Left monotone increase in risk: An overview and some new results

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Introduction

Counter-intuitive result of partial insurance

Let 2 Expected Utility Maximizers D1 and D2, both risk averse, such that D2 is more risk averse than D1, it can be the case that :

- the less risk averse D1 accept to pay an amount of c to exchange the more risky Y for the less risky X (in the classical sense of Rothschild and Stiglitz)
- the more risk averse D2 accept to pay only c' smaller than c for the same exchange

Introduction

This paradoxical result leads Jewitt(1989) to a new definition of increasing risk, where, for any 2 risk averse Expected Utility Maximizers with one more risk averse than the other, such counter-intuitive results cannot happen.

It turns out that in fact, this notion of increasing risk is model-free and has a lot of nice properties that can make it challenging the classical notion of increasing risk, the Mean Preserving Increase in Risk one, or MPIR (R&S).

Outline

1. Formal definition
2. Different characterizations of this stochastic order
3. Generating process justifying the name of "Left monotone increase in risk"
4. Corresponding definition of left monotone risk aversion and its properties
5. Left monotone increase in risk and deductible contracts
6. Promising notion of left monotone increase (or reduction) of inequalities (kind of Rawlsian notion)

New notion of increase in risk

The following definition is the exact translation in terms of Expected Utilities to restrict the notion of increase in risk in order to avoid such counter-intuitive result.

Definition of "location-independent risk" (Jewitt, 1989)

Y is location-independent riskier than X if:

$$\text{for any increasing and concave functions } u \text{ and } \eta : R \to R$$
$$\int u(x-c)dF_X(x) = \int u(y)dF_Y(x) \implies$$
$$\int \eta(u(x-c))dF_X(x) \geq \int \eta(u(y))dF_Y(x)$$

Properties

In this definition, the new notion of increase in risk seems to be linked with the EU model but in fact,

- this notion is model-free
- this stochastic order is interesting by itself and has a lot of nice properties similar to the ones of Rothschild&Stiglitz(1970) increase in risk
- this notion will be useful for many economic applications: deductible, call, minimum prices, inequalities...

Useful Property

Concave transformation of location-independent riskier :

If Y is location-independent riskier than X and

$g : R \to R$ is increasing and concave,

then $g(Y)$ is location-independent riskier than $g(X)$.

Definition of "Left Monotone increase in risk"

This notion of Location-Independent Riskier has been defined by Jewitt for any 2 random variables but, we define here the restriction to r.v. with equal means.

Definition

Y is a left monotone increase in risk of X if
(1) Y is location-independent riskier than X ;
(2) E(X)=E(Y).

From now on, we will always suppose E(X)=E(Y)

Different characterizations of left monotone increase in risk

Let $F_X^{-1}(t) = \inf \{x \in R : F_X(x) \geq t\}$

Characterization 1 in terms of integrals

Y is a left monotone increase in risk of X, if and only if

$$\int_{-\infty}^{F_Y^{-1}(p)} F_Y(x)dx \geq \int_{-\infty}^{F_X^{-1}(p)} F_X(x)dx \text{ for every } p \text{ in } [0,1]$$

(Jewitt 1989, Lansberger and Meilijson, 1994)

- Notice that in the integral definition above, the upper limits of integration are quantiles corresponding to equal cumulative probabilities p.
- If, instead, the upper limits of integration are equal, the corresponding integral condition becomes the Definition of MPIR

Characterization in terms of "more weight in lower tail"

Definition

The random variable Y has more weight in the lower tail (defined by p) than random variable X if

$$E\{y(p) - Y \ / \ Y \leq y(p)\} \geq E\{x(p) - X \ / \ X \leq x(p)\}$$

where $x(p)$ is the $100p^{th}$ percentile of X

Characterization 2 in terms of more weight in lower tail

Y is a left monotone increase in risk of X if Y has more weight in the lower tail than X

Example of Left monotone reduction of risk

Example

Y initial distribution ;
X Left monotone reduction in risk defined by :

$$
\begin{array}{l}
X = Y - \pi \quad \text{if} \quad Y > d \\
X = d - \pi \quad \text{if} \quad Y \leq d
\end{array}
$$

with conditions on π and d s.t. $E(X) = E(Y)$

- Deductible (Y is the possible loss)
- Call
- Minimum prices
  Eeckhoudt and Hansen, 1980.
- Reduction of inequalities

Generating Process for discrete random variables:

Special transfer à la Pigou Dalton

A canonical example :

Let X be a discrete random variable with
$L(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$,
where $x_1 < x_2 < x_3 < x_4$.
and let Y be :
$L(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)(1)$

We go from X to Y by a transfer à la Pigou-Dalton, that preserves the mean of the distribution : In this "spread of losses", the minimal outcome is always spread out, but not necessarily the maximal outcome .

Generating process for discrete random variables:

For every couple (X, Y) of discrete random variables, with $E(X) = E(Y)$ s.t. Y is a left monotone increase in risk of X,

Y can be reached from X by the finite sequence of transfers as in (1).

Generating process for all random variables

If we define a special "single crossing " of c.d.f. as follows:

Definition of Left monotone single crossing:

G is a single crossing of F such that $G^{-1}(v) - F^{-1}(v)$ is non-decreasing in the interval where it is negative.

The horizontal distance between the cdf is non-decreasing when negative

Generating process for all random variables

The left monotone increase in risk is
the transitive closure of left monotone single-crossing

(Landsberger & Meilijson, 1994)
This property justifies the denomination of left monotone increase in risk.

Characterization in terms of "quantiles"

Characterization 3 in terms of quantiles

X is left monotone less risky than Y if and only if the function of p:

$$\frac{1}{p}\int_0^p \left[F_X^{-1}(t) - F_Y^{-1}(t)\right] dt$$

is positive and decreasing in $p \in (0, 1]$.

The additional expected gain of X upon Y for the p% smallest gains is positive and decreasing in p,

where as, for the classical increase in risk, this difference is only positive.

Characterization in terms of "addition of a noise"

A function f from [0, 1] to [0, 1] is ceasaro-increasing if and only

$$f(u) \geq \frac{1}{u} \int_0^u f(v)dv.$$

this means that it exceeds always the average of its values to the left of the point.It is a very weak form of monotonicity.

Characterization 4: in terms of addition of a noise:

Y is left monotone increase in risk of X if

$$Y \underset{d}{=} X + Z$$

- where $X = F^{-1}(U)$
- where $E(Z) = 0$ and $Z = Z(U)$ is Cesaro-increasing.

Characterization in terms of "comparing functionals"

Let F denote the set of increasing functions f from $[0, 1]$ to $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$.

Definition:

A function $f \in F$ is star-shaped at m, if

$$\frac{f(m)-f(p)}{m-p}$$

is an increasing function of p on $[0, m) \cup (m, 1]$.

In particular, a function $f \in F$ is star-shaped at 1 , if

$$\frac{1-f(p)}{1-p}$$

is an increasing function of p on $[0, 1)$.

Characterization in terms of "comparing functionals"

Characterization 5: in terms of comparing functionals

Y is a left monotone increase in risk of X if and only if for every $f \in F$, star-shaped at 1,

$$\int f(P\{X > x\})dx \geq \int f(P\{Y > x\})dx$$

If we interpret the functional $V(X) = \int f(P\{X > x\})dx$ as characterizes a Yaari DM with perception function f, the characterization becomes:

Y is a left monotone increase in risk of X if and only if all the Yaari's Decision Makers with probability perception function star shaped at 1 prefer X to Y

Left monotone risk aversion

A decision maker is left monotone risk averse if and only if he is averse to left monotone increase in risk. Formally:

Definition of Left monotone risk aversion

A decision maker is left monotone risk averse if and only if for every X and Y s.t. Y is a left monotone increase in risk of X, then $X \succeq Y$

Comparison with other notions of Risk aversion (RA)

Strong Risk Aversion $\Rightarrow$ Left monotone Risk Aversion $\Rightarrow$ Monotone Risk Aversion $\Rightarrow$ Weak Risk Aversion

Characterization in EU model

In EU, all the different notions of RA are characterized by u concave.

Characterization in RDU model

A RDU DM characterized by (u, f) is Left Monotone Risk Averse if and only if

u is concave and f is star shaped at 1

Remark: a RDU having a function f, star shaped at 1 has an infinite derivative at 1, can be LMRA with a NON-concave u

Optimality of a deductible contract for a left monotone risk averse agent: Model free results

Generalization of Arrow(1965), Gollier&Schlesinger(1996)

Result 1

If a DM is left monotone risk averse, the optimal contract, for a given premium based on the expected value, is always a deductible contract

(Vergnaud 1997)

Result 2

Left monotone risk aversion is the weakest notion of risk aversion for which such a deductible contract is optimal

(Chateauneuf, Cohen, Vergnaud(2010))

Examples for different notions of "reduction of risk"

We can also interpret these distributions as income distributions and reductions as reductions of inequalities

Pr	1/5	1/5	1/5	1/5	1/5	
Y	-2000	-1000	0	+1000	+2000	
$X1$	-2000	0	0	0	+2000	R&S
$Z1$ $= Y - X1$	0	-1000	0	+1000	0	reduct. of risk
Y	-2000	-1000	0	+1000	+2000	
$X2$	-1250	-750	0	0	+2000	Left Monotone
$Z2 = Y - X2$	-750	-250	0	+100	0	reduct. of risk