset's payoff increases (resp. decreases) the optimal level of the choice variable*
2. *For all* (y, z) *we have* $(-1)^N U^{(1,N)}(y, z) \leq 0$ *(resp.* ≥ 0*),* $(-1)^N U^{(0,N)}(y, z) \geq 0$ *(resp.* ≤ 0*) and* $(-1)^N \left(zU^{(0,N+1)}(y, z) + NU^{(0,N)}(y, z)\right) \geq 0$ *(resp.* ≤ 0*).*
3. *The preferences represented by U display Nth-degree multiplicative-risk attraction (resp. aversion) in the direction of* $\left(\rho_y, \rho_z\right) \in \mathbb{R}_- \times \mathbb{R}_+$.

Thank you!!!