

Asymmetric Contests with Interdependent Valuations

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Abstract

I show that a unique equilibrium exists in a two-player all-pay auction with asymmetric independent discrete signal distributions and asymmetric interdependent valuations. The proof is constructive, and the construction is simple to implement as a computer program. For special cases, which include some private value settings, common value settings, and symmetric players, I derive additional properties and comparative statics. I also characterize the set of equilibria when a reserve price is introduced.

1 Introduction

This paper investigates a contest model in which two asymmetric contestants compete for a prize by expending resources. Each contestant has some private information that may affect both contestants' valuation for the prize, and contestants are asymmetric in that their private information may be drawn from different distributions and impact their valuations differently. For example, consider a research and development race in which the firm with the highest-quality product enjoys a dominant market position. Each firm may

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be informed about different attributes of the market, which together determine the value of winning. This value may differ between the firms, because the profit associated with a dominant market position may depend on firm-specific characteristics such as production costs and marketing expertise.

Section 2 models the contest as an asymmetric all-pay auction with independent signals and interdependent valuations. Each player's signal is drawn from a finite set of ordered signals according to a probability distribution with full support. The sets of signals and corresponding probability distributions may differ between the players. After observing his signal, each player decides how much to bid, and the player with the higher bid wins the prize. The value of the prize is a player-specific function of both players' signals. The only restriction is that this function increase in the player's own signal. This model includes complete information, private values, common values, and one informed and one uninformed player as special cases.

Section 3 contains the main result of the paper, which is a constructive characterization of the unique equilibrium. This characterization uses the finiteness of players' type spaces to employ insights from the analysis of complete-information contest models. Each player, for each of his types, chooses his bid from an interval, and higher types choose bids from higher intervals. This ordering of intervals means that, by proceeding from the top, the equilibrium can be constructed in a finite number of steps. In each step, one type of player 1 competes against one type of player 2. In the resulting interval of competition, players compete as in a complete-information all-pay auction with valuations that correspond to the competing types. Once one player has exhausted the probability mass associated with his lowest type, any remaining probability mass of the other player is expended as an atom at 0. This simple procedure is easy to implement as a computer program.¹ Section 3.1 enumerates the possible equilibrium orderings of players' bidding intervals, which depend on players' valuations and probability distributions.

Section 4 applies the construction result to examine a few special cases. First, a complete characterization is provided when each player has one or two types (excluding the

¹A Matlab implementation of this procedure is available on my website, <http://faculty.wcas.northwestern.edu/~rsi665/>.

case of complete information, which has been well studied by, for example, Hillman and Riley (1989)). This characterization generalizes those of Konrad (2004, 2009) and Szech (2010), who examined two-player all-pay auctions with private values and one or two types. Second, a closed-form solution is provided when players have private values, one player is known to be stronger than the other, and only one player has private information. When the privately-informed player is the strong one, his low types enjoy a higher payoff increase relative to the corresponding complete-information contest than do his high types. The reverse inequalities hold weakly when the privately-informed player is the weak one, and hold strictly when the probability of low types is sufficiently high. Thus, which types enjoy higher “information rents” depends on whether the informed player is strong or weak. Third, a partial characterization is provided when the value of the prize is common to both players. This characterization shows that players’ equilibrium strategies are identical from an ex-ante perspective. Players’ payoffs may differ, however, because each player may condition his bid on his private information, which may differ between the players. A closed form solution is provided when, in addition to common values, only one player has private information. Fourth, a symmetric closed-form solution is provided when players are “quasi-symmetric,” in that their ex-ante information structures are identical and their valuations for winning are the same whenever they observe the same signals.

Section 5 uses the equilibrium construction result to derive a candidate equilibrium when players’ types are drawn from continuous distributions. The continuous distributions are approximated by increasingly finer discrete distributions, and the limit of the equilibria of the corresponding contests deliver a differential equation that identifies a candidate equilibrium. When players have private values, the candidate equilibrium coincides with that of Amann and Leininger (1996), who considered an asymmetric two-player all-pay auction with private values and continuous type distributions. They characterized the unique equilibrium candidate within the class of differentiable equilibria, and did not consider discrete distributions or interdependent valuations.²

²They did not prove that the candidate equilibrium is indeed an equilibrium, or that it is unique within the class of all equilibria. These lacunae can most likely be filled by the tools developed in Lizzeri and Persico (2000).

Section 6 extends the model by adding a reserve price, which corresponds to the minimum investment necessary to win the contest. A player who bids below the reserve price loses, regardless of what the other player bids. While the contest may have multiple equilibria, there exists a bid such that in any equilibrium, players' bidding behavior above the reserve price coincides with their bidding behavior above this bid in the contest without a reserve price. I characterize the set of equilibria, which are payoff equivalent, and show that players' payoffs decrease in the reserve price. Any two equilibria differ in the behavior of at most one type of each player, so when the probability of each type is small the difference between any two equilibria is small. This is consistent with Lizzeri and Persico's (2000) result that with a continuum of types and a sufficiently high reserve price there exists a unique equilibrium. In contrast to their analysis, I place no restrictions on the reserve price, and provide a constructive procedure for characterizing the set of equilibria. Another difference is that they require a non-atomic distribution, whereas I stipulate a finite number of signals.

Beyond Lizzeri and Persico's (2000) and Amann and Leininger's (1996) work, few papers have studied auction-like contests with incomplete information and asymmetries in players' valuations and information structure, even though these features arise in many real-world competitions with sunk investments. A notable exception is Parreiras and Rubinchik's (2010) work, which characterized some equilibrium properties in an asymmetric all-pay auction with private values and more than two players. These and most other papers that analyze auction-like contests assume a continuum of signals and non-atomic atomic distributions. The assumption of a finite number of signals made here shows that certain insights and techniques used in the analysis of complete-information all-pay auctions apply when there is incomplete information. This provides a novel connection between complete and incomplete information all-pay auctions. The model also includes complete-information all-pay auctions as a special case, in contrast to its usual treatment as a limiting case in models with non-atomic distributions. The finiteness assumption is also useful for the explicit analysis of examples and applications, and facilitates equilibrium characterization with a non-restricted reserve price.³

³This characterization includes, of course, complete-information all-pay auctions with a reserve price,

2 Model

There are two players and one prize. Each player $i = 1, 2$ observes a private signal s_i in S_i , where S_i is a finite ordered set. The signals in S_i are distributed according to a commonly-known probability distribution F_i , where $F_i(s_i) > 0$ is the probability of signal s_i . The distributions F_1 and F_2 are statistically independent. Player i 's valuation for the prize is $V_i : S_1 \times S_2 \rightarrow \mathbb{R}_{++}$, which is strictly increasing in player i 's signal, but need not be increasing or monotonic in the other player's signal. After observing their signals, the players compete in an all-pay auction: they simultaneously chooses how much money to bid, forfeit their bid, and the player with the higher bid wins the prize (in case of a tie, any procedure can be used to allocate the prize between the players). Thus, player i 's payoff if he observes signal s_i and players' bids are b_1 and b_2 is

$$u_i(s_i, b_1, b_2) = P_i(b_1, b_2) \sum_{s_{-i} \in S_{-i}} (F_{-i}(s_{-i}) V_i(s_i, s_{-i})) - b_i,$$

where $-i$ refers to player $3 - i$,

$$V_i(s_i, s_{-i}) = \begin{cases} V_1(s_1, s_2) & \text{if } i = 1 \\ V_2(s_1, s_2) & \text{if } i = 2 \end{cases},$$

and

$$P_i(b_1, b_2) = \begin{cases} 1 & \text{if } b_i > b_{-i}, \\ 0 & \text{if } b_i < b_{-i}, \\ \text{any value in } [0, 1] & \text{if } b_i = b_{-i}, \end{cases},$$

such that $P_1(b_1, b_2) + P_2(b_1, b_2) = 1$.

3 Equilibrium

Denote a mixed strategy of player i by $G_i : S_i \times \mathbb{R} \rightarrow [0, 1]$, where $G_i(s_i, x)$ is the probability that player i bids at most x when he observes s_i (so $G_i(s_i, \cdot)$ is a CDF for every signal s_i). Abusing notation, I will sometimes treat G_i as a function of one variable, $G_i(x) = \sum_{s_i \in S_i} F_i(s_i) G_i(s_i, x)$, so G_i is also a CDF. An equilibrium is a pair $\mathbf{G} = (G_1, G_2)$, such

which, to the best of my knowledge, have not been previously studied.

that given that player i plays the mixed strategy G_i , the CDF $G_{-i}(s_{-i}, \cdot)$ assigns measure 1 to player $-i$'s set of best responses when he observes signal s_{-i} , for every signal s_{-i} . I say that a player has an *atom* at x if the player bids x with positive probability when he observes one of his signals.

One difficulty in solving for equilibrium is that a player's valuation for the prize may depend on the other player's signal, which can be inferred (at least partially) from the other player's equilibrium bid. This does not happen in a private-value model. Of course, even with private values a player's probability of winning with a given bid depends on the other player's strategy. The additional complication here is that the player's valuation for the prize also depends on the other player's strategy (through the equilibrium mapping between the other player's signal and his strategy). The key to solving for equilibrium is to show that a simple structure governs this dependency.

The remainder of the section characterizes the unique equilibrium. I begin with a few preliminaries.⁴

Lemma 1 *In any equilibrium \mathbf{G} , (i) there is no bid at which both players have an atom, (ii) there is no positive bid at which either player has an atom, (iii) if $x > 0$ is not a best response for player i for any signal, then no bid $y \geq x$ is a best response for either player for any signal, and (iv) both players have best responses at 0 or arbitrarily close to 0.⁵*

Proof. For (i), suppose that player 1 chose x with positive probability when he observed s_1 , and player 2 did the same when observing s_2 . Because $F_1(s_1) > 0$, player 2 could do strictly better by choosing a bid slightly above x , so x cannot be a best response for player 2 when observing s_2 , a contradiction. For (ii), suppose that player i chose $x > 0$ with positive probability when he observed s_i . By an argument similar to the one used to prove (i), the other player would not have best responses on some positive-length interval with upper bound x for any signal. But then player i could do strictly better by bidding slightly below x , so x cannot be a best response for player i when observing s_i , a contradiction. For

⁴Similar equilibrium properties arise in many complete-information models of competition, such as those of Bulow and Levin (2006) and Siegel (2009, 2010).

⁵Parts (iii) and (iv) rely on the assumption that there are two players.

(iii), note that (ii) proved that each player's CDF is continuous above 0 for any signal he observes. Therefore, if $x > 0$ is not a best response for player i at any signal, the same is true for all bids in a some maximal neighborhood of x . This implies that the other player also does not choose any bids in this neighborhood. But then, again by continuity, no player would have a best response at the top of this neighborhood. For (iv), suppose that 0 is not a best response for one of the players and that player does not have best responses arbitrarily close to 0. This means that the player does not have best responses in some interval with lower endpoint 0. By (iii), the player does not have any best-responses, so \mathbf{G} is not an equilibrium. ■

Denote by $u_i(s_i, x)$ player i 's expected payoff when he observes signal s_i and bids x and the other player uses strategy G_{-i} . Choose an equilibrium \mathbf{G} , and denote by $BR_i(s_i)$ player i 's best responses when he observes signal s_i and the other player uses strategy G_{-i} .

Lemma 2 *If $s'_i > s_i$, then for any x in $BR_i(s_i)$ and y in $BR_i(s'_i)$, we have $y \geq x$.*⁶

Proof. Choose x in $BR_i(s_i)$ and y in $BR_i(s'_i)$. Suppose $x > y$. By part (ii) of Lemma 1, neither player has an atom at $x > 0$. And player $-i$ does not have an atom at y , otherwise y would not be in $BR_i(s'_i)$. Therefore,

$$u_i(s_i, x) - u_i(s_i, y) = \sum_{s_{-i} \in S_{-i}} (F_{-i}(s_{-i}) V_i(s_i, s_{-i}) (G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y))) - (x - y) \geq 0, \quad (1)$$

where the last inequality follows from $u_i(s_i, x) \geq u_i(s_i, y)$, because x is in $BR_i(s_i)$. This last inequality and $x - y > 0$ imply that $G_{-i}(s_{-i}, x) - G_{-i}(s_{-i}, y) > 0$ for at least one signal s_{-i} . This shows that the value of (1) strictly increases if s_i is replaced with s'_i , because $V_i(s'_i, s_{-i}) > V_i(s_i, s_{-i})$ for every signal s_{-i} . Therefore, $u_i(s'_i, x) > u_i(s'_i, y)$, which means that y is not in $BR_i(s'_i)$, a contradiction. ■

The following corollary of Lemmas 1 and 2 describes the structure of players' best response sets in any equilibrium.

Corollary 1 *For every player i and every signal s_i , $BR_i(s_i)$ is an interval. For any two consecutive signals $s'_i > s_i$, the upper bound of $BR_i(s_i)$ is equal to the lower bound of*

⁶The lemma relies on the assumption that players' signal distributions are statistically independent.

$BR_i(s'_i)$. Moreover,

$$\sup_{s_1 \in S_1} BR_1(s_1) = \sup_{s_2 \in S_2} BR_2(s_2) \quad \text{and} \quad \inf_{s_1 \in S_1} BR_1(s_1) = \inf_{s_2 \in S_2} BR_2(s_2) = 0. \quad (2)$$

Proof. By part (iii) of Lemma 1 and Lemma 2, $BR_i(s_i)$ is an interval. By part (iii) of Lemma 1, $BR_i(s_i) \cap BR_i(s'_i)$ is not empty, and Lemma 2 shows that this intersection can include only the boundaries of the best-response sets. Parts (iii) and (iv) of Lemma 1 imply (2). ■

Figure 1 depicts an equilibrium structure consistent with Corollary 1, where T denotes the common upper bound of players' best response sets.⁷

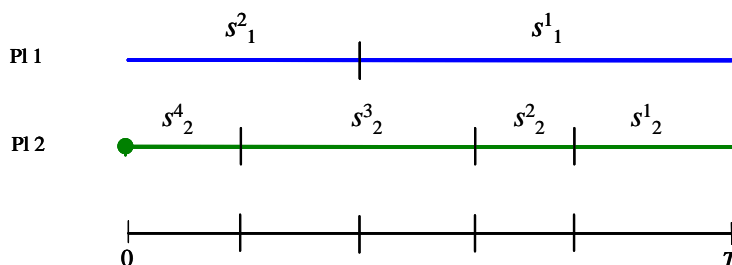


Figure 1: A possible equilibrium structure of players' best response sets when player 1 has two signals, player 2 has four signals, and player 2 has an atom at 0

This structure shows that the equilibrium can be found by using an iterative procedure. To see this, let s_i^k be the k^{th} signal in S_i when signals are ordered from highest to lowest, so $s_i^{k+1} < s_i^k$. Consider the coarsest partition of $[0, T]$ that includes both partitions of $[0, T]$ into players' best response sets (henceforth: the joint partition). In Figure 1, the joint partition is depicted on the bottom line. Consider two bids $x < y$ in the top interval of this partition. Both x and y are best responses for player 1 when his type is s_1^1 , and therefore

⁷Note that because $T > 0$ (at most one player has an atom at 0) and players' strategies are continuous above 0 (part (ii) of Lemma 1), T is a best response for both players' highest types.

lead to the same expected payoffs

$$\begin{aligned} & \sum_{s_2 < s_2^1} F_2(s_2) V_1(s_1^1, s_2) + F_2(s_2^1) G_2(s_2^1, y) V_1(s_1^1, s_2^1) - y \\ = & \sum_{s_2 < s_2^1} F_2(s_2) V_1(s_1^1, s_2) + F_2(s_2^1) G_2(s_2^1, x) V_1(s_1^1, s_2^1) - x, \end{aligned} \quad (3)$$

which can be rewritten as

$$\frac{G_2(s_2^1, y) - G_2(s_2^1, x)}{y - x} = \frac{1}{F_2(s_2^1) V_1(s_1^1, s_2^1)}.$$

Taking $y - x$ to 0 shows that in the top interval $G_2(s_2^1, \cdot)$ is differentiable with density

$$g_2(s_2^1, x) = \frac{1}{F_2(s_2^1) V_1(s_1^1, s_2^1)}.$$

Similarly, in the top interval $G_1(s_1^1, \cdot)$ is differentiable with density

$$g_1(s_1^1, x) = \frac{1}{F_1(s_1^1) V_2(s_1^1, s_2^1)}.$$

(Note that these densities generalize the ones that arise in the equilibrium of the complete-information all-pay auction (Hillman and Riley (1989)), which are, respectively, $1/V_1$ and $1/V_2$, where V_i is player i 's commonly-known valuation for the prize.)

Having identified the densities of players' strategies in the top interval of the joint partition, we can find the length of this interval. For this, note that because $BR_i(s_i)$ is an interval, the top interval of the joint partition ends when one of the two players exhausts the probability mass associated with his highest signal. Therefore, the length of the top interval is

$$\min \{ F_2(s_2^1) V_1(s_1^1, s_2^1), F_1(s_1^1) V_2(s_1^1, s_2^1) \}, \quad (4)$$

with the player whose density determines the length of the interval exhausting the probability mass associated with his highest signal. Players' densities in the next interval are calculated in a similar fashion, with the player(s) who has exhausted the probability mass associated with his highest signal "moving" to his second highest signal. This process is iterated, calculating the length of each interval and players' densities in each interval. Suppose we are at the k^{th} interval of the joint partition, after player 1 has exhausted the probability mass associated with his k_1 highest signals and player 2 has exhausted the

probability mass associated with his k_2 highest signals. The equivalent of Equation 3 is then

$$\begin{aligned} & \sum_{s_2 < s_2^{k_2+1}} F_2(s_2) V_1(s_1^{k_1+1}, s_2) + F_2(s_2^{k_2+1}) G_2(s_2^{k_2+1}, y) V_1(s_1^{k_1+1}, s_2^{k_2+1}) - y \\ = & \sum_{s_2 < s_2^{k_2+1}} F_2(s_2) V_1(s_1^{k_1+1}, s_2) + F_2(s_2^{k_2+1}) G_2(s_2^{k_2+1}, x) V_1(s_1^{k_1+1}, s_2^{k_2+1}) - x, \end{aligned}$$

which leads to densities

$$g_2(s_2^{k_2+1}, x) = \frac{1}{F_2(s_2^{k_2+1}) V_1(s_1^{k_1+1}, s_2^{k_2+1})} \text{ and } g_1(s_1^{k_1+1}, x) = \frac{1}{F_1(s_1^{k_1+1}) V_2(s_1^{k_1+1}, s_2^{k_2+1})}. \quad (5)$$

When computing the length of this interval, the probability mass associated with the signals $s_1^{k_1+1}$ and $s_2^{k_2+1}$ expended on higher intervals must be taken into account (at most one of these signals will have probability mass expended on higher intervals, by definition of the joint partition).

When one of the players has exhausted the probability mass associated with his lowest signal, the remaining mass of the other player must be an atom, and this atom must be at bid 0 (part (ii) of Lemma 1). This atom may include the mass associated with several signals. If both players exhaust their probability mass simultaneously, then the point of exhaustion is also 0 (part (iv) of Lemma 1). By going from 0 upwards, the equilibrium can be constructed from players' densities on each interval. The following result shows that the resulting pair of strategies is the unique equilibrium.

Proposition 1 *There is a unique equilibrium, constructed by the procedure above. In this equilibrium, each player's strategy is continuous above 0 and piecewise uniform. At most one player has an atom 0.*

Proof. Because the procedure described above relies on necessary conditions for equilibrium, the pair of strategies resulting from the procedure is the unique candidate for an equilibrium. To show that it is indeed an equilibrium, it suffices to show that every type of each player chooses best responses with probability 1. Suppose that player 1 observes s_1^k , and denote by l_k and t_k the upper and lower bounds of the interval on which

$g_1(s_1^k, \cdot) > 0$ (as identified by the procedure above). By construction, player 2's strategy is continuous at all positive bids and player 1 obtains the same payoff at every bid $(l_k, t_k]$. Moreover, if player 2 does not have an atom at l_k , then player 1 obtains the same payoff at l_k as well. If player 2 does have an atom at l_k , then $l_k = 0$ is not a profitable deviation for player 1. To complete the proof, therefore, it suffices to show that player 1 does not have profitable deviations lower than l_k or higher than t_k . To this end, denote by $0 = l_K, t_K, \dots, l_{k+1}, t_{k+1}, l_k, t_k, \dots, l_2, t_2, l_1, t_1 = T$ the partition of $[0, T]$ identified by the procedure that corresponds to player 1's types. That is, $g_1(s_i^j, x) > 0$ for every $j \leq K$ and x in (l_j, t_j) . Suppose that player 1 has a profitable deviation below l_k when he observes s_i^k , and let $[l_i, t_i]$ be the highest interval below $[l_k, t_k]$ that contains a profitable deviation y . Because $t_i = l_{i-1}$, $y < t_i$. By construction, player 1 obtains the same payoff at y and t_i when he observes s_1^i .⁸ Therefore, because $s_1^k > s_1^i$, player 1 obtains strictly more at $t_i = l_{i-1}$ than at y when he observes s_i^k (this follows from (1) and the argument that follows it in the proof of Lemma 2). If $i - 1 = k$, this shows that y is not a profitable deviation. If $i - 1 < k$, then $[l_{i-1}, t_{i-1}]$ contains a profitable deviation, contradicting the definition of $[l_i, t_i]$. This shows that there are no profitable deviations below $[l_k, t_k]$. A similar argument shows that there are no profitable deviations above t_k , by considering the lowest interval above $[l_k, t_k]$ that contains a hypothesized profitable deviation, and noting that bids above T are strictly dominated by slightly lower bids. Therefore, player 1 does not have profitable deviations. The same argument shows that player 2 chooses best responses with probability 1 as well.

■

3.1 Equilibrium Ordering

The procedure for constructing the equilibrium shows that players' types exhaust their probability masses in an order that depends on their valuation functions and probability distributions. That is, if player 1 has n_1 types and player 2 has n_2 types, then the equilibrium induces an ordering $(s^1, \dots, s^{n_1+n_2})$ of the elements in $S_1 \cup S_2$, such that the probability mass associated with s^j is expended on an interval of bids whose lower bound

⁸If $y = 0$ and player 2 has an atom at 0, then choose a slightly higher y as the profitable deviation.

is (weakly) lower than those of the intervals of bids that correspond to types s^1, \dots, s^{j-1} . And if the last type in the ordering, $s^{n_1+n_2}$, is a type of player i , then the lower bound of the interval of bids of the last type of player $-i$ in the ordering is 0. The equilibrium payoff of this last type of player $-i$ is 0, as is the equilibrium payoff of all the types that appear later in the ordering (all of whom belong to player i). For example, the ordering that corresponds to Figure 1 is $(s_2^1, s_2^2, s_1^1, s_2^3, s_1^2, s_2^4)$, the lower bound of the interval of bids of type s_1^2 of player 1 is 0, and equilibrium payoff of type s_1^2 of player 1 and type s_2^4 of player 2 is 0.

In any such ordering, and for any pair of types of a player, the higher type appears before the lower type. Thus, the number of equilibrium ordering of players' types that can be generated by varying players' valuation functions and probability distributions is at most $(n_1 + n_2)! / (n_1! n_2!)$: this is the number of orderings of $n_1 + n_2$ elements, n_1 of which are identical and the other n_2 of which are identical. Conversely, it is easy to see that each ordering of n_1 identical elements and n_2 identical elements corresponds to an equilibrium ordering of players' types for some valuation functions and probability distributions.⁹

4 Special Cases

4.1 Two Types for Player 1, One Type for Player 2

The $(2 + 1)! / 2! 1! = 6/2 = 3$ possible equilibrium orderings are (i) (s_1^1, s_2^1, s_1^2) , (ii) (s_1^1, s_2^1, s_1^2) , and (iii) (s_2^1, s_1^1, s_1^2) .

In (i), player 1 exhausts the mass associated with both his types before player 2 exhausts the mass associated with his single type. Therefore, starting from the top, player 1's type s_1^1 exhausts his mass before player 2's type s_2^1 , so we must have

$$g_1(s_1^1, \cdot) = \frac{1}{F_1(s_1^1) V_2(s_1^1, s_2^1)} > \frac{1}{V_1(s_1^1, s_2^1)} = g_2(s_2^1, \cdot),$$

⁹If signal s_i of player i immediately follows signal s_{-i} of player $-i$ in the ordering (so the probability mass associated with s_{-i} is exhausted before that associated with s_i), then by increasing $V_i(s_i, s_{-i})$ the order of the two signals can be reversed.

or

$$V_1(s_1^1, s_2^1) > F_1(s_1^1) V_2(s_1^1, s_2^1). \quad (6)$$

The length of the top interval is therefore

$$\frac{1}{g_1(s_1^1, \cdot)} = F_1(s_1^1) V_2(s_1^1, s_2^1).$$

In the second interval, player 1's type s_1^2 exhausts his mass before player 2's type s_2^1 exhausts his remaining mass of

$$1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{V_1(s_1^1, s_2^1)}.$$

Together with (5) this implies that

$$\left(1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{V_1(s_1^1, s_2^1)}\right) \underbrace{V_1(s_1^2, s_2^1)}_{\text{The reciprocal of player 2's density}} \geq \underbrace{F_1(s_1^2) V_2(s_1^2, s_2^1)}_{\text{The reciprocal of player 1's density}}. \quad (7)$$

Fixing player 1's probability distribution and player 2's valuation function, (6) and (7) are satisfied when $V_1(s_1^1, s_2^1)$ and $V_1(s_1^2, s_2^1)$ are large enough.

In (ii), player 2 exhausts the mass associated with his single type after player 1 exhausts the mass associated with his high type, so (6) holds, but before player 1 exhausts the mass associated with his low type, so the reverse of (7) holds. Fixing player 1's probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ is small enough.

In (iii), player 2 exhausts the mass associated with his single type before player 1 exhausts the mass associated with his high type. Therefore, the reverse of (6) holds, and the length of the top (and only) interval is $V_1(s_1^1, s_2^1)$. Fixing player 1's probability distribution and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is small enough.

The three possible equilibrium configurations are illustrated in Figure 2.

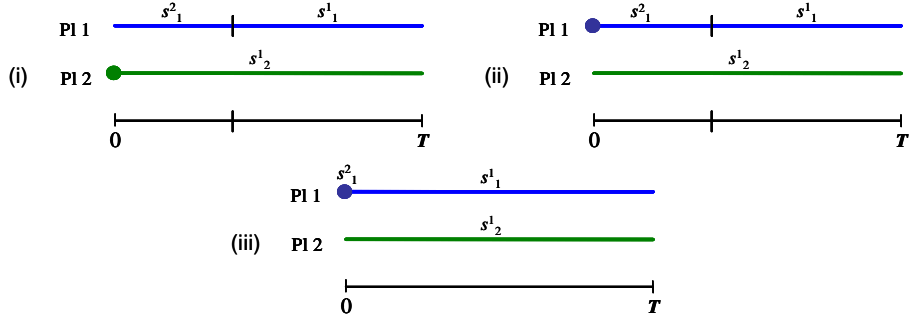


Figure 2: The possible equilibrium configurations when player 1 observes one of two signals and player 2 has one signal as a function of $V_1(s_1^1, s_2^1)$ and $V_1(s_1^2, s_2^1)$, fixing player 1's probability distribution and player 2's valuation function

4.2 Two Types for Each Player

The $(2+2)!/(2!2!) = 24/4 = 6$ possible equilibrium orderings are (i) $(s_1^1, s_1^2, s_2^1, s_2^2)$, (ii) $(s_1^1, s_2^1, s_1^2, s_2^2)$, (iii) $(s_1^1, s_2^1, s_2^2, s_1^2)$, (iv) $(s_2^1, s_2^2, s_1^1, s_1^2)$, (v) $(s_2^1, s_1^1, s_2^2, s_1^2)$, and (vi) $(s_2^1, s_1^1, s_1^2, s_2^2)$.

In (i), player 1 exhausts the mass associated with both his types before player 2 exhausts the mass associated with his high type. Therefore, starting from the top, player 1's type s_1^1 exhausts his mass before player 2's type s_2^1 , so we must have

$$F_2(s_2^1) V_1(s_1^1, s_2^1) > F_1(s_1^1) V_2(s_1^1, s_2^1), \quad (8)$$

and the length of the top interval is $F_1(s_1^1) V_2(s_1^1, s_2^1)$. In the second interval, player 1's type s_1^2 exhausts his mass before player 2's type s_2^1 exhausts his remaining mass of

$$1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{F_2(s_2^1) V_1(s_1^1, s_2^1)}.$$

This implies that

$$\left(1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{F_2(s_2^1) V_1(s_1^1, s_2^1)}\right) F_2(s_2^1) V_1(s_1^2, s_2^1) \geq F_1(s_1^2) V_2(s_1^2, s_2^1). \quad (9)$$

Fixing players' probability distributions and player 2's valuation function, (8) and (9) are satisfied if $V_1(s_1^1, s_2^1)$ and $V_1(s_1^2, s_2^1)$ are large enough.

In (ii), player 2 exhausts the mass associated with his high type after player 1 exhausts the mass associated with his high type, so (8) holds, but before player 1 exhausts the mass

associated with his low type, so the strict reverse of (9) holds. The length of the second interval of the joint partition is therefore

$$\left(1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{F_2(s_2^1) V_1(s_1^1, s_2^1)}\right) F_2(s_2^1) V_1(s_1^2, s_2^1),$$

and player 1 exhausts

$$\left(1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{F_2(s_2^1) V_1(s_1^1, s_2^1)}\right) \frac{F_2(s_2^1) V_1(s_1^2, s_2^1)}{F_1(s_1^2) V_2(s_1^2, s_2^1)} < 1 \quad (10)$$

of the mass associated with his low type in the second interval. In the third interval, player 1 exhausts the remaining mass associated with his low type before player 2 exhausts the mass associated with his low type. This implies that

$$\left(1 - \left(1 - \frac{F_1(s_1^1) V_2(s_1^1, s_2^1)}{F_2(s_2^1) V_1(s_1^1, s_2^1)}\right) \frac{F_2(s_2^1) V_1(s_1^2, s_2^1)}{F_1(s_1^2) V_2(s_1^2, s_2^1)}\right) F_1(s_1^2) V_2(s_1^2, s_2^2) < F_2(s_2^2) V_1(s_1^2, s_2^2). \quad (11)$$

Fixing players' probability distributions and player 2's valuation function, (8) and the strict reverse of (9) are satisfied when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ is small enough. And (10) and (11) are satisfied when $V_1(s_1^2, s_2^1)$ is small enough and $V_1(s_1^2, s_2^2)$ is large enough.

In (iii), players 1 and 2 behave as in (ii) in the first two intervals, but in the third interval player 2 exhausts the remaining mass associated with his low type before player 1 exhausts the remaining mass associated with his low type. Therefore, (8), the strict reverse of (9), (10), and the reverse of (11) hold. Fixing players' probability distributions and player 2's valuation function, this happens when $V_1(s_1^1, s_2^1)$ is large enough and $V_1(s_1^2, s_2^1)$ and $V_1(s_1^2, s_2^2)$ are small enough.

The orderings (iv), (v), and (vi) are the symmetric counterparts of (i), (ii), and (iii). That is, (iv), (v), and (vi) are obtained from (i), (ii), and (iii) by switching the indices of players 1 and 2. The equilibrium configurations that correspond to (i), (ii), and (iii) are illustrated in Figure 3.

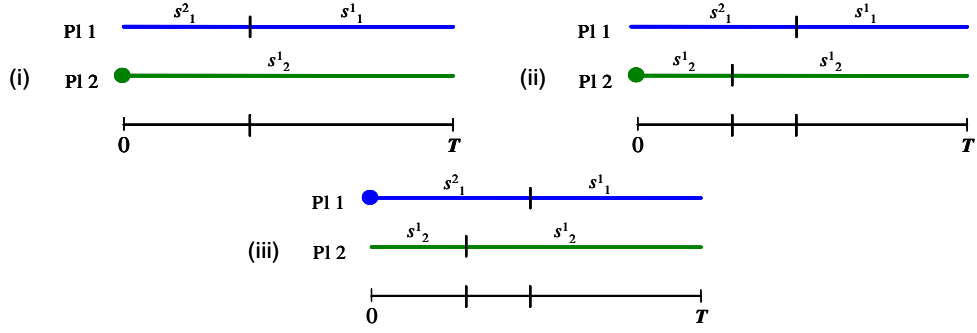


Figure 3: The equilibrium configurations that correspond to orderings (i), (ii), and (iii) when each player observes one of two signals, as a function of $V_1(s_1^1, s_2^1)$, $V_1(s_1^2, s_2^1)$, and $V_1(s_1^2, s_2^2)$, fixing players' probability distributions and player 2's valuation function

4.3 Private Values, Only the Strong Player Is Informed

Suppose that players have private values, player 2 has no private information (so he only has one type, s_2), and player 1 is “stronger,” in that his valuation for the prize is always higher than that of player 2. Without loss of generality, let each player's type equals his valuation for the prize, so $s_i = V_i(s_i, s_{-i})$. That player 1 is stronger means that $s_1 \geq s_2$ for any signal s_1 of player 1.

The equilibrium can be described in closed form. The number of intervals in the joint partition is n , the number of player 1's possible signals. Denote by s_1^j player 1's j^{th} signal when his signals are ordered from high to low. When player 1 observes signal s_1^j , he chooses a bid from an interval of length $F_1(s_1^j) s_2$ according to a uniform distribution with density $1/F_1(s_1^j) s_2$. On the same interval, player 2 chooses a bid according to a uniform distribution with density $1/s_1^j$. Because $\sum_{k=1}^n F_1(s_1^k) s_2 = s_2$, the equilibrium bidding range is $[0, s_2]$. The equilibrium densities are

$$g_1(s_1^j, x) = \begin{cases} \frac{1}{F_1(s_1^j) s_2} & \text{if } x \text{ is in } \left[s_2 \sum_{k=j+1}^n F_1(s_1^k), s_2 \sum_{k=j}^n F_1(s_1^k) \right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2(s_2, x) = \begin{cases} \frac{1}{s_1^j} & \text{if } x \text{ is in } \left[s_2 \sum_{k=j+1}^n F_1(s_1^k), s_2 \sum_{k=j}^n F_1(s_1^k) \right] \\ 0 & \text{otherwise} \end{cases}.$$

In addition, player 2 chooses 0 with probability $1 - s_2 \sum_{k=1}^n F_1(s_1^k) / s_1^k \geq 0$.¹⁰

Compare this equilibrium to the one of the complete-information all-pay auction in which player 2's valuation is s_2 and player 1's valuation is s_1^j for some $j \leq n$. In the complete-information contest, player 1 mixes uniformly on $[0, s_2]$ with density $1/s_2$, and player 2 bids 0 with probability $1 - s_2/s_1^j$ and mixes uniformly on $[0, s_2]$ with density $1/s_1^j$. In both contests, players choose bids from $[0, s_2]$, player 1's unconditional bid distribution is the same (it is uniform with density $1/s_2$), and player 2's payoff is 0. Denote by Δ_1^j the difference between player 1's payoff in the incomplete-information contest when he observes signal s_1^j , and his payoff in the complete-information contest when his valuation is s_1^j . This difference is non-negative, because by bidding s_2 in the incomplete-information contest player 1 can obtain $s_1^j - s_2$, which is his payoff in the complete-information contest. Moreover, $\Delta_1^1 = 0$, because s_2 is a best response for player 1 in the incomplete-information contest when he observes signal s_1^1 . But Δ_1^j strictly increases in j and, in particular, is positive for $j > 1$. To see why, denote by $\bar{s}_1^j = s_2 \sum_{k=j}^n F_1(s_1^k)$ the upper bound of the bidding interval of player 1's type s_1^j in the incomplete-information contest. By bidding \bar{s}_1^j in the incomplete-information contest, player 1 wins with probability $1 - s_2 \sum_{k=1}^{j-1} F_1(s_1^k) / s_1^k$. By bidding \bar{s}_1^j in the complete-information contest, player 1 wins with probability

$$1 - \frac{s_2}{s_1^j} + \frac{\bar{s}_1^j}{s_1^j} = 1 - \frac{s_2 - \bar{s}_1^j}{s_1^j} = 1 - \frac{s_2 \left(1 - \sum_{k=j}^n F_1(s_1^k)\right)}{s_1^j} = 1 - \frac{s_2 \sum_{k=1}^{j-1} F_1(s_1^k)}{s_1^j}.$$

The difference between these probabilities is

$$1 - s_2 \sum_{k=1}^{j-1} \frac{F_1(s_1^k)}{s_1^k} - \left(1 - s_2 \sum_{k=1}^{j-1} \frac{F_1(s_1^k)}{s_1^j}\right) = s_2 \sum_{k=1}^{j-1} F_1(s_1^k) \left(\frac{1}{s_1^j} - \frac{1}{s_1^k}\right),$$

and this difference multiplied by s_1^j equals Δ_1^j , so

$$\Delta_1^j = s_2 \sum_{k=1}^{j-1} F_1(s_1^k) \left(1 - \frac{s_1^j}{s_1^k}\right). \quad (12)$$

¹⁰The inequality follows from

$$s_2 \sum_{k=1}^n \frac{F_1(s_1^k)}{s_1^k} \leq \frac{s_2}{s_1^n} \sum_{k=1}^n F_1(s_1^k) = \frac{s_2}{s_1^n} \leq 1.$$

If player 1 has at least two types (so the first inequality is strict) or $s_1^n > s_2$ (so the second inequality is strictly), then the atom is of positive measure. (Equivalently, if player 1 has a type strictly higher than s_2 .)

The right-hand side of (12) increases in j (because s_1^j decreases in j), so the increase in payoff relative to the complete-information contest is higher for lower types of player 1. This increase in payoff can be interpreted as the information rent that type s_1^j of player 1 obtains in excess of the “economic rent” that accrues to him because of his higher valuation.

Figure 4 depicts the unique equilibrium when player 1’s valuation is 3 or 5 with equal probabilities, and player 2’s valuation is 2.

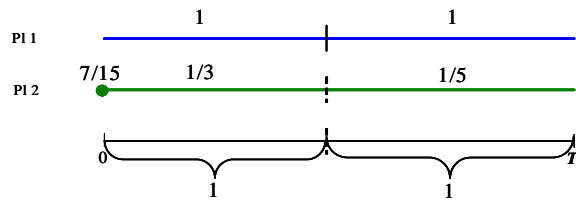


Figure 4: Equilibrium densities and player 2’s atom

The equilibrium bidding range is $[0, 2]$, just like in the complete-information contest. Player 1’s payoff when his valuation is 5 is 3, just like in the complete-information contest, but his payoff when his valuation is 3 is $7/5$, higher than his payoff of 1 in the complete-information contest. Player 2’s payoff is 0, just like in the complete-information contest.

4.4 Private Values, Only the Weak Player Is Informed

Suppose that players have private values, player 2 has no private information (so he only has one type, s_2), and player 1 is “weaker,” in that his valuation for the prize is always lower than that of player 2. Without loss of generality, let each player’s type equals his valuation for the prize, so $s_i = V_i(s_i, s_{-i})$. That player 1 is weaker means that $s_1 \leq s_2$ for any signal s_1 of player 1.

The equilibrium can be described in closed form. Denote by s_1^j player 1’s j^{th} signal when his signals are ordered from high to low, and by n the number of player 1’s signals. Suppose that when the equilibrium is constructed player 1 exhausts his probability mass first. This implies that when player 1 observes signal s_1^j , he chooses a bid from an interval of length $F_1(s_1^j) s_2$ according to a uniform distribution with density $1/F_1(s_1^j) s_2$. On this interval, player 2 chooses a bid according to a uniform distribution with density $1/s_1^j$.

Because $s_2 \sum_{k=1}^n F_1(s_1^k) = s_2$, the equilibrium bidding range would be $[0, s_2]$, on which player 2 would expend mass $s_2 \sum_{k=1}^n F_1(s_1^k) / s_1^k \geq 1$, with equality only if player 1 has one type and this type is s_2 .¹¹ Therefore, player 2 exhausts his probability mass before player 1 does (so player 2 does not have an atom at 0). The equilibrium bidding range is determined by the type of player 1 in whose interval player 2 exhausts his probability mass. This is type m , which is given by

$$m = 1 + \max \left\{ j : s_2 \sum_{k=1}^j \frac{F_1(s_1^k)}{s_1^k} < 1 \right\}.$$

Every type s_1^j , $j = 1, \dots, m-1$ of player 1 exhausts his density on an interval of length $F_1(s_1^j) s_2$, as described above. On these intervals player 2 expends mass $s_2 \sum_{k=1}^{m-1} F_1(s_1^k) / s_1^k < 1$. Let $\mu = 1 - s_2 \sum_{k=1}^{m-1} F_1(s_1^k) / s_1^k$. Type s_1^m chooses bids from an interval on which player 2 exhausts his remaining mass of μ . Therefore, the length of this interval is μs_1^m , so the equilibrium bidding range is $[0, \mu s_1^m + s_2 \sum_{k=1}^{m-1} F_1(s_1^k)]$. The equilibrium densities are

$$g_1(s_1^j, x) = \begin{cases} \frac{1}{F(s_1^m) s_2} & \text{if } j = m \text{ and } x \text{ is in } [0, \mu s_1^m] \\ \frac{1}{F(s_1^j) s_2} & \text{if } x \text{ is in } \left[\mu s_1^m + s_2 \sum_{k=j+1}^{m-1} F_1(s_1^k), \mu s_1^m + s_2 \sum_{k=j}^{m-1} F_1(s_1^k) \right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2(s_2, x) = \begin{cases} \frac{1}{s_1^m} & \text{if } x \text{ is in } [0, \mu s_1^m] \\ \frac{1}{s_1^j} & \text{if } x \text{ is in } \left[\mu s_1^m + s_2 \sum_{k=j+1}^{m-1} F_1(s_1^k), \mu s_1^m + s_2 \sum_{k=j}^{m-1} F_1(s_1^k) \right] \\ 0 & \text{otherwise} \end{cases}.$$

In addition, type m of player 1 chooses 0 with probability $1 - \mu s_1^m / F_1(s_1^m) s_2$, and types $j > m$ of player 1 choose 0 with probability 1.

Compare this equilibrium to the one of the complete-information all-pay auction in which player 2's valuation is s_2 and player 1's valuation is s_1^j for some $j \leq n$. In the

¹¹The inequality follows from

$$s_2 \sum_{k=1}^n \frac{F_1(s_1^k)}{s_1^k} \geq \frac{s_2}{s_1^1} \sum_{k=1}^n F_1(s_1^k) = \frac{s_2}{s_1^1} \geq 1.$$

If player 1 has at least two types (so the first inequality is strict) or $s_1^1 < s_1$ (so the second inequality is strictly), then the inequality is strict. (Equivalently, if player 1 has a type strictly lower than s_2 .)

complete-information contest, player 2 mixes uniformly on $[0, s_1^j]$ with density $1/s_1^j$, and player 1 bids 0 with probability $1 - s_1^j/s_2$ and mixes uniformly on $[0, s_1^j]$ with density $1/s_2$. In both contests, player 1's unconditional bid distribution above 0 is the same (it is uniform with density $1/s_2$). But in the incomplete-information contest the equilibrium bidding range is $[0, \mu s_1^m + s_2 \sum_{k=1}^{m-1} F_1(s_1^k)]$, and the upper bound of this range is in $(s_1^n, s_1^1]$ (because the density of player 2's bid distribution is $1/s_1^1$ on some interval, and may be higher elsewhere, but nowhere higher than $1/s_1^1$). Therefore, player 2's payoff in the incomplete information contest is at least as high as in the complete-information contest in which $s_1^j = s_1^1$, but lower than in the complete-information contest in which $s_1^j = s_1^n$. Player 1's payoff in the complete-information contest is 0. Denote by Δ_1^j player 1's payoff in the incomplete-information contest when he observes signal s_1^j . Clearly, $\Delta_1^j = 0$ for $j \geq m$. Moreover, Δ_1^j strictly decreases in j for $j \leq m$ and, in particular, is positive for $j < m$. This is because by bidding the top of type s_1^j 's bidding interval, type s_1^{j-1} obtains a strictly higher payoff than type s_1^{j-1} does (he wins with the same probability, but his valuation for the prize is higher). This increase in payoff can be interpreted as the information rent that type s_1^j of player 1 obtains in excess of his "economic rent" of 0. In contrast to the case analyzed above, in which the strong player was informed, here higher types have only weakly higher information rents - for types to make strictly positive information rents, the probability of lower types has to be sufficiently high.

Figure 5 demonstrates this by considering two contests, which differ in the probability that player 1's type is low.

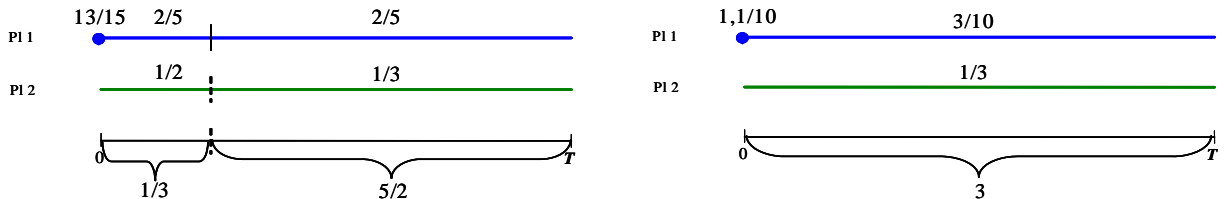


Figure 5: Equilibrium densities and player 1's atom

The left-hand side of Figure 5 depicts the unique equilibrium when player 1's valuation is 2 or 3 with equal probability, and player 2's valuation is 5. The equilibrium bidding range is $[0, 17/6]$, larger than that of the complete-information contest in which player 1's valuation

is 2, $[0, 2]$, and smaller than that of the complete-information contest in which player 1's valuation is 3, $[0, 3]$. Player 1's payoff when his valuation is 2 is 0, just like in the complete-information contest, but his payoff when his valuation is 3 is $1/6$, higher than his payoff of 0 in the complete-information contest. The probability that player 1's valuation is low is high enough for his high type to obtain a positive information rent. Player 2's payoff is $13/6$, higher than his payoff in the complete-information contest in which player 1's valuation is 3, and lower than his payoff in the complete-information contest in which player 1's valuation is 2. The right-hand side of Figure 5 depicts the unique equilibrium when player 1's valuation is 2 with probability $1/3$ and 3 with probability $2/3$, and player 2's valuation is 5. The equilibrium bidding range is $[0, 3]$, just like in the complete-information contest in which player 1's valuation is 3. Player 1's payoff is 0 regardless of his valuation. The probability that player 1's valuation is low is not high enough for his high type to obtain a positive information rent. Player 2's payoff is 2, just like in the complete-information contest in which player 1's valuation is 3.

4.5 Common Values

Suppose that the value of the prize is common to both players, and denote this common value function by $V(\cdot) = V_1(\cdot) = V_2(\cdot)$. In equilibrium, the unconditional distribution of players' bids is the same, regardless of the information structure and the function V . To see why, note that for almost any x in $(0, T]$ we have

$$G_i(x) = \sum_{s_i \in S_i} F_i(s_i) G_i(s_i, x) \Rightarrow g_i(x) = F_i(s_i(x)) g_i(s_i(x), x),$$

where g_i is the density of G_i , and $s_i(x)$ is the signal of player i for which x is a best response. In conjunction with (5), this means that for almost any x in $(0, T]$ we have

$$g_1(x) = g_2(x) = \frac{1}{V(s_1(x), s_2(x))}.$$

In particular, both players exhaust the same unconditional probability mass on $(0, T]$. And since at most one player has an atom at 0, this mass must be 1, so no player has an atom at 0. Therefore, the lowest type of each player has a payoff of 0. However, other types' payoffs, and therefore the ex-ante expected payoffs, may differ between the players.

That players' strategies are identical from an ex-ante perspective is reminiscent of Engelbrecht-Wiggans, Milgrom, and Weber's (1983) result, who showed that this property also holds in the equilibrium of a common-value first-price auction in which only one bidder is informed about the value of the object.

4.6 Common Values, One Informed Player

Suppose that the prize is of common value, that player 1 knows the common value, and that player 2 only knows its distribution. This means that player 2 has only one type. Without loss of generality, let player 1's type equal the common value, so $s_1^j = V_1(s_1^j, s_2^1) = V_2(s_1^j, s_2^1)$. In this case, the equilibrium can be described in closed form. The number of intervals in the joint partition equals the number of player 1's possible signals, n , and no player has an atom at 0. When player 1 observes signal s_1 , he chooses a bid from an interval of length $F_1(s_1) s_1$ according to a uniform distribution with density $1/F_1(s_1) s_1$. On the same interval, player 2 chooses a bid according to a uniform distribution with density $1/s_1$. Denote by s_1^j player 1's j^{th} signal when his signals are ordered from high to low. The equilibrium densities are

$$g_1(s_1^j, x) = \begin{cases} \frac{1}{F_1(s_1^j) s_1^j} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^n F_1(s_1^k) s_1^k, \sum_{k=j}^n F_1(s_1^k) s_1^k \right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_2(s_2^1, x) = \begin{cases} \frac{1}{s_1} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^n F_1(s_1^k) s_1^k, \sum_{k=j}^n F_1(s_1^k) s_1^k \right] \\ 0 & \text{otherwise} \end{cases} .$$

Player 2's payoff is 0 (no player has an atom at 0). Because $G_1 = G_2$ (as explained in Section 4.5), both players win the prize with the same probability. Player 1's payoff is positive, however, because he places higher bids, and therefore wins more often, when the prize is more valuable.

4.7 Quasi-Symmetric Players

A contest is quasi-symmetric if $S = S_1 = S_2$, $F = F_1 = F_2$, and $V_1(s, s) = V_2(s, s)$ for every s in S .¹² In a quasi-symmetric contest the equilibrium is symmetric and can be described in closed form. The number of intervals in the joint partition equals the number of signals in S , n , and no player has an atom at 0. Let $V(s) = V_1(s, s) = V_2(s, s)$. Then, when a player observes signal s , he chooses a bid from an interval of length $F(s)V(s)$ according to a uniform distribution with density $1/F(s)V(s)$. Denote by s^j the j^{th} signal when signals are ordered from high to low. The equilibrium density $g = g_1 = g_2$ is

$$g(s^j, x) = \begin{cases} \frac{1}{F(s^j)V(s^j)} & \text{if } x \text{ is in } \left[\sum_{k=j+1}^n F(s^k)V(s^k), \sum_{k=j}^n F(s^k)V(s^k) \right] \\ 0 & \text{otherwise} \end{cases}.$$

The equilibrium is efficient, because higher types choose bids from higher intervals and the equilibrium is symmetric.

5 Approximating Continuous Type Distributions

5.1 Private Values and Uniform Distributions

To approximate private values drawn independently from the uniform distribution on $[0, 1]$, suppose that for some $n > 1$ each player's valuation is independently drawn from the set $S^n = \{j/n\}_{j=1}^n$ according to a uniform probability distribution. For every n , the equilibrium is symmetric, and the joint partition is comprised of n intervals. The density of each player's strategy in the j^{th} interval is

$$\frac{1}{\frac{j}{n} \frac{1}{n}} = \frac{n^2}{j},$$

so the length of the j^{th} interval is j/n^2 . Type j/n chooses bids from the interval

$$\left[\sum_{k=1}^{j-1} \frac{k}{n^2}, \sum_{k=1}^j \frac{k}{n^2} \right] = \left[\frac{j(j-1)}{2n^2}, \frac{j(j+1)}{2n^2} \right] = \left[\frac{j^2 - j}{2n^2}, \frac{j^2 + j}{2n^2} \right].$$

¹²For a quasi-symmetric contest to be symmetric we must have that $V_1(s, s') = V_2(s', s)$ for every s and s' in S .

For any x in $[0, 1]$, consider a sequence $\{j_n/n\}_{n=1}^\infty$ with j_n/n in \mathcal{S}^n and $j_n/n \rightarrow x$. We have that

$$\frac{j_n^2 - j_n}{2n^2}, \frac{j_n^2 + j_n}{2n^2} \rightarrow \frac{x^2}{2},$$

which is the equilibrium bid of type x in the limiting all-pay auction (see, for example, Krishna (2002) and Amann and Leininger (1996)).

5.2 Connection to Amann and Leininger (1996)

Using an approximation similar to the one in Section 5.1, we can heuristically derive a candidate equilibrium for the asymmetric all-pay auction with independent private values and a continuum of signals, which was analyzed by Amann and Leininger (1996). Consider distributions on $[0, 1]$, H_1 for player 1 and H_2 for player 2, with corresponding positive continuous densities h_1 and h_2 . For any $n > 1$, consider a sequence of n values in $(0, 1]$, $\alpha^n < \alpha^{n-1} < \dots < \alpha^1 = 1$ such that $\alpha^j - \alpha^{j+1} < \frac{1}{n}$ for any $j \leq n-1$. To approximate H_1 with a finite distribution, let player 1's valuation be α^j with probability $\int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx$. For player 2, consider a sequence of n values in $(0, 1]$, $k(\alpha^n) < k(\alpha^{n-1}) < \dots < k(\alpha^1) = 1$, and approximate player 2's distribution by letting player 2's valuation be $k(\alpha^j)$ with probability $\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x) dx$. Given $\alpha^1, \dots, \alpha^n$, choose the sequence $k(\alpha^1), \dots, k(\alpha^n)$ such that in equilibrium type α^j of player 1 and type $k(\alpha^j)$ of player 2 choose bids from the same interval. This is useful because then each type of each player chooses bids using one density (whenever one player exhausts the mass associated with one of his types so does the other player). To do this, it suffices to choose $k(\alpha^j)$ such that the length of the intervals from which type α^j of player 1 and type $k(\alpha^j)$ of player 2 choose bids are the same. For this, it suffices that the densities according to which the players choose their bids are the same. From (5), these densities are

$$\frac{1}{k(\alpha^j) \int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx}$$

for player 1 and

$$\frac{1}{\alpha^j \int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x) dx}$$

for player 2. Equating these densities, we obtain

$$k(\alpha^j) \int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx = \alpha^j \int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x) dx. \quad (13)$$

The equality (13) shows how to uniquely identify $k(\alpha^1), \dots, k(\alpha^n)$ by proceeding inductively from 1 to n : set $k(\alpha^1) = \alpha^1 = 1$ and, given α^j, α^{j+1} , and $k(\alpha^j)$, use (13) to solve for $k(\alpha^{j+1})$. Now, (13) implies that

$$\frac{\int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx}{\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x) dx} = \frac{\alpha^j}{k(\alpha^j)},$$

so

$$\frac{\frac{\int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx}{\alpha^j - \alpha^{j+1}}}{\frac{\int_{k(\alpha^{j+1})}^{k(\alpha^j)} h_2(x) dx}{k(\alpha^j) - k(\alpha^{j+1})}} = \frac{\alpha^j (k(\alpha^j) - k(\alpha^{j+1}))}{k(\alpha^j) (\alpha^j - \alpha^{j+1})}. \quad (14)$$

As n grows large, fixing α^j as an element in the sequence of player 1's types, (14) heuristically "becomes"

$$\frac{h_1(\alpha^j)}{h_2(k(\alpha^j))} = \frac{\alpha^j}{k(\alpha^j)} k'(\alpha^j),$$

which is precisely (1) in Amann and Leininger's (1996). As in Amann and Leininger (1996), in the limit, $k(\alpha^j)$ is the type of player 2 that submits a bid equal to that of type α^j of player 1.¹³

We can say more about the limiting bid of type α^j of player 1. For simplicity, assume that for any n player 1's lowest type does not have an atom at 0. Then, the length of the bidding interval of type α^j of player 1 is $k(\alpha^j) \int_{\alpha^{j+1}}^{\alpha^j} h_1(x) dx$, so the lower endpoint of the bidding interval of type α^j is

$$\sum_{l=n}^{j+1} k(\alpha^l) \int_{\alpha^{l+1}}^{\alpha^l} h_1(x) dx. \quad (15)$$

As n grows large, fixing α^j as a point in the sequence of player 1's types, by definition of the Lebesgue integral (15) heuristically "becomes"

$$\int_0^{\alpha^j} k(x) h_1(x) dx,$$

¹³With interdependent valuations, a similar heuristic derivation leads to the expression

$$\frac{h_2(k(x))}{h_1(x)} = \frac{1}{k'(x)} \frac{V_2(x, k(x))}{V_1(x, k(x))}.$$

Interdependent values were not considered by Amann and Leininger (1996).

which is precisely the formula in Amann and Leininger (1996) that describes the bid of type a^j of player 1 (on page 8, following (2)).¹⁴

In particular, if $H_1 = H_2$, then $k(\alpha^i) = \alpha^i$ (directly from (13)), so

$$\int_0^{\alpha^j} k(x) h_1(x) dx = \int_0^{\alpha^j} x h_2(x) dx = E[x|x < \alpha^j] H_2(\alpha^j).$$

6 Equilibrium with a Reserve Price

Suppose that a reserve price $r > 0$ is introduced. That is, a player loses if he bids below r , regardless of what the other player bids. In this case, an equilibrium is comprised of two regions. Up to r , the players bid 0 or r , and at most one player bids r . Above r , much of the previous analysis applies: for each player's type that bids above r , the set of best responses above r is an interval, these intervals are higher for higher types, and the union of these intervals for each player across his types is also an interval. Therefore, an equilibrium with a reserve price can be obtained from the one without a reserve price by identifying a bid b , such that any higher bid $b + x$ without a reserve price corresponds to the bid $r + x$ with a reserve price, and bids below b without a reserve price correspond to 0 or r with a reserve price. The bid b is unique,¹⁵ but the mapping of bids lower than b may lead to multiple equilibria. These equilibria differ only in that some of the bids lower than b correspond to bidding 0 in one equilibrium and r in another equilibrium. All equilibria are, however, payoff equivalent. The bottom part of Figure 6 depicts an equilibrium structure consistent with the introduction of a reserve price to the contest whose equilibrium structure is depicted in the top part of Figure 6.

¹⁴With interdependent valuations, a similar heuristic derivation leads to the expression

$$\int_0^x V_2(y, k(y)) h_1(y) dy.$$

¹⁵It is the highest bid such that in the equilibrium without a reserve price, for at least one player, the gross winnings at that bid of the (lowest) type for whom the bid is a best response are no higher than r .

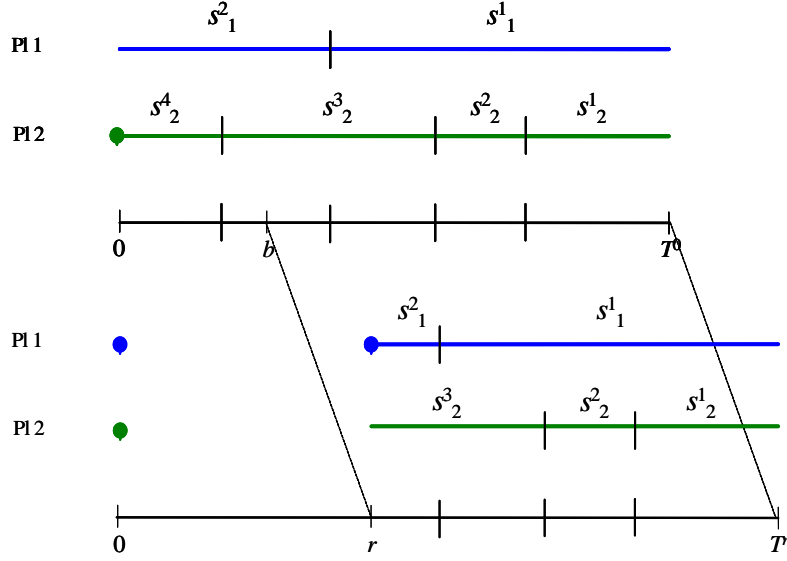


Figure 6: A possible equilibrium configuration of players' atoms and best response sets when player 1 has two signals and player 2 has four signals, without a reserve price (top), and with a reserve price (bottom)

I now describe players' equilibrium strategies in greater detail. Denote by $\mathbf{G}^0 = (G_1^0, G_2^0)$ the unique equilibrium of the contest with a reserve price of 0, i.e., without a reserve price. For a bid x in $(0, T^0]$, where T^0 is the common supremum of players' best responses in \mathbf{G}^0 , denote by $s_i(x)$ player i 's type for which x is a best response in \mathbf{G}^0 .¹⁶ Let

$$v_i^0(x) = \sum_{s_{-i} \in S_{-i}} F_{-i}(s_{-i}) G_{-i}^0(s_{-i}, x) V_i(s_i(x), s_{-i})$$

denote player i 's expected (gross) winnings without a reserve price if his type is $s_i(x)$, he bids x , and the other player plays \mathbf{G}_{-i}^0 . Note that $v_i^0(\cdot)$ strictly increases on $(0, T^0]$,¹⁷ and

$$v_i^0(T^0) = \sum_{s_{-i} \in S_{-i}} F_{-i}(s_{-i}) V_i(s_i^1, s_{-i}),$$

¹⁶If x is a best response for two types of player i , which happens only if x is an endpoint of the interval of bids for some type of player i , denote by $s_i(x)$ the lower of the two types.

¹⁷ $v_i^0(x)$ is piecewise differentiable with slope 1 wherever it is differentiable, and jumps upward wherever it is not differentiable.

where s_i^1 is player i 's highest signal. Let

$$b_i = \max \{x \in (0, T^0] : v_i^0(x) \leq r\}$$

if this set is non-empty, and $b_i = 0$ otherwise.¹⁸ Note that b_i weakly increases in r , and $b_i = T^0$ if and only if $v_i^0(T^0) \leq r$. In addition, $b_i \leq r$.¹⁹ Also, $b_1 > 0$ or $b_2 > 0$, because at least one player does not have an atom at 0. The following lemma characterizes the set of equilibria when r is large.

Proposition 2 *Suppose that $b_i = T^0$. Then, for every p in $[0, 1]$, the following pair of strategies is an equilibrium. Every type of player i bids 0. Type s_{-i} of player $-i$ bids 0 if*

$$\sum_{s_i \in S_i} F_i(s_i) V_{-i}(s_i, s_{-i}) - r < 0, \quad (16)$$

and bids r if the reverse inequality holds. If (16) holds with equality (which happens for at most one type s_{-i}), then type s_{-i} of player $-i$ bids 0 with probability p , and r with probability $1 - p$. All these equilibria are payoff equivalent. Moreover, every equilibrium is such a pair of strategies for some i for which $b_i = T^0$.

Proof. First, note that in any equilibrium both players choose bids only from $\{0, r\}$. Indeed, because $b_i = T^0$ implies that $v_i^0(T^0) \leq r$, and since $v_i^0(T^0)$ is the highest possible (gross) winnings for player i , he does not bid more than r . Therefore, player $-i$ does not bid more than r (for any such bid a slightly lower bid is better). Clearly, neither player chooses bids from $(0, r)$. To see that the proposed pairs of strategies are optimal, note that player i obtains at most 0 by bidding r , so bidding 0 is optimal for him. Therefore, player $-i$ wins with probability 1 when bidding r , so the left hand side of (16) describes his payoff when he bids r . This implies that the proposed strategies for player $-i$ are optimal and lead to the same payoffs. To see that every equilibrium is such a pair of strategies, note that in equilibrium at most one player bids r with positive probability (as in part

¹⁸Because $s_i(\cdot)$ is left continuous, b_i is well defined.

¹⁹If $b_i > r$, then $r < T^0$ (because $b_i \leq T^0$), so $v_i^0(b_i) \leq r$ implies that $v_i^0(r) < r$ (because $v_i^0(\cdot)$ strictly increase on $(0, T^0]$). But $v_i^0(x) - x \geq 0$ for any x in $(0, T^0]$, because x is a best response for type $s_i(x)$ of player i .

(i) of Lemma 1). Therefore, in any equilibrium in which player $-i$ bids r with positive probability, every type of player i bids 0, and any such equilibrium is a pair of strategies as specified above. In any equilibrium in which player i bids r with positive probability, every type of player $-i$ bids 0, so $b_{-i} = T^0$ (otherwise bidding slightly above r would be a profitable deviation for the highest type of player $-i$). The pairs of strategies described above, with $-i$ instead of i , describe all the equilibria in which every type of player $-i$ bids 0. ■

Proposition 2 describes the set of equilibria when $b = T^0$, where $b = \max\{b_1, b_2\} > 0$. I now turn to the case $b < T^0$.

Lemma 3 *If $b < T^0$, then in any equilibrium the union of each player's best response sets across his types includes bids higher than r .*

Proof. If the claim is false, then in any equilibrium both players choose bids only from $\{0, r\}$, and at most one player chooses r with positive probability (as in part (i) of Lemma 1). Therefore, every type of some player i has a payoff of 0. But because $b_i < T^0$, we have $v_i^0(T^0) > r$, so by bidding slightly above r player i 's highest type can win with certainty and obtain a positive payoff, a contradiction. ■

Choose an equilibrium $\mathbf{G}^r = (G_1^r, G_2^r)$ of the contest with a reserve price. Denote by $BR_i^{r+}(s_i)$ player i 's best responses higher than r when he observes signal s_i and the other player uses strategy G_{-i}^r . Denote by S_i^{r+} the set of player i 's signals for which $BR_i^{r+}(s_i)$ is not empty. The next lemma shows that the set of players' best responses higher than r have a structure similar to that of the best response sets in the equilibrium of the contest without a reserve price.

Lemma 4 *Suppose that $b < T^0$. For every player i and every signal s_i in S_i^{r+} , $BR_i^{r+}(s_i)$ is an interval. Also, if s_i is in S_i^{r+} and $s'_i > s_i$, then all of player i 's best responses when he observes signal s'_i are higher than r . For any two consecutive signals $s'_i > s_i$ in S_i^{r+} , the upper bound of $BR_i^{r+}(s_i)$ is equal to the lower bound of $BR_i^{r+}(s'_i)$. Moreover,*

$$\begin{aligned} \sup \cup_{s_1 \in S_1^{r+}} BR_1^{r+}(s_1) &= \sup \cup_{s_2 \in S_2^{r+}} BR_2^{r+}(s_2) \text{ and} \\ \inf \cup_{s_1 \in S_1^{r+}} BR_1^{r+}(s_1) &= \inf \cup_{s_2 \in S_2^{r+}} BR_2^{r+}(s_2) = r. \end{aligned} \tag{17}$$

Proof. A proof similar to that of Lemma 1 shows that no player has atoms above r , at most one player has an atom at r , the union of each player's set of best responses higher than r across his types is an interval, and these intervals have the same upper bound and the same lower bound of r . A proof similar to that of Lemma 2 shows that for any two signals $s'_i > s_i$, such that s_i is in S_i^{r+} , and any x in $BR_i^{r+}(s_i)$ and y that is a best response for s'_i , we have $y \geq x$. This implies the remainder of the claim, as in the proof of Corollary 1. ■

Denote by T^r the common supremum in (17). Lemma 4 shows that the construction procedure described in Section 3 applies to bids in $(r, T^r]$. Therefore, above r any equilibrium coincides with the equilibrium without a reserve price starting from some point. The next lemma shows that this point is b , as in Figure 6.

Lemma 5 *For every player i , every signal s_i in S_i , and every $x \geq 0$, we have*

$$G_i^r(s_i, r+x) = G_i^0(s_i, b+x). \quad (18)$$

Proof. Because the construction procedure described in Section 3 applies to bids in $(r, T^r]$, the statement of the lemma holds with some y in place of b in (18). Suppose that $y < b$, so that $y < b_i$ for some player i . Because $y < b_i$ and $v_i^0(\cdot)$ strictly increases on $(0, T^0]$, we have that $v_i^0(y + \varepsilon) < r$ for small $\varepsilon > 0$. Consider type s_i of player i , who bids slightly above y in \mathbf{G}^0 . By bidding slightly above r in \mathbf{G}^r , this type's (gross) winnings are less than r , so his payoff is negative. Therefore, \mathbf{G}^r is not an equilibrium. Now suppose that $y > b$. Because $y > b_1$ and $y > b_2$, we have $v_1^0(y) > r$ and $v_2^0(y) > r$. Therefore, the payoffs in \mathbf{G}^r of types $s_1(y)$ and $s_2(y)$ (the lowest types that bid y in \mathbf{G}^0) are positive. And because $G^r(s_1(y), r) = G^0(s_1(y), y) > 0$ (where the inequality follows from $y > b > 0$ and the definition of $s_1(y)$) and, similarly, $G^r(s_2(y), r) > 0$, types $s_1(y)$ and $s_2(y)$ each have an atom at 0 and/or r . But because at most one player has an atom at r , either type $s_1(y)$ or type $s_2(y)$ (or both) have an atom at 0, leading to a payoff of 0 in \mathbf{G}^r , a contradiction.

■

Lemma 5 pins down \mathbf{G}^r above r . To completely characterize the set of equilibria, it remains to specify how players choose bids from $\{0, r\}$, as in Proposition 2.

Proposition 3 *Suppose that $b < T^0$, and $b_i = b$ for player i . Then, for every p in $[0, 1]$, the following pair of strategies is an equilibrium. Every type $s_i < s_i(b)$ of player i bids 0, and type $s_i(b)$ of player i bids 0 with probability $G_i^0(s_i(b), b)$. Every type $s_i \geq s_i(b)$ of player i chooses bids higher than r according to (18). Type $s_{-i} < s_{-i}(b)$ of player $-i$ bids 0 if*

$$\sum_{s_i \in \mathcal{S}_i} F_i(s_i) G_i^0(s_i, b) V_{-i}(s_i, s_{-i}) - r < 0, \quad (19)$$

and bids r if the reverse inequality holds. If (19) holds with equality (which happens for at most one type s_{-i}), then type s_{-i} of player $-i$ bids 0 with probability p , and r with probability $1 - p$. Type $s_{-i}(b)$ of player $-i$ chooses with probability $G_i^0(s_i(b), b)$ bids from $\{0, r\}$ according to (19), as specified above for lower types. Every type $s_{-i} \geq s_{-i}(b)$ of player $-i$ chooses bids higher than r according to (18) (with $-i$ in place of i). All these equilibria are payoff equivalent. Moreover, every equilibrium is such a pair of strategies for some i for which $b_i = b$.

Proof. Similarly to the proofs of Propositions 1 and 2, it is straightforward to verify that the proposed pairs of strategies are equilibria. To see that every equilibrium is such a pair of strategies, recall that Lemma 5 pins down players' equilibrium behavior above r , and at most one player bids r with positive probability. Therefore, similarly to the proof of Proposition 2, any equilibrium in which player $-i$ bids r with positive probability is a pair of strategies as specified above. If there is an equilibrium in which player i bids r with positive probability, then $b_{-i} = b$. This is because $b_{-i} < b$ implies that type $s_i(b)$ of player 2 can obtain a positive payoff by bidding slightly above r , but because at most one player has an atom at r , type $s_i(b)$ of player $-i$ must bid 0 (and get 0) with probability $G_i^0(s_i(b), b) > 0$, a contradiction. Therefore all the equilibria in which player i bids r with positive probability are given by the pairs of strategies described above, with $-i$ instead of i . ■

Propositions 2 and 3 imply that introducing a reserve price makes both players weakly worse off. This is because $b_i \leq r$, as stated above, so $b \leq r$, which means that above

r players' strategies are "higher" versions of their strategies without a reserve price, as depicted in Figure 6. Thus, players face tougher competition with a reserve price, which lowers their payoffs. The next corollary generalizes this observation.

Corollary 2 *The equilibrium payoff of every type of every player weakly decreases in the reserve price $r \geq 0$.*

Proof. It suffices to show that the payoff of every type of every player at any given bid decreases in r when the other player plays his equilibrium strategy. For bids in $[0, r)$ this is true, because the payoff there is 0. For bids $x \geq r$, it suffices to show that $G_i^r(s_i, x)$ weakly decreases in r for every s_i in S_i and $i = 1, 2$, because this implies that the (gross) winnings at x for player $-i$ weakly decrease in r . Because the equilibrium above r is given by (18), it suffices to show that T^r weakly increases in r or, equivalently, that the increase in b resulting from an increase in r is no higher than the increase in r . But this follows from the definitions of $v_i^0(x)$ and b_i : $v_i^0(x)$ is piecewise differentiable with slope 1 wherever it is differentiable, and jumps upward wherever it is not differentiable, so an increase in r leads to a weakly lower increase in b_i , and therefore to a weakly lower increase in b . ■

Propositions 2 and 3 show that multiple equilibria may exist. This occurs when (16) or (19) hold with equality, which happens for at most one type of each player (because every player's valuation for the prize strictly increases in his type). The equilibria differ only in the probabilities with which that particular type bids 0 and r . Therefore, when the probability of each type is small (so the number of types is large), the difference between any two equilibria is small.²⁰ This observation is consistent with Lizzeri and Persico's (2000) result, which implies that with a continuum of types, each of which occurs with probability 0, and a sufficiently high reserve price there is a unique equilibrium.²¹ Their result does not apply when there is no reserve price, or when the reserve price is low. In contrast, Propositions 2 and 3 characterize the set of equilibria for any reserve price.

²⁰For example, the distance between any two equilibria is small according to the metric induced by the sup norm.

²¹The reserve price must be high enough to exclude a positive measure of types from bidding, regardless of the other bidder's type.

To gain some intuition for the effects of a reserve price, consider a complete-information all-pay auction for a prize of common value V . Without a reserve price, $T^0 = V$ and each player mixes uniformly with density $1/V$ on $[0, V]$. Suppose a reserve price is introduced. Because $v_i^0(x) = x$ for $x \leq V$, we have $b_i = b = \min\{r, V\}$. If $r > V$, then both players bid 0 (in this case $b = V$, so Proposition 2 and (16) hold). If $r = V$, then one player bids 0 and the other player mixes between 0 and V (Proposition 2 holds and (16) holds with equality). If $r < V$, then on (r, V) both players mix uniformly with density $1/V$; one of the players bids 0 with his remaining probability, r/V , and the other player bids 0 with probability pr/V and V with probability $(1-p)r/V$, for some p in $[0, 1]$ (Proposition 3 holds and (19) holds with equality).

In a complete-information all-pay auction with asymmetric valuations, there is a unique equilibrium even with a reserve price, as long as the reserve price is not equal to the higher of the two players' valuations. To see this, denote by V_i player i 's valuation for the prize, and let $V_1 > V_2$. Without a reserve price, $T^0 = V_2$, player 1 mixes uniformly with density $1/V_2$ on $[0, V_2]$, and player 2 chooses 0 with probability $(V_1 - V_2)/V_1$ and mixes uniformly with density $1/V_1$ on $(0, V_2)$. For the equilibrium with a reserve price, note that $v_1^0(x) = V_1 - V_2 + x$ and $v_2^0(x) = x$ for $x \leq V_2$. Therefore, $b_1 = \max\{0, r - (V_1 - V_2)\}$ for $r < V_1$ and $b_1 = V_2$ for $r \geq V_1$, and $b_2 = r$ for $r < V_2$ and $b_2 = V_2$ for $r \geq V_2$. This implies that $b = b_2$. If $r > V_1$, then both players bid 0 (Proposition 2 and (16) hold). If $r = V_1$, then player 2 bids 0 and player 1 mixes between 0 and V_1 (Proposition 2 holds and (16) holds with equality for $i = 2$). If r is in $[V_2, V_1)$, then player 2 bids 0 and player 1 bids r (Proposition 2 holds and (16) holds with the reverse inequality for $i = 2$). If $r < V_2$, then on (r, V_2) both players mix uniformly with their respective densities, $1/V_2$ and $1/V_1$; player 1 bids r with his remaining probability, r/V_2 , and player 2 bids 0 with his remaining probability, $(r + V_1 - V_2)/V_1$ (Proposition 3 holds and (19) holds with the reverse inequality for $i = 2$).

In contrast to the complete information case, when players have private information it may be that $b < r$, even when $b < T^0$, as depicted in Figure 6. To see this, consider a private value setting in which each player's valuation for the prize is 1 or 2 with equal probabilities. Without a reserve price, $T^0 = 3/2$, the low type of each player mixes uniformly with density

2 on $[0, 1/2]$, and the high type of each player mixes uniformly with density 1 on $[1/2, 3/2]$.

This implies that $v_i^0(x) = \begin{cases} x & \text{if } x \leq 1/2 \\ x + 1/2 & \text{if } 1/2 < x \leq 3/2 \\ 2 & \text{if } x > 3/2 \end{cases}$. Therefore, for $r \leq 1/2$ we have

$b = b_i = r$, as in the complete-information case. But for r in $[1/2, 1]$, we have $b = 1/2$, and for r in $(1, 2]$ we have $b = r - 1/2$.

7 Conclusion

This paper has investigated an asymmetric two-player all-pay auction. The novel features are a finite number of signals for each player, asymmetric distributions and interdependent valuations, and a non-restricted reserve price. The constructive characterization of the set of equilibria has shown that there is a unique equilibrium without a reserve price, and that with a reserve price all equilibria are payoff equivalent and differ in the behavior of at most one type for each player. A closed-form equilibrium characterization has been given for some special cases.

One direction for future research is to apply the equilibrium construction results to additional special cases in order to derive comparative statics and closed-form equilibrium characterizations. These can be used to investigate models of real-world competitions, such as the research and development setting described in the Introduction. Another direction is to extend the model to more than two players and signals that are not independent. This seems to be a non-trivial task, because much of the equilibrium construction is driven by these assumptions.

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