The Comparative Statics of Changes in Risk for Most Decision Makers

Rachel J. Huang, Pai-Ta Shih, Larry Y. Tzeng

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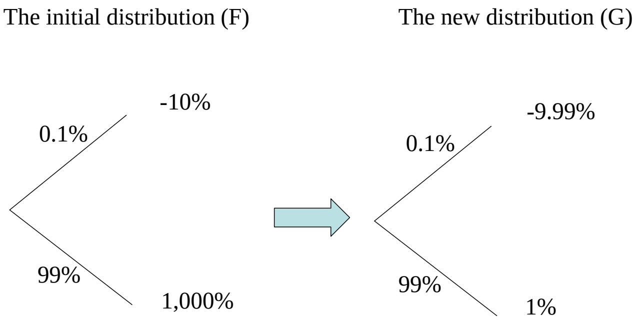
1. Introduction

Motivation

- The effect of changes in risk on optimal decisions has been well studied for decades since Rothschild and Stiglitz's (1971) pioneering work.
- Gollier (1995) found the necessary and sufficient condition under a change in risk for all risk-averse individuals to have unambiguous comparative statics.
  - In a static portfolio choice problem, Gollier (1995) concluded that *all* risk-averse investors will reduce their investment in a risky asset if and only if the distribution of the risky asset becomes centrally riskier (CR), i.e., the location-weighted probability mass function under the new cumulative distribution function (CDF) is uniformly less than that under the initial CDF times some scalars.

Example 1

- Assume that an investor is considering investing in a risk-free asset and a risky asset where the risk-free rate of return is zero.
  - The net return on the risky asset:



The location-weighted probability mass function

- The location-weighted probability mass function under F and G are

$$T_F(x) = \int_{\underline{x}}^{x} t dF(t) = \begin{cases} 0 \text{ if } x < -10\% \\ -0.1\% \text{ if } -10\% \leq x < 1000\% \\ 989.9\% \text{ if } x \geq 1000\%, \end{cases}$$

and

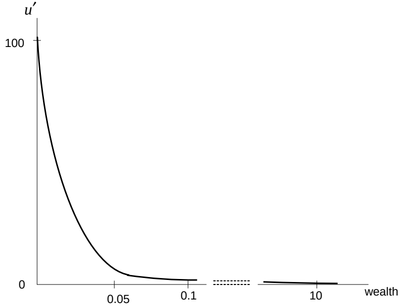
$$T_G(x) = \int_{\underline{x}}^{x} t dG(t) = \begin{cases} 0 \text{ if } x < -9.99\% \\ -9.99\% \text{ if } -9.99\% \leq x < 1\% \\ 0.8901\% \text{ if } x \geq 1\%. \end{cases}$$

Central Riskiness

- Gollier (1995) had shown that all risk-averse investors will reduce their investment in the risky asset if and only if there exists a real scalar γ such that
$$T_G(x) \leq \gamma T_F(x), \; \forall x.$$
- He further indicated that the above necessary and sufficient condition could be restated as $\inf_{\{x|T_F(x)<0\}} \frac{T_G(x)}{T_F(x)} \geq \sup_{\{x|T_F(x)>0\}} \frac{T_G(x)}{T_F(x)}$.
- In this case, we have
$$\inf_{\{x|T_F(x)<0\}} \frac{T_G(x)}{T_F(x)} = \frac{0.8901\%}{-0.1\%} < \sup_{\{x|T_F(x)>0\}} \frac{T_G(x)}{T_F(x)} = \frac{0.8901\%}{989.9\%}.$$
- However, will you reduce the investment in the risky asset?

All v.s. most risk-averse individuals

- In the above case, the individuals with utility function $u(z) = 1 - e^{-100z}$ would increase their investment in risky assets from 0.91% to 20.87% while facing such a change in risk.



- This type of utility function leads to a different decision from that of **most** decision makers mainly because the preferences are extreme and pathological as argued by Leshno and Levy (2002).

The purpose

- In this paper, we would like to extend Gollier (1995) by limiting the set of decision makers by assuming that individuals exhibit non-pathological preferences characterized by Leshno and Levy (2002).
- Leshno and Levy (2002) shed light on indicating the inability of stochastic dominance and mean-variance rules to distinguish two options which could be easily compared by most decision makers.
- They further establish a new concept of stochastic orders, i.e., almost stochastic dominance rules (ASD), for the decision makers with non-pathological preferences to rank distributions.

Contributions

- Our paper contributes to the literature by providing the unambiguous comparative statics for most decision makers characterized by non-pathological preferences as in Leshno and Levy (2002).
  - Our paper, however, differs from Leshno and Levy (2002) in that, for most decision makers, we find the conditions for the unambiguous comparative statics while they provided conditions to rank distributions.
- Since the set of the decision makers in this paper is smaller than that in Gollier (1995), who considers all risk-averse decision makers, we could obtain a weaker distribution condition than those in Gollier (1995). Thus, our condition could be applied to more cases in the real world.

2. Portfolio Problem

Assumptions

- Let us first discuss a standard static portfolio choice problem.
- Assume that the investor with one dollar respectively chooses to invest α and $1 - \alpha$ in the risky asset and the risk-free asset.
- Let $\bar{r}$ denote the risk-free return and $\tilde{x} \in [\underline{x}, \overline{x}]$ denote the return on the risky asset where $\tilde{x}$ follows a probability density function $f(x)$.
- Assume $E_f(x) > \bar{r}$.

The objective function

- The objective function of an expected utility maximizer

$$\max_{\alpha} \quad Eu = \int_{\underline{x}}^{\overline{x}} u\left(\alpha x + (1-\alpha)\bar{r}\right) f(x) dx. \tag{2}$$

- The corresponding first-order condition (FOC) for the optimal α_f is

$$\int_{\underline{x}}^{\overline{x}} (x - \bar{r})\, u'\left(\alpha_f x + (1-\alpha_f)\bar{r}\right) f(x) dx = 0. \tag{3}$$

A change in risk

- Let the probability density function of $\tilde{x}$ shifts from $f(x)$ to $g(x)$.
- The FOC will become

$$\int_{\underline{x}}^{\overline{x}} (x - \bar{r})\, u'\left(\alpha_g x + (1 - \alpha_g)\bar{r}\right) g(x) dx = 0. \tag{5}$$

- When the SOC holds, $\alpha_g \leq \alpha_f$ if and only if

$$\int_{\underline{x}}^{\overline{x}} (x - \bar{r})\, u'\left(\alpha_f x + (1 - \alpha_f)\bar{r}\right) g(x) dx \leq 0. \tag{6}$$

- The necessary and sufficient condition can be restated as that $\exists \gamma \in R$ such that

$$\int_{\underline{x}}^{\overline{x}} (x - \bar{r})\, u'\left(\alpha_f x + (1 - \alpha_f)\bar{r}\right) \left[g(x) - \gamma f(x)\right] dx \leq 0. \tag{7}$$

The main equation

- Or,

$$\int_{\underline{x}}^{\overline{x}} u'\left(\alpha_f x + (1-\alpha_f)\bar{r}\right)\left[t_g(x) - \gamma t_f(x)\right] dx \leq 0, \qquad (8)$$

where

$$(x - \bar{r})\, f(x) = t_f(x) \text{ and } (x - \bar{r})\, g(x) = t_g(x).$$

A restriction on the marginal utility

- Define $U_1(\varepsilon_1)$ as follows (Leshno and Levy, 2002):

$$U_1(\varepsilon_1) = \left\{ u : u'(z) > 0 \text{ and } u'(z) \leq \inf\{u'(z)\} \left[\frac{1}{\varepsilon_1} - 1\right], \forall z \right\}, \tag{9}$$

  where $\varepsilon_1 \in \left(0, \frac{1}{2}\right)$.
- The marginal utility is positive and the ratio between the maximum and the minimum value of the marginal utility is bounded by $\frac{1}{\varepsilon_1} - 1$.
- If ε_1 approaches $\frac{1}{2}$, then only the linear utility functions will be in the set of $U_1(\varepsilon_1)$. If ε_1 approaches zero, then the set $U_1(\varepsilon_1)$ is the same as the set of all decision makers with $u'(z) > 0$.

Proposition 1. Given the SOC, we have $\alpha_g \leq \alpha_f$ for all $u \in U_1(\varepsilon_1)$ if and only if $\exists \gamma \in R$ such that

$$\int_{\Omega_1(\gamma)} [t_g(x) - \gamma t_f(x)]\, dx \leq \varepsilon_1 \|t_g(x) - \gamma t_f(x)\| . \quad (12)$$

- Note that

$$\Omega_1(\gamma) = \{x : t_g(x) - \gamma t_f(x) > 0\} .$$

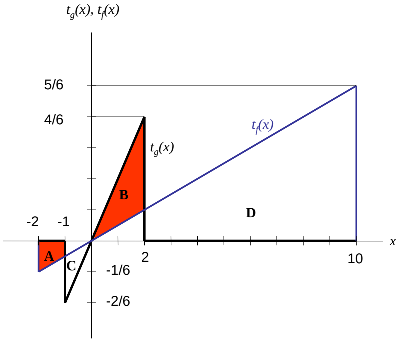
  Thus, if $x > \bar{r}$, then x is in the set of $\Omega_1(\gamma)$ if $g(x) - \gamma f(x) > 0$. If $x < \bar{r}$, then x is in the set of $\Omega_1(\gamma)$ if $g(x) - \gamma f(x) < 0$.

- Denote

$$\|t_g(x) - \gamma t_f(x)\| = \int_{\underline{x}}^{\overline{x}} |t_g(x) - \gamma t_f(x)|\, dx.$$

Example 2

- Let $\bar{r} = 0$ and $f(x)$ as a uniform distribution in $[-2, 10]$. Assume that $g(x)$ also follows a uniform distribution in $[-1, 2]$.
- The condition $\int_{\Omega_1(\gamma)} [t_g(x) - \gamma t_f(x)]\, dx \leq \varepsilon_1 \| t_g(x) - \gamma t_f(x) \|$ is equivalent to $\frac{A+B}{A+B+C+D} \leq \varepsilon_1$.



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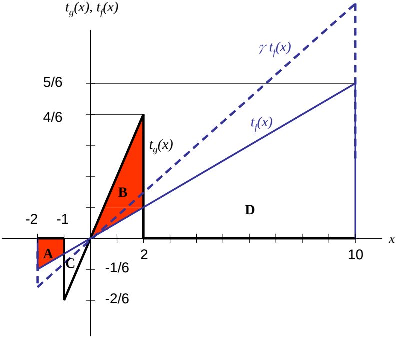

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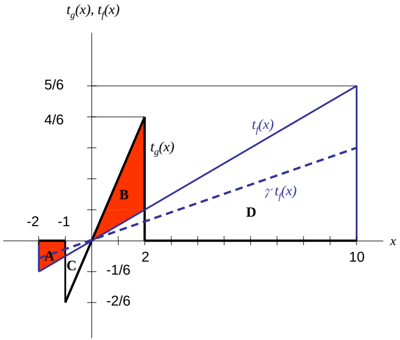


- $\alpha_g \leq \alpha_f$ for all $u \in U_1$ ($\varepsilon_1 = 5.9\%$)
  - Leshno and Leibovitch's (2010) experiments suggest that the maximium critical value of ε_1 is 5.9%.
  - We can find a $\gamma = 4$ such that

$$\Omega_1(\gamma) = \{x : -2 \leq x \leq -1\}.$$

  - It is easy to find that Condition (12) is satisfied, i.e.,

$$\frac{\int_{\Omega_1(\gamma_1)} [t_g(x) - \gamma_1 t_f(x)]\, dx}{\|t_g(x) - \gamma_1 t_f(x)\|} = \frac{\frac{1}{2}}{\frac{1}{2} + 16} = 3.03\% \leq 5.9\%.$$

In other words, individuals with $\varepsilon_1 = 5.9\%$ will decrease their investment in the risky asset when the distribution of the risky assets shifts from f to g.

A restriction on the slope of marginal utility

- Define $U_2(\varepsilon_2)$ as follows:

$$U_2(\varepsilon_2) = \left\{ \begin{array}{c} u : u'(z) > 0,\ u''(z) < 0 \\ \text{and } \ -u''(z) \leq \inf\{-u''(z)\} \left[\frac{1}{\varepsilon_2} - 1\right], \forall z \end{array} \right\}. \quad (13)$$

Integration by parts

- Integrating equation (8) by parts and obtain

$$\begin{aligned} & \int_{\underline{x}}^{\overline{x}} u'\left(\alpha_f x + (1-\alpha_f)\bar{r}\right)\left[t_g(x) - \gamma t_f(x)\right] dx \\ = \; & u'\left(\alpha_f \overline{x} + (1-\alpha_f)\bar{r}\right)\left[T_g(\overline{x}) - \gamma T_f(\overline{x})\right] \\ & - \int_{\underline{x}}^{\overline{x}} \alpha_f u''\left(\alpha_f x + (1-\alpha_f)\bar{r}\right)\left[T_g(x) - \gamma T_f(x)\right] dx, \end{aligned} \tag{14}$$

where

$$T_f(x) = \int_{\underline{x}}^{x} t_f(s) ds \tag{15}$$

and

$$T_g(x) = \int_{\underline{x}}^{x} t_g(s) ds. \tag{16}$$

Some other notations

- Let

$$\Omega_2(\gamma) = \{x : T_g(x) - \gamma T_f(x) > 0\}, \tag{17}$$

and

$$\| T_g(x) - \gamma T_f(x) \| = \int_{\underline{x}}^{\overline{x}} | T_g(x) - \gamma T_f(x) | \, dx. \tag{18}$$

Proposition 2

Proposition 2. For all $u \in U_2(\varepsilon_2)$, $\alpha_g \leq \alpha_f$ if and only if f dominates g by ε_2-Almost Central Riskiness.

Definition 1. (Almost Central Riskiness, ACR). For $0 < \varepsilon_2 < \frac{1}{2}$, f dominates g by ε_2-Almost Central Riskiness if $\exists \gamma \in R$ such that

$$\int_{\Omega_2(\gamma)} \left[T_g(x) - \gamma T_f(x) \right] dx \leq \varepsilon_2 \left\| T_g(x) - \gamma T_f(x) \right\|, \quad (19)$$

and $T_g(\bar{x}) - \gamma T_f(\bar{x}) \leq 0$.

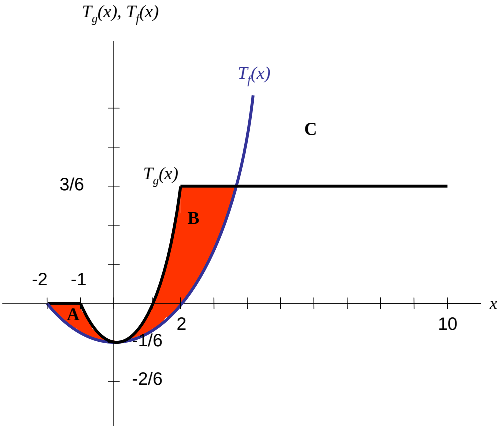
- Note that the condition $T_g(\bar{x}) - \gamma T_f(\bar{x}) \leq 0$ can be written as

$$\gamma \geq \frac{E_g(x) - \bar{r}}{E_f(x) - \bar{r}}. \quad (20)$$

It provides a lower bound for γ.

CR v.s. ACR: Example 2

- CR: Find a γ such that A and B disappear.
- ACR: Find a γ such that $\frac{A+B}{A+B+C} \leq \varepsilon_2$.



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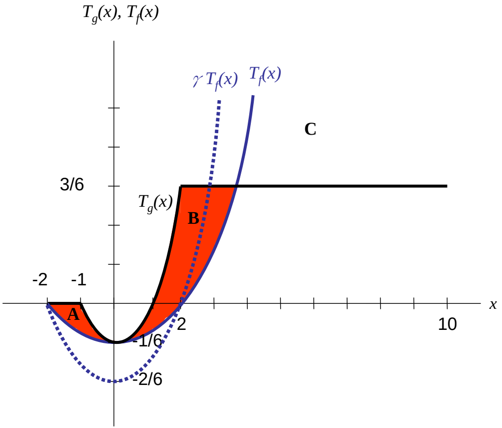

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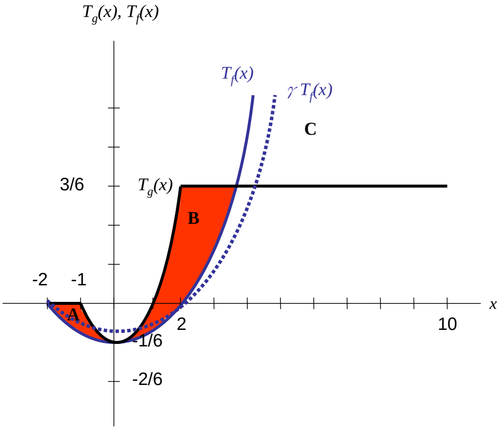

3. Non-pathological Higher-order Preferences

Some definitions

- $U_N = \left\{ u : (-1)^{n+1} u^{(n)} \geq 0,\ n = 1, 2, ..., N \right\}.$
- $U_N(\varepsilon_N) =$
  $\left\{ u \in U_N\colon\ (-1)^{N+1} u^{(N)}(x) \leq \inf \left\{ (-1)^{N+1} u^{(N)}(x) \right\} \left[\frac{1}{\varepsilon_N} - 1 \right],\ \forall x \right\}.$
- $T_f^{(n)}(x) = \int_{\underline{x}}^{x} T_f^{(n-1)}(s)ds$ and $T_f^{(1)}(x) = T_f(x)$. Similiarly, we can define $T_g^{(n)}(x)$.
- $\Omega_N(\gamma) = \left\{ x : T_g^{(N-1)}(x) - \gamma T_f^{(N-1)}(x) > 0 \right\}.$

Proposition 3

- For all $u \in U_N(\varepsilon_N)$, $\alpha_g \leq \alpha_f$ if and only if $\exists \gamma \in R$ such that

$$\begin{aligned} & \int_{\Omega_N(\gamma)} \left[T_g^{(N-1)}(x) - \gamma T_f^{(N-1)}(x) \right] dx \\ \leq \; & \varepsilon_N \left\| T_g^{(N-1)}(x) - \gamma T_f^{(N-1)}(x) \right\|, \end{aligned} \tag{25}$$

and $T_g^{(n)}(\overline{x}) - \gamma T_f^{(n)}(\overline{x}) \leq 0$, $n = 1, 2, ..., N-1$.

4. A Model with a General Payoff Function

Assumptions

- Assume that the payoff function of the DM is $z(x, \alpha)$.
- For simplicity, assume that $\frac{\partial z(x,\alpha)}{\partial x} = z_x(x, \alpha) > 0$ and $z_{\alpha\alpha}(x, \alpha_f) \leq 0$as in Gollier (1995).

Proposition 4

- We have $\alpha_g \le \alpha_f$

1. for all $u \in U_1(\varepsilon_1)$ if and only if $\exists \gamma \in R$ such that

$$\int_{\Omega_1(\gamma)} \left[t_g(x;\alpha_f) - \gamma t_f(x;\alpha_f)\right] dx \le \varepsilon_1 \left\| t_g(x;\alpha_f) - \gamma t_f(x;\alpha_f) \right\| .$$

2. for all $u \in U_2(\varepsilon_2)$ if and only if $\exists \gamma \in R$ such that

$$\begin{aligned} & \int_{\Omega_2(\gamma)} z_x(x,\alpha_f)\left[T_g(x;\alpha_f) - \gamma T_f(x;\alpha_f)\right] dx \\ \le \quad & \varepsilon_2 \left\| z_x(x,\alpha_f)\left[T_g(x;\alpha_f) - \gamma T_f(x;\alpha_f)\right] \right\| \end{aligned}$$

and $T_g(\overline{x};\alpha_f) - \gamma T_f(\overline{x};\alpha_f) \le 0$.

5. Conclusion

- In this paper, we have provided the necessary and sufficient condition for the unambiguous comparative statics for most decision makers who exhibit non-pathological preferences and are economically important.
- We have further analyzed the conditions for higher-order preferences.
- In addition, we have generalized our results to the cases with non-linear payoff functions.