Optimal Auction with Resale*

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Abstract

In the existing auction literature, an auction is designed as if its allocation were final. Thus, resale may defeat the purpose for which an auction is designed. This paper considers optimal auctions when resale among bidders cannot be banned. Here any owner of a good can choose a mechanism to sell it to others, taking into account the possibility that whoever buys the good can also choose a mechanism to resell it. This paper finds an equilibrium that generates a tower of nested post-sale optimal auctions: The initial owner labels the bidders by $1, \ldots, n$ and auctions off the good to them via a mechanism I design; the winner, say w, then auctions off the good to bidders $w+1, \ldots, n$ via a similar mechanism; the process continues until an auction results in no sale or when bidder n wins. The equilibrium gives the initial owner the same expected profit as the highest level he could obtain in the corresponding environment where resale can be costlessly banned.

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1 Introduction

Although auction design has recently become a leading field in economics, it assumes that bidders cannot trade among themselves after an auction. Consequently, it is not clear whether auction design has any significant impact when secondary markets are present. If a bidder can obtain the good from resale, why would he care about winning in the auction? If secondary markets will eventually deliver the good to a person who values it most, why should a social planner care about auction design? It is difficult to fend off this criticism, since resale occurs in many sectors (U.S. Treasury bill auctions, the trading of pollution permits, the subcontracting of loans among banks, etc.). Even if a seller can impose legal restrictions on resale, resale can take the form of post-auction mergers among the bidders.

Optimal-auction theory is perhaps the most significant part of auction design, and yet the most vulnerable to resale. The theory is significant because it predicts that efficiency is not guaranteed if a seller is free to choose a profit-maximizing auction mechanism. The theory is vulnerable because resale causes a tension between a seller's incentive to manipulate the allocation of a good—as the theory has demonstrated—and the bidders' ability to undo the manipulation via resale.

We can see this tension from an example adopted from Ausubel and Cramton [1]: Bidders A and B pursue an indivisible good; its dollar-value to bidder A is uniformly distributed on $[0, 10]$, and that to bidder B is commonly known as \$2. An auction that sells the good to the highest bidder can generate only \$2 as revenue. Alternatively, the seller can intensify the bidding competition by *subsidizing* bidder B, i.e., selling him the good even when the other bidder's valuation is higher. According to the optimal-auction theory, a seller-optimal mechanism is to sell the good to A at price \$6 and, if rejected, sell it to B at price \$2. If resale were banned, then this mechanism would have generated a revenue \$3.6. If resale cannot be banned, however, this mechanism fails: Bidder A rejects, and bidder B buys the good at price \$2 and resells to A at price \$6. The seller gets only a revenue \$2. Due to the resale option, subsidization fails to intensify bidding competition: Instead of competing with each other, the bidders tacitly collude in the mechanism and trade subsequently. The mechanism yields suboptimal revenue, since the seller could have received \$2.5 from making bidder A a take-it-or-leave offer at price \$5.

Focusing on the aforementioned tension, this paper is aimed at providing a theory of optimal auctions when resale cannot be banned. In order to concentrate on the impact of resale, I change only the no-resale assumption in the mainstream framework and leave its assumption about a seller's commitment ability unchanged. The environment is a multistage auction-resale game with one good and n bidders ($n = 1, 2, \ldots$). At each stage, the current owner of the good commits to a mechanism to auction it off to the bidders. If the mechanism does not sell the good, then the current owner becomes the final owner and the game ends. (This captures the assumption of a seller's commitment ability.) Otherwise, he trades with the *winner*. Then the next stage begins, the winner becomes the owner, and he commits to an auction mechanism to resell the good to the other bidders. (This captures the assumption that resale cannot be banned.) The game continues until someone becomes the final owner of the good. To focus on the impact of resale, I assume that bidders' valuations (or *types*) of being the final owner of the good are private and stochastically independent, so that the common-value component that resale may endogenously induce can be easily identified. To keep the environment closed, I assume that the information available to the economy is constant throughout the process although individuals may update from the actions of others.

Difficulties arise from the dynamic aspect of the above auction-resale game. At every stage of the game, the current owner chooses an optimal mechanism by anticipating how subsequent owners will resell the good. The corresponding task of mechanism design in the traditional framework is handled by the revelation principle and the no-resale assumption: By the revelation principle, equilibrium-feasible mechanisms are effectly contained in the set of Bayesian incentive-feasible direct revelation mechanisms (DRM), which turn out to be tractable. By the no-resale assumption, any Bayesian incentive-feasible DRM is equilibrium-feasible. In the auction-resale game, while the first half of the traditional approach remains true, its second half does not follow, because equilibrium-feasibility in a dynamic game requires stronger conditions than Bayesian incentive-feasibility.

My method is to identify a first-best allocation for a current owner and then construct an equilibrium-feasible mechanism that implements the allocation. Finding a first-best allocation for the initial owner is easy; it is the *Myerson allocation*, or the allocation, characterized by Myerson [18], that optimizes for a seller among Bayesian incentive-feasible allocations (Lemma 2.1). Implementing it by an equilibrium-feasible mechanism is hard. This task would have been easy if an owner can implement the Myerson allocation within the current stage and resale does not occurs at equilibrium. However, this is impossible (Remark 3.1). Thus, at any equilibrium that implements the Myerson allocation, resale must occur with a positive probability, and we must trace the updating of beliefs, which causes the complication that the environment for a current mechanism-designer (the current owner) is determined by the mechanisms and outcomes in earlier stages. (This fact in turn makes it nontrivial to identify the first-best allocation for secondary sellers.)

With resale unavoidable, a seller designing an optimal mechanism needs to choose a winner-selection rule such that a subsequent seller, after updating from his winning status, will find it optimal to implement an allocation optimal from the viewpoint of the current seller. If such a rule is available, then the current seller needs only to ensure (i) that a winner learns nothing new other than his winning status (to cut off unwanted information linkage between auction and resale) and (ii) that all players' posterior beliefs are commonly known (to avoid intractable continuation games with diverse beliefs or privately informed mechanism-designers). The seller can ensure condition (i) by keeping a winner's payment independent of the bids submitted by those that will become the winner's bidders. He can ensure condition (ii) by publicly disclosing a winner's bid at the end of the stage.

The result is an equilibrium demonstrating a tower of nested post-sale optimal auctions, where middlemen arise from the final users of the good (Theorem 1). The initial seller indexes the bidders by $1, \ldots, n$ according the ranking induced by the Myerson allocation.

Starting from bidder 1, the seller successively asks each bidder i to compete with those bidders j indexed above i; this *leading bidder* i is subsidized in the sense that he is regarded as defeating j even if i's bid is lower than j's by a margin within a certain range. The first leading bidder who defeats all the rivals indexed above him is picked as the winner. The winner, say w, finds it optimal to resell the good only to bidders $w+1, \ldots, n$ via a similar mechanism, with the same indices and winner-selection rule. This process of auction and resale continues until either a mechanism results in no sale or bidder n wins.[1]

Remarkably, this equilibrium gives the initial owner as much expected profit as the highest level he could obtain in the corresponding environment where resale can be costlessly banned, and the equilibrium implements the Myerson allocation despite resale. That is also true for every subsequent owner on the equilibrium path. In fact, the Myerson allocation in the environment for an owner is the same as the allocation resulting from selling the good to the winner w and having w implement the Myerson allocation in the environment for w.

This paper contributes a reconciliation between auction design and resale, because it shows that a seller's first-best outcome when he can costlessly ban resale can be achieved when he cannot ban resale at all. Since the received theory of optimal auction has demonstrated that a seller's first-best outcome is often inefficient, my result implies that, despite bidders' free access to resale, an auction designed by a sufficiently smart seller can still undermine efficiency. Consequently, the identity of the initial owner of a good has a significant impact on social welfare.

By constructing the equilibrium explicitly, this paper also contributes a new recipe of seller-optimal auctions (Subsection 4.1). This recipe is the only discovered theoretical construct that achieves optimality when resale cannot be banned; in this setting, the traditional design of optimal auctions fails, as remarked previously. Being explicit, this recipe may be a useful benchmark for auction-designers in sectors with secondary markets.

This paper is an attempt to extend the theory of auction design from static to dynamic environments. To my knowledge, such extension efforts consist of three branches. The first focuses on the possibility of pre-bidding collusion such as Laffont and Martimort [14]. A second branch focuses on the possibility that a seller cannot commit to a selling mechanism; examples are McAfee and Vincent [15] and Jehiel and Moldovanu [10]; related to this branch is the literature on mechanism design with renegotiation (e.g., Hart and Tirole [9]). The third branch, to which this paper belongs, focuses on the resale among bidders. Another paper in this branch is Ausubel and Cramton [1]. Constructing three interesting examples, Ausubel and Cramton have initiated the question about optimal auction with resale. Their paper assumes that resale always achieves Pareto efficiency. My paper studies resale by solving the continuation game of all subsequent resales.

The other related analytical works have provided some positive analyses of auctions followed by resales. Bikhchandani and Huang [2] analyze the information linkage between auction and resale in a common-value model. Bose and Deltas [3] address the question

[1] This feature of nested post-sale auctions somewhat resembles a real episode documented by Porter [19].

whether a seller would exclude the final consumers from an auction. Haile [5, 6] and Gupta and Lebrun [4] analyze first- and second-price auctions by focusing on equilibria that fully reveal private information before resale. Haile [7] and Jehiel and Moldovanu [11] prove that resale can lead to the non-existence of such equilibria. Kamien, Li, and Samet [13] and Kamien and Li [12] show that the revenue raised at an auction depends critically on who has the bargaining power at resale. The analytical works based on complete-information models include Milgrom [17], Kamien, Li, and Samet [13], Kamien and Li [12], and Jehiel and Moldovanu [10]. Haile [8] provides an empirical analysis of the effects of resale. In most of these papers, the allocation rule in the resale stage is assumed to be exogenous. The exceptions are Milgrom [17], Jehiel and Moldovanu [10], and my paper.

The rest of the paper is organized as follows. Section 2 formulates the auction-resale environment into a multistage game of mechanism-designers. As the players choose mechanisms, the equilibrium concept is complicated and hence spelled out in Subsection 2.1. Subsection 2.3 contains all the assumptions about the distributions of players' types. Section 3 prepares for my construction of the equilibrium by sketching the main idea of the construct (Subsections 3.3 and 3.4). Section 4 presents the main result. In Subsection 4.1, I construct a mapping that assigns an optimal mechanism to every seller at every environment on the equilibrium path. In Subsection 4.2 I construct an equilibrium for the entire auction-resale game by induction. The appendix contains all the postponed proofs. The index of symbols and words is in the last page.

2 The Model

I shall formulate the environment governing the choice of mechanisms and players' actions as the following multistage auction-resale game. It concerns the allocation of an indivisible good, initially owned by one player and pursued by n other players ($n = 1, 2, \ldots$). At each stage, the current owner of the good commits to a mechanism to auction it off to the current bidders. If the mechanism does not sell the good, then the current owner becomes the final owner and the multistage game ends. Otherwise, he trades with the *winner* and exits the game. Then the next stage begins, the winner becomes the owner, and he commits to an auction mechanism to resell the good to the other bidders. The game continues until someone becomes the final owner of the good.

Mechanisms. At each stage of the auction-resale game, the current owner of the good is allowed to choose any mechanism subject to three constraints. First, a bidder is allowed not to participate. Second, the mechanism is feasible with respect to the indivisible good. Third, no possible allocation is contingent on any event that may occur after the current stage. This last constraint is important; it implies that a current owner cannot prohibit resale among bidders after his trade with a winner of the current stage.

As usual, a *mechanism* can be viewed as a mapping from a profile of bidders' actions in the current stage to a mandate at the end of the stage. In our setup, such a mandate consists of three items: (i) a lottery that may pick a winner among the bidders, (ii) a configuration of monetary transfers, (iii) a *public announcement* of the actions taken by some players during the current stage. Items (i) and (ii) are the same as those in the standard model of auction design. Item (iii) is relevant due to the dynamic aspect of the auction-resale game. Since all the three items are independent of the actions taken by the players after the current stage, this formulation of mechanism captures the assumption that resale cannot be banned.

The above formulation is based on two crucial assumptions. One is that resale cannot be banned, as already presented above. The other assumption is that sellers have commitment ability: At each stage, the current owner chooses a mechanism; once chosen, the mechanism is operated by a neutral guardian and no one can tamper with it; hence a current owner can commit to withholding the good forever if his mechanism results in no sale. Combined together, the two assumptions say that a seller can commit to his own action while he cannot mandate the actions of others.

Information Structure. The auction-resale game is subject to the following exogenous information structure. At each stage, a bidder's action and his net payment to others are not observed by the other bidders, unless the public announcement exposes them. At the end of the current stage, the identity of the winner and the contents of the public announcement, if any, are the only new common knowledge.

Preferences. As usual in the auction literature, I assume that a player is risk-neutral and his utility is a quasi-linear function of his monetary payment and his consumption of the good. Specifically, suppose a player i's total net payment to others throughout the auction-resale game is p (with $-p$ denoting the net receipts from others), then his payoff is $t_i - p$ if he is the final owner of the good, and is $-p$ if otherwise. Here the level t_i of his utility from being the final owner of the good is a given constant to him and is called his *type*. At the beginning of the auction-resale game, a bidder's type is his private information and is regarded as a random variable independently drawn from a commonly known distribution. Different bidders' types can be drawn from different distributions. The initial owner's type is assumed to be common knowledge at the beginning of the game.

2.1 The Notion of Equilibrium

We wish to analyze the auction-resale game by perfect Bayesian equilibrium (PBE). As players choose mechanisms in our auction-resale game, however, the notion of PBE runs into a technical difficulty: its condition of sequential rationality requires that a PBE specify the consequence after a current owner has chosen a mechanism that has no continuation equilibrium. One might consider ruling out such mechanisms when we model the auction-resale game. Unfortunately, they cannot be identified a priori, because the existence of a continuation equilibrium depends on the belief system of a PBE of the entire game.

To resolve this Catch-22 problem, I shall weaken the notion of sequential rationality slightly. The idea is to identify a set of *exceptional events* exempt from the condition of sequential rationality. This resolution is analogous to the standard condition of belief-updating in a PBE, which imposes no restriction on a belief system at a set of equilibrium-dependent exceptional events. Roughly speaking, I will classify any event in which a player has chosen an action whose continuation game has no "equilibrium" as an *exceptional event* for sequential rationality. We need to be careful here because "equilibrium" and "exceptional event" are intertwined notions. I therefore define them by recursion on the length of our multistage game, which is the number n of bidders, since by assumption every seller exits after his current stage.

Definition 1 *If an auction-resale game is single-staged, then an equilibrium of the game is defined to be any Bayesian Nash equilibrium (BNE) of the game, as the notion of BNE is well-formed. Pick any natural number k and suppose that the notion of equilibrium is well-formed for any auction-resale game whose number of stages is at most k. If such a game does not have any equilibrium, then we say that it is* equilibrium-infeasible*. For any auction-resale game $\mathcal{G}$ with $k+1$ stages, the notion of equilibrium-infeasibility is hence well-formed for its continuation games. An* equilibrium *of the game $\mathcal{G}$ is defined to be a pair of strategy profile $\mathcal{S}$ and belief system $\mathcal{B}$ with two properties: (a) $\mathcal{B}$ follows Bayes's rule relative to $\mathcal{S}$, except at any history where the decision-maker can conclude that someone else has deviated from $\mathcal{S}$ at the immediately preceding history; (b) $\mathcal{S}$ is sequentially rational relative to $\mathcal{B}$, except at any history where either (b1) a seller has chosen a mechanism whose continuation game is equilibrium-infeasible or (b2) the current seller has deviated from $\mathcal{S}$ and the deviation that gives him less expected payoff than his equilibrium move from the standpoint when the deviation was made.*

We have hence recursively defined a notion of equilibrium for the auction-resale game with any finite number of bidders.[2] With this equilibrium concept, we do not need to specify the consequence after an owner has chosen a mechanism whose continuation game is equilibrium-infeasible (exceptional events (b1)), nor do we need to specify an owner's optimal mechanism if in the past he made a deviant and "dumb" move (exceptional events (b2)).

An *allocation* is defined to be a function that associates a final ownership to every possible profile of bidders' types, where a *final ownership* means a lottery that picks a final owner of the good from the set of all players (including sellers). An allocation is said to be *equilibrium-feasible* if there is a mechanism M at the initial stage of the auction-resale game such that the continuation game of M has an equilibrium that generates the allocation. In contrast, an allocation is said to be *BNE-feasible* if there is a mechanism M at the initial stage of the auction-resale game such that the continuation game of M has a BNE that generates the allocation.

[2]One can refine the notion of equilibrium by putting restrictions on the exceptional events of sequential rationality, say, by requiring that the configuration of beliefs at an exceptional event be commonly known among all the involved players.

2.2 The Relation to the Received Theory of Auction Design

BNE-feasible allocations have been the focus of the received theory of auction design. Myerson [18] has characterized the allocations that maximize the initial owner's expected profit among all the BNE-feasible allocations. I shall call any such profit-maximizing allocation *Myerson's allocation* and the amount of expected profit for the initial owner in such an allocation *Myerson's level*.

In our auction-resale game, the fact that an allocation is BNE-feasible does not mean that there is an equilibrium-feasible mechanism that generates it. To guarantee that such a mechanism always exists, we need to remove all the resale stages of the auction-resale game;[3] Removing the resale options means that resale can be costlessly banned. By assuming that the stages for possible resale are exogenously available, I replace this traditional assumption with the assumption "resale cannot be banned." The next lemma relates the mainstream framework to mine.

Lemma 2.1 *Myerson's level is an upper bound of the initial owner's expected profits that are equilibrium-feasible. A mechanism reaches Myerson's level for the initial owner if its continuation game has an equilibrium that generates Myerson's allocation and gives zero surplus to the lowest type of each bidder.*

Both statements of this lemma follows from the fact that an equilibrium is a BNE, and the second statement also uses the revenue equivalence theorem for BNE-feasible allocations. An equilibrium is a BNE because the condition of sequential rationality in our equilibrium concept (Definition 1) is stronger than the condition of rationality in BNE.

2.3 The Assumptions of the Distributions

Let us label the initial owner as player 0. For every bidder $i \in \{1, \ldots, n\}$, let F_i denote the distribution function of i's type, f_i denote the associated density function, and T_i denote the support. The first assumption is standard.

Assumption 1 (Hazard Rate) *For each $i \in \{1, \ldots, n\}$, the support T_i is convex and bounded from below and, if T_i is a nondegenerate interval, the density function f_i is positive and continuous on T_i and differentiable in its interior, and $(1 - F_i(t_i))/f_i(t_i)$ is a decreasing function of t_i on T_i.*

For each bidder $i \in \{1, \ldots, n\}$, define the *virtual utility* of i by

$$V_i(t_i) := t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}, \ \forall t_i \in T_i. \tag{1}$$

[3] Equivalently, we can assume that the initial owner can stay after stage one and that all complete contracts are available to him.

As well-known, Assumption 1 implies that the virtual utility function V_i is strictly increasing on T_i if T_i is a nondegenerate interval. In fact, we know more (proved in Appendix A):

Lemma 2.2 *If $(1 - F_i(x))/f_i(x)$ is a decreasing function of x on T_i, then for every $a \in T_i$ the expression $(F_i(a) - F_i(x))/f_i(x)$ is a decreasing function of x on $T_i \cap (\infty, a]$.*

With V_i strictly monotone, one can show that all Myerson's allocations are identical for almost all profile of bidders' types, and these allocations can be defined by the property that, for almost all profile $(t_i)_{i=1}^n$ of bidders' types, the final owner is selected in descending order of the virtual utilities $V_i(t_i)$ across i. It is also well-known that, when the distributions F_i are different across i, Myerson's allocation may assign the good to a bidder who values it less than another bidder. In this case, we can think of the allocation as *subsidizing* the former bidder against the other bidder. In our auction-resale game, Myerson's allocation is difficult to implement, because a subsidized bidder may upset the allocation via resale.

Thus, there is a tension between a seller's incentive to subsidize some bidders and bidders' ability to defeat the purpose of the subsidy via resale. To focus on this tension, I make the next assumption so that Myerson's allocation has a ranking over the bidders on whom to subsidize against whom.

Assumption 2 (Uniform Ranking) *The bidders can be indexed by $1, \ldots, n$ so that for all $i, j \in \{1, \ldots, n\}$, if $i > j$ and $x \in T_j$ then $x \in T_i$, $V_i(x) \leq V_j(x)$, and $V_j(x) \in V_i[T_i]$.*

Thus, for any $i, j \in \{1, \ldots, n\}$, $i > j$ means that Myerson's allocation subsidizes bidder j against bidder i: with a positive probability j is the final owner of the good while his type is less than bidder i's type. I will stick to the index system in Assumption 2 from now on.

For all $i, j \in \{1, \ldots, n\}$ such that $i > j$, define a mapping $\alpha_{ij} : T_j \to T_i$ by the equation

$$F_i \circ \alpha_{ij}(t_j) = F_i \circ V_i^{-1} \circ V_j(t_j) + f_i \circ V_i^{-1} \circ V_j(t_j)[V_i^{-1} \circ V_j(t_j) - t_j], \ \forall t_j \in T_j. \qquad (2)$$

Here the notation $\circ$ denotes the composition between functions; the function $V_i^{-1} \circ V_j$ exists because of Assumption 1 and $V_j[T_j] \subseteq V_i[T_i]$ (Assumption 2). The next lemma, proved in Appendix A, asserts that the mapping α_{ij} is well-defined.

Lemma 2.3 *By Assumptions 1 and 2, for all $i, j \in \{1, \ldots, n\}$ such that $i > j$, the function α_{ij} is well-defined by Eq. (2) and $\alpha_{ij}(x) \geq V_i^{-1}(V_j(x))$ for all $x \in T_j$.*

When there are more than two bidders, I will construct an equilibrium of the auction-resale game consisting of a nested tower of optimal auctions. To facilitate this recursive structure, I make the next three assumptions, whose roles will be clear later.

Assumption 3 *For all $i, j \in \{1, \ldots, n\}$ with $i > j$, the function α_{ij} is strictly increasing.*

Hence the inverse α_{ij}^{-1} of α_{ij} exists, so the next assumption is well-formed.

Assumption 4 *For all $i, j, k \in \{1, \ldots, n\}$ and for all $t_k \in T_k$, if $i > k > j$ then $\alpha_{ij} \circ \alpha_{kj}^{-1}(t_k) \geq V_i^{-1} \circ V_k(t_k)$.*

Assumption 5 *For any $w \in \{1, \ldots, n\}$, any $i, j \in \{w+1, \ldots, n\}$, and any $t_w \in T_w$, if $V_i(t_i) \geq$ (resp. $=$) $V_j(t_j)$ then $f_i(V_i^{-1}(V_w(t_w)))/f_i(t_i) \geq$ (resp. $=$) $f_j(V_j^{-1}(V_w(t_w)))/f_j(t_j)$.*

For example, suppose each bidder i's type is uniformly distributed on $[0, a_i]$, with $a_1 \leq \cdots \leq a_n$. Then we have:

$$\begin{aligned}
V_i(t_i) &= t_i - a_i; \\
V_i^{-1} \circ V_j(t_j) &= t_j + (a_i - a_j)/2; \\
\alpha_{ij}(t_j) &= t_j + a_i - a_j; \\
\alpha_{ij} \circ \alpha_{kj}^{-1}(t_k) &= t_k + (a_i - a_k)/2.
\end{aligned}$$

It is easy to verify that this example satisfies all the above assumptions.

The following fact will be useful. It is proved in Appendix A.

Lemma 2.4 *Suppose Assumptions 1 through 4. Pick any bidders i, j, k such that $i > k > j$. If $\alpha_{ij}(t_j) < t_i$ and $\alpha_{kj}(t_j) \geq t_k$, then $V_i(t_i) > V_k(t_k)$ and $t_i > t_k$.*

3 Preliminary Analyses

I shall construct an equilibrium of the auction-resale game. It will turn out that in this equilibrium the initial owner receives an expected profit equal to Myerson's level, which we already know is an upper bound of the equilibrium-feasible welfare for the initial owner (Lemma 2.1). This section prepares for the construction of this equilibrium.

3.1 Myerson's Allocation Needs Resale

Since Myerson's level is an upper bound of the feasible welfare for the initial owner, it would be easy to construct an equilibrium if the initial owner could implement Myerson's allocation during stage one so that no resale occurs afterwards. However, that is impossible unless the bidders' virtual utility functions are identical, as the following remark asserts.

Remark 3.1 *Suppose Assumption 1 and that for some distinct bidders $i, j \in \{1, \ldots, n\}$ there is an interior point x of $V_i^{-1} \circ V_j[T_j]$ such that $V_i(x) < V_j(x)$. Then it is impossible to generate Myerson's allocation such that the probability of resale is zero.*

Proof. By the hypotheses, Remark 3.1 in Zheng [20] implies that there is a positive probability with which Myerson's allocation picks bidder j as the final owner while his type is less than bidder i's type. Corollary 3.1 in [20] hence implies our remark here.[4] ■

Here is the intuition for the above remark (paraphrased from [20]). Suppose that there is a mechanism at stage one that has a continuation equilibrium generating Myerson's allocation with zero occurrence of resale. Then the allocation must have been implemented by the end of stage one. By hypotheses of the remark, this allocation subsidizes some bidder j against another bidder i. Consider the event that j wins at the end of stage one. From his winning status j learns that there is a positive probability with which the gain of trade between him and bidder i is positive. If winner j sticks to the equilibrium, there would be no resale and his payoff is zero; if j deviates (say making a take-it-or-leave offer to i at a price of j's type plus a tiny amount), he would get a positive expected profit from resale, hence a contradiction. The only complication skipped by the above argument is that winner j might have obtained additional news other than his winning status. The proof of Lemma 3.1 in [20] has taken that possibility into account.

Remark 3.1 implies that, although Myerson's allocation may be a first-best scenario for a seller, the seller cannot use it as a recipe. If a seller chooses a mechanism that uses Myerson's allocation as the rule to select a winner, then at equilibrium the rule must be violated with a positive probability.

3.2 Troubles with the Traditional Revelation-Principle Method

From the viewpoint of any current owner of the good, the task is to optimize on the set of equilibrium-feasible allocations and find a mechanism to implement an optimum. The corresponding task in the traditional framework is handled by the revelation principle and the assumption of no resale: The traditional framework focuses on BNE-feasible allocations; the revelation principle turns them into tractable mathematical objects; given any optimum among them, the no-resale assumption allows the owner to implement the optimal allocation by a direct revelation mechanism that uses the allocation as the winner-selection rule.

Applying this traditional method to our auction-resale environment, however, is difficult if not impossible. Since the resale option adds more incentive conditions than BNE-feasibility, the usual version of the revelation principle may include allocations infeasible in our environment. Furthermore, even if we have an equilibrium-feasible allocation, the current owner still need not know how to implement it. For example, Myerson's allocation is equilibrium-feasible (shown later), but Remark 3.1 implies that a seller cannot implement it without resorting to some bidders acting as middlemen.

[4] Zheng [20] assumes that there are only two bidders, but it is trivial to extend Remark 3.1 and Corollary 3.1 there to the case with n bidders.

3.3 The Main Idea for Constructing an Equilibrium

My method in this paper is to reach Myerson's level by directly constructing a mechanism that has a continuation equilibrium generating Myerson'a allocation and giving zero surplus to the lowest type of each bidder. If such a mechanism exists, then Lemma 2.1 implies that it is optimal for the owner.

Let me illustrate the main idea by the case with two bidders whose types' supports are $[0, \bar{t}_i]$ for $i = 1, 2$. To avoid the trivial case where there is no gain of trade between the initial owner (player 0) and the bidders, let us assume that $\bar{t}_i > t_0$ for each $i \in \{1, 2\}$. With Assumptions 1 and 2, one can calculate Myerson's allocation, illustrated by Figure 1. Here the rectangle $OBDH$ is the space of possible profiles of bidders' types. According to

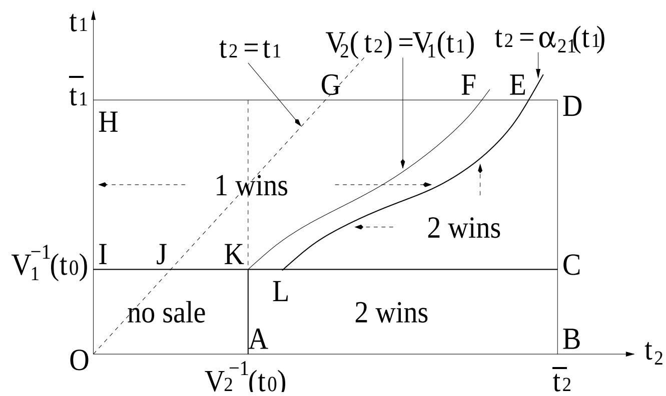

Figure 1: The Case with 2 bidders

Myerson's allocation, the final owner should be the initial owner on the area $OAKI$, bidder 1 on the area $IKFH$, and bidder 2 on the area $KABDF$. The curve FK corresponds to the equation $V_1(t_1) = V_2(t_2)$, which is the *dividing curve* between the bidders conditional on the initial owner's selling the good. Notice that the allocation subsidizes bidder 1 against bidder 2 on the region $GJKF$. As we have seen after Remark 3.1, it is this region that makes it hard to implement Myerson's allocation, for the subsidized bidder may resell the good to the other bidder, thereby upsetting the allocation.

To implement Myerson's allocation, we want to design a mechanism such that the subsidized bidder would find it optimal to resell according to Myerson's allocation in the event that he wins. Let us try the winner-selection rule depicted in Figure 1: it is the same as Myerson's allocation except that the dividing curve FK is replaced by the curve EL to its right, so that bidder 1 wins on the bigger region $HILE$. The new curve EL is represented by the equation $t_2 = \alpha_{21}(t_1)$ (recall α_{21} from (2)). When bidder 1 wins under this winner-selection rule and if the only news he gets is that he has won, bidder 1's updated belief about bidder 2's type is the conditional distribution $F_2(\cdot|t_2 \leq \alpha_{21}(t_1))$. Consequently, if he is choosing a resale price as a take-it-or-leave offer to bidder 2 at stage two, bidder 1 will find it optimal to post the price $V_2^{-1} \circ V_1(t_1)$, which one can show by the definition of α_{21}. Then the final owner between bidders 1 and 2 is chosen according to the dividing curve

FK of Myerson's allocation!

Let us therefore consider the following mechanism: Both bidders bid independently. If bidder 1's bid $\hat{t}_1$ is below his minimum bid $V_1^{-1}(t_0)$, bidder 1 loses and the initial owner makes a take-it-or-leave offer to bidder 2 at price $V_2^{-1}(t_0)$ (bidder 2's minimum bid). If the profile of bids lies on the region $ELCD$, bidder 1 loses and the initial owner makes a take-it-or-leave offer to bidder 2 at price $V_2^{-1} \circ V_1(\hat{t}_1)$. If the profile of bids lies on $HILE$, then bidder 1 wins and makes a payment $\wp_1(\hat{t}_1)$. To ensure that the bidder-turned winner 1 learns no news other than his winning status, his payment $\wp_1(\hat{t}_1)$ depends only on his own bid, and the public announcement at the end of stage one is silent about bidder 2's bid. To ensure that a winner's resale mechanisms are confined to posting prices, we disclose his bid in the public announcement of stage one, so that at equilibrium he is not a privately informed principal at stage two. To ensure that the lowest type of bidder 1 gets zero surplus, we construct his payment function $\wp_1$ so that his equilibrium expected payoff, taking into account his equilibrium profit from resale, is equal to his surplus in Myerson's allocation.

Here is a continuation equilibrium of the above mechanism that generates Myerson's allocation: Each bidder i reports his true type t_i. If bidder 1 wins, he updates as described above and posts a resale price at $V_2^{-1} \circ V_1(t_1)$. If bidder 2 wins, he updates that his type is higher than bidder 1's ($t_2 \geq \alpha_{21}(t_1) > t_1$) and keeps the good. If a winner posts any resale price, the loser accepts or rejects it truthfully. If a winner chooses any other equilibrium-feasible resale mechanism, then the loser acts according to the associated continuation equilibrium. If bidders follow this equilibrium, the final outcome is Myerson's allocation and the initial owner's welfare reaches Myerson's level.

To see why bidder 2 is truthful at stage one, notice that it is unprofitable for him to win the good now and resell it to bidder 1, since, as one can show, 2's payment upon winning is more than his revenue from reselling the good to bidder 1. Thus, bidder 2's decision at stage one, if pivotal, is whether to be the final owner now or later; he is indifferent between these options because his payments is the same across the stages. The demonstration of bidder 1's incentive requires a calculation of his expected payoff, so let us postpone it to the general case. At this point let us just observe that at equilibrium bidder 1 will not be privately informed when he chooses a resale mechanism after winning at stage one. The reason is that at equilibrium he bids truthfully at stage one and his bid is disclosed in the stage-one public announcement.

The above is a sketch of the main idea of how to implement Myerson's allocation. In the rest of the paper, I will formalize the idea for the general n-bidder case.

3.4 The Self-Replicating Auction-Design Environments

With more than two bidders, the design of a resale mechanism is more challenging, since a posted-price resale mechanism may be suboptimal for a secondary seller. To solve his mechanism-design problem, we need to know the environment given to a secondary seller.

For a seller, the most essential structure of his environment is the ranking across his potential buyers induced by their virtual utility functions. That is because this ranking determines Myerson's allocation, the first-best outcome for the seller. As a secondary seller's environment depends on the auction mechanisms and their outcomes in earlier stages, as well as the belief system of the entire auction-resale game, pinning down the law of motion for the virtual utilities throughout the auction-resale game is obviously difficult. Fortunately, owing to Assumptions 3, 4, and 5, this structure is preserved from one seller to another on an equilibrium path. I will prove this recursive feature in this subsection.

Let us consider stage one of the auction-resale game and the following rule:

for any profile $(\hat{t}_i)_i$ of bids

the winner is the lowest indexed bidder w such that $\hat{t}_i \leq \alpha_{iw}(\hat{t}_w)$ for all $i > w$. (3)

Suppose that everyone has bid truthfully, a bidder w becomes the winner by the rule (3), and his bid (hence type t_w) is disclosed in the current stage. Let us consider the environment inherited by the winner w. We have:

Lemma 3.1 *Suppose Assumptions 1 through 5. If bidder w is selected by the winner-selection rule (3) and if every bidder has been truthful during the winner-selection process, then w's type (resp. virtual utility) is higher than the type (resp. virtual utility) of any bidder $i < w$.*

Proof. The lemma is vacuously true if $w = 1$. If $w > 1$, then by the winner-selection rule (3), there must be the lowest indexed bidder i_1 such that $\alpha_{i_1 1}(t_1) < t_{i_1}$. As $\alpha_{i_1 1}(t_1) \geq V_{i_1}^{-1}(V_1(t_1))$ (Lemma 2.3) and $V_{i_1}^{-1}(V_1(t_1)) \geq t_1$ (Assumption 2), we know that $V_{i_1}(t_{i_1}) > V_1(t_1)$ and $t_{i_1} > t_1$. For all k between 1 and i_1, the choice of i_1 implies that $\alpha_{k1}(t_1) \geq t_k$, hence Lemma 2.4 implies $t_{i_1} > t_k$ and $V_{i_1}(t_{i_1}) > V_k(t_k)$. Thus, $t_{i_1} > t_k$ for all bidders $k < i_1$. If $\alpha_{j i_1}(t_{i_1}) \geq t_j$ for all $j > i_1$, then $w = i_1$ and the lemma is proved. Otherwise, we pick the lowest indexed bidder i_2 such that $\alpha_{i_2 i_1}(t_{i_1}) < t_{i_2}$ and as before we have $t_{i_2} > t_j$ and $V_{i_2}(t_{i_2}) > V_j(t_j)$ for all j between i_1 and i_2, including i_1; transitivity gives these two inequalities for all $j < i_2$. The lemma is therefore proved by induction. ■

Thus, a winner w wants to resell the good only to those bidders $i > w$. (Even if a lower-indexed bidder is willing to pay more than his true type because of an expectation of resale profit, the winner can obtain at least the same profit by being a reseller himself.) Suppose that every such bidder learns only w's type t_w and his winning status. Then for every bidder $i > w$, the posterior belief about i's type is commonly known among the players $\{w, w+1, \ldots, n\}$ and is represented by the conditional distribution function $F_i(\cdot | t_i \leq \alpha_{iw}(t_w))$. We can then define the *posterior* virtual utility function $V_{i|t_w}$ of every bidder $i \in \{w+1, \ldots, n\}$: for all $t_i \leq \alpha_{iw}(t_w)$, define

$$V_{i|t_w}(t_i) := t_i - \frac{1 - F_i(t_i | t_i \leq \alpha_{iw}(t_w))}{f_i(t_i | t_i \leq \alpha_{iw}(t_w))}. \tag{4}$$

The next lemma says that across i the ranking across i induced by $V_{i|t_w}$ is the same as the ranking across i induced by V_i. The proof is in Appendix A.

Lemma 3.2 *Suppose Assumptions 1 and 5. For any $w \in \{1, \ldots, n\}$, any $t_w \in T_w$, and any $i, j \in \{w+1, \ldots, n\}$,*

a. *$V_{i|t_w}(t_i) >$ (resp. $=$) t_w iff $V_i(t_i) >$ (resp. $=$) $V_w(t_w)$, and*

b. *$V_{i|t_w}(t_i) >$ (resp. $=$) $V_{j|t_w}(t_j)$ iff $V_i(t_i) >$ (resp. $=$) $V_j(t_j)$.*

By Lemma 3.2, if all the prior virtual utility functions satisfy Assumption 2, then all the posterior virtual utility functions satisfy the same assumption, with $V_{i|t_w}$ playing the role of V_i. By Lemma 2.2, every function $V_{i|t_w}$ is strictly increasing, hence its inverse $V_{i|t_w}^{-1}$ exists. Thus, for every $w \in \{1, \ldots, n\}$, every $t_w \in T_w$, and every $i, j \in \{w+1, \ldots, n\}$ such that $i > j$, we can define $\alpha_{ij|t_w}$ by Eq. (2), with $V_{k|t_w}$ playing the role of V_k for all k. Moreover, we have:

Lemma 3.3 *By Assumptions 1 and 2, $V_{i|t_w}^{-1} \circ V_{j|t_w} = V_i^{-1} \circ V_j$ and $\alpha_{ij|t_w} = \alpha_{ij}$ for every $w \in \{1, \ldots, n\}$, every $t_w \in T_w$, and every $i, j \in \{w+1, \ldots, n\}$ such that $i > j$.*

This lemma is proved in Appendix A. It follows that the functions $\alpha_{ij|t_w}$ satisfy Assumptions 3 and 4. Therefore, a secondary seller w's mechanism-design environment has a similar structure as the previous seller's environment. Myerson's allocations in both environments have the same ranking across the bidders, and both sellers use the same α functions if they select winners according to the rule (3). The next proposition summarizes the findings of this subsection.

Proposition 3.1 *Suppose that player $k \in \{0, 1, \ldots, n-1\}$ is the current owner and that among the players in $\{k, k+1, \ldots, n\}$ the prior distribution of every $i > k$ is commonly known to be F_i. Suppose that k sells the good to a winner w chosen from $\{k+1, \ldots, n\}$ by the rule (3) and suppose that everyone in $\{k+1, \ldots, n\}$ has been truthful during the selection and has publicly observed w's type. Then the winner w wants to sell the good only to those players $j > w$; further, if the prior distributions F_i ($\forall i > k$) satisfy Assumptions 1 through 5, then the posterior distributions of players $j > w$ satisfy these assumptions, the Myerson allocation in k's environment is the same as the allocation resulting from selling the good to w and having w implement the Myerson allocation in w's environment. The winner-selection rule (3) for w's case uses the same α functions as the rule for k's case.*

4 Equilibrium: A Tower of Nested Optimal Auctions

With the recursive structure discovered in the previous section, we are now ready to construct an equilibrium of the auction-resale game. I will first construct a mechanism for each player in the case that he becomes an owner of the good. I will then construct an equilibrium where each owner uses this mechanism, each bidder is honest and obedient, and each owner's welfare is his first-best outcome—the Myerson level in the environment he inherits.

4.1 The Mechanism

Let $k \in \{0, 1, \ldots, n-1\}$ denote a current owner of the good and suppose that among the players in $\{k, k+1, \ldots, n\}$ the prior distribution of every $i > k$ is commonly known to be F_i and it satisfies Assumptions 1 through 5. Hence the virtual utility function V_i for each $i > k$ and the function α_{ij} for every $i, j \in \{k+1, \ldots, n\}$ with $i > j$ are well-defined. Let t_k denote k's type (not necessarily commonly known). Denote the configuration of these data by $(k, t_k, (F_i)_{i=k+1}^n)$. Let me first define a mechanism $\mathcal{Z}(k, t_k, (F_i)_{i=k+1}^n)$ by the following algorithm and then explain the design. Here l denotes a "leading bidder" to be explained later, and w denotes the player (k or a bidder) who wins the good at the end of the current stage.

$w := k$; $\underline{b}_i := \max\{\inf T_i, V_i^{-1}(t_k)\}$ $(\forall i = k+1, \ldots, n)$; $l := k+1$;
while $l \leq n$ **do**
  secretly inform l of $\underline{b}_l$;
  each $i \in \{l, \ldots, n\}$ bids $\hat{t}_i$ independently across i and time;
  if $\hat{t}_l < \underline{b}_l$
    then $\underline{b}_j := \max\{\underline{b}_j, V_j^{-1}(V_l(\hat{t}_l))\}$ $(\forall j > l)$
      $l := l + 1$;
    else if $\alpha_{il}(\hat{t}_l) \geq \hat{t}_i$ $(\forall i > l)$
      then
        trade with l at price $\wp_l(\hat{t}_l | \underline{b}_l)$ by Eq. (7)
        publicly announce $\hat{t}_l$
        $w := l$ and **halt**;
      else
        $i_1 :=$ the lowest indexed $i > l$ with $\alpha_{il}(\hat{t}_l) < \hat{t}_i$
        $\underline{b}_j := \max\{\underline{b}_j, \max\{V_j^{-1}(V_i(\hat{t}_i)) : i = l, \ldots, i_1 - 1\}\}$ $(\forall j > l)$
        $l := i_1$

4.1.1 The Winner-Selection Rule

In order to induce a recursive structure, we want to use the winner-selection rule (3). However, if we try to employ this rule by a one-shot procedure, a winner would need to beat simultaneously those rivals indexed below him and those indexed above him, and his calculation would be complicated. The mechanism defined above hence employs this rule sequentially: At each round, a *leading bidder* $l \in \{k+1, \ldots, n\}$ is designated, with a pre-specified minimum bid $\underline{b}_l$ and a payment function $\wp_l(\cdot | \underline{b}_l)$; bidders in $\{l, \ldots, n\}$ bid independently. If l's bid is below $\underline{b}_l$, then skip him and designate $l+1$ as the leading bidder and repeat the process, where a bidder i's bid $\hat{t}_i$ is allowed to be different from his previous ones. If l's bid $\hat{t}_l \geq \underline{b}_l$ and $\alpha_{il}(\hat{t}_l) \geq \hat{t}_i$ for all $i > l$, then sell the good to l at the price $\wp_l(\hat{t}_l | \underline{b}_l)$ and publicly announce his bid $\hat{t}_l$. If $\hat{t}_l \geq \underline{b}_l$ and yet $\alpha_{il}(\hat{t}_l) < \hat{t}_i$ for some $i > l$, then find such an i with the lowest index, designate him as the leading bidder (skipping all bidders indexed below him), and repeat the process. Whenever a bidder i is skipped, update the minimum-bid requirement $\underline{b}_j$ for each $j > i$ by taking the higher number between $\underline{b}_j$ and $V_j^{-1}(V_i(\hat{t}_i))$. The updating of minimum bids is to ensure that a bidder wins in our mechanism only if he can beat the bidders indexed below him in the Myerson allocation. That is explained by the next lemma, which follows directly from the updating procedure:

Lemma 4.1 *For any $i \in \{k+1, \dots, n\}$, if every bidder $j < i$ has been truthfully reporting j's type t_j in the mechanism $\mathcal{Z}(k, t_k, (F_i)_{i=k+1}^n)$ up to the point when bidder i is designated the leading bidder, then the minimum bid $\underline{b}_i$ for i is $\rho_i((t_j)_{j=k+1}^{i-1})$, where*

$$\rho_i\left((t_j)_{j=k+1}^{i-1}\right) := \max\{\inf T_i, V_i^{-1}(t_k), \max\{V_i^{-1}(V_j(t_j)) : j = k+1, \dots, i-1\}\}. \tag{5}$$

4.1.2 The Payment Scheme

As we want owner k to obtain the Myerson level in his environment $(k, t_k, (F_i)_{i=k+1}^n)$, the payment scheme should ensure that every bidder's equilibrium expected payoff—viewed at the beginning of the current stage—be equal to his surplus in the Myerson allocation in the environment $(k, t_k, (F_i)_{i=k+1}^n)$.

Let us therefore fix our attention to the Myerson allocation of this environment. For every $i \in \{k+1, \dots, n\}$ and for every profile $(x_j)_{j=k+1}^n$ of objects, denote $x_{<i} := (x_j)_{j=k+1}^{i-1}$, $x_{>i} := (x_j)_{j=i+1}^n$, and $x_{-i} := (x_{<i}, x_{>i})$. Denote $q_i^*((t_j)_{j=k+1}^n)$ for the probability with which bidder i becomes the final owner in this allocation given the profile $(t_j)_{j=k+1}^n$ of types. In this allocation, to beat bidders indexed lower than him, bidder i's type must be at least $\rho_i(t_{<i})$. For any $\underline{b}_i$ in the range of ρ_i and for any $t_i \in T_i$, define:

$$\begin{aligned} q_i^*(t_i, t_{>i}|\underline{b}_i) &:= \mathrm{E}_{t_{<i}} \mathbf{1}_{\rho_i(t_{<i})=\underline{b}_i}(t_{<i}) q_i^*(t_i, t_{<i}, t_{>i}); \\ U_i^*(t_i|\underline{b}_i) &:= \int_{\inf T_i}^{t_i} \mathrm{E}_{t_{>i}} q_i^*(z, t_{>i}|\underline{b}_i) dz. \end{aligned} \tag{6}$$

Here E_X denotes the expected-value operator of functions of the random variable X with the probability measure induced by the priors F_j, and $\mathbf{1}_Y$ denotes the indicator function for the event Y.

Pick any $i \in \{k+1, \dots, n\}$, any $\underline{b}_i$, and any $t_i \in T_i$. Let $Q_i(t_i|\underline{b}_i)$ be zero if $t_i < \underline{b}_i$ and be the probability of the event $\{t_{>i} : (\forall j > i)\ \alpha_{ji}(t_i) \geq t_j\}$. Notice that $Q_i(t_i|\underline{b}_i)$ is bidder i's expected probability of winning in the mechanism given that he is the leading bidder, his bid is t_i, and the minimum-bid requirement is $\underline{b}_i$. Let $\pi_i^*(t_i)$ denote the Myerson level of the environment inherited by i if i has been selected by the rule (3) and everyone has been truthful during the selection. The quantity $\pi_i^*(t_i)$ is well-defined by Proposition 3.1. Let us construct the payment scheme $\wp_i(\cdot|\underline{b}_i) : T_i \to R$ by:

$$Q_i(\hat{t}_i|\underline{b}_i)\wp_i(\hat{t}_i|\underline{b}_i) := Q_i(\hat{t}_i|\underline{b}_i)\left[\hat{t}_i + \pi_i^*(\hat{t}_i)\right] - U_i^*(t_i|\underline{b}_i). \tag{7}$$

The reason for such construction will be clear in Step 2 of the next section.

4.2 The Equilibrium

We are now ready for the main result.

Theorem 1 *Suppose Assumptions 1 through 5. Then the mechanism* $\mathcal{Z}(0, t_0, (F_i)_{i=1}^n)$ *is equilibrium-feasible, implements the Myerson allocation, and maximizes the initial owner's expected profit among all equilibrium-feasible mechanisms, with this maximum profit equal to the Myerson level of the environment.*

Proof. The theorem will be proved by induction on the number n of bidders. The case for $n = 1$ is trivial, for the mechanism $\mathcal{Z}(0, t_0, (F_i)_{i=1}^n)$ becomes posting a price $\max\{V_i^{-1}(t_0), \inf T_i\}$ to the bidder i. Pick any $N = 1, 2, \ldots$ and suppose that the theorem is true whenever $n \leq N$. Let us prove the theorem when $n = N + 1$. By Lemma 2.1, it suffices to construct a continuation equilibrium of the mechanism $\mathcal{Z}(0, t_0, (F_i)_{i=1}^{N+1})$ that (i) generates Myerson's allocation and (ii) gives zero surplus to the lowest type of each bidder. The rest of the proof is organized as follows. First, I propose a strategy profile $\mathcal{S}^*$ for such a continuation equilibrium. Second, I show that both tasks (i) and (ii) are fulfilled if everyone follows this $\mathcal{S}^*$. Then I prove that, if everyone has been truthful throughout stage one, following $\mathcal{S}^*$ thereafter constitutes a continuation equilibrium starting at stage two. One is then lead to demonstrate a bidder's incentive of truthful bidding in stage one, provided that others abide to $\mathcal{S}^*$. The demonstration is done by analyzing a bidder's incentive when he is a leading bidder and his incentive when he is not. Once this analysis is done, the proof will be complete.

Step 1 *A strategy profile* $\mathcal{S}^*$ *as a candidate for a continuation equilibrium of* $\mathcal{Z}(0, t_0, (F_i)_{i=1}^{N+1})$*:*

> Throughout stage one, each bidder bids truthfully. If a bidder k has been truthful and won at the stage, then he auctions of the good to bidders $k+1, \ldots, N+1$ via the mechanism $\mathcal{Z}(k, t_k, (F_i(\cdot|t_i \leq \alpha_{ik}(t_k)))_{i=k+1}^{N+1})$. If a winner k does choose this resale mechanism, then every bidder in $\{k+1, \ldots, N+1\}$ who was truthful acts according to the strategy profile prescribed by the continuation equilibrium of this resale mechanism in the environment $(k, t_k, (F_i(\cdot|t_i \leq \alpha_{ik}(t_k)))_{i=k+1}^{N+1})$. (This continuation equilibrium exists by the induction hypothesis.) If a winner chooses a different and equilibrium-feasible mechanism, then every bidder behaves according to the associated equilibrium.

Applying the equilibrium concept (Definition 1) to cases exempt from the sequential rationality condition, the above profile does not specify bidders' actions if a seller chooses a deviant equilibrium-infeasible mechanism; nor does $\mathcal{S}^*$ specify a seller's mechanism if he was not truthful and won, since we will show that truth-telling yields a greater expected payoff than lying.

Step 2 *Tasks (i) and (ii) are fulfilled if every player follows the strategy profile $\mathcal{S}^*$.*

Given truth-telling at stage one, the mechanism $\mathcal{Z}(0, t_0, (F_i)_{i=1}^{N+1})$ selects a winner w by (3), who will use the resale mechanism $\mathcal{Z}(w, t_w, (F_i(\cdot|t_i \leq \alpha_{iw}(t_w)))_{i=w+1}^{N+1})$ according to $\mathcal{S}^*$. By Proposition 3.1, the posterior distributions $(F_i(\cdot|t_i \leq \alpha_{iw}(t_w)))_{i=w+1}^{N+1}$ satisfy Assumptions 1 through 5. Thus, the induction hypothesis implies that this resale mechanism will implement the Myerson allocation in the posterior environment $(w, t_w, (F_i(\cdot|t_i \leq \alpha_{iw}(t_w)))_{i=w+1}^{N+1})$. Proposition 3.1 then implies that selling the good to w implements the Myerson allocation in the initial environment. Hence task (i) is fulfilled.

For task (ii), it suffices to prove that, calculated at the beginning of stage one, a type-t_i bidder i's expected payoff $U_i(t_i)$ when everyone follows the strategy profile $\mathcal{S}^*$ is equal to $\mathrm{E}_{\underline{b}_i} U_i^*(t_i|\underline{b}_i)$, where U_i^* is defined in (6). This is sufficient because $U_i^*(t_i|\underline{b}_i) = 0$ if $t_i \leq \underline{b}_i$ (the probability $q_i^*(t_i, t_{>i}|\underline{b}_i) = 0$ in (6) if $t_i \leq \underline{b}_i$) and the lowest type $\inf T_i$ of i can never exceed any $\underline{b}_i$ taken from the range of ρ_i (Eq. (5)). To calculate $U_i(t_i)$, let us first calculate a type-t_i bidder i's expected payoff $u_i(\hat{t}_i, t_i)$ from bidding $\hat{t}_i$, when he is leading and secretly informed of his minimum bid requirement $\underline{b}_i$; this would give us $U_i(t_i)$ because $U_i(t_i) = \mathrm{E}_{\underline{b}_i} u_i(t_i, t_i|\underline{b}_i)$, as he learns nothing new until he is told to lead. Hence we calculate:

$$u_i(\hat{t}_i, t_i|\underline{b}_i) = Q_i(\hat{t}_i|\underline{b}_i) \left[t_i - \wp_i(\hat{t}_i|\underline{b}_i) + \pi_i(\hat{t}_i, t_i) \right], \tag{8}$$

where $\pi_i(\hat{t}_i, t_i)$ denotes i's optimal expected profit from possible resale conditional on his winning at the current stage. Eq. (8) results from the fact that the bidder has zero payoff if he does not win in the current stage when everyone else follows the strategy profile $\mathcal{S}^*$ (Proposition 3.1). Recall that $\pi^*(t_i)$ denotes the Myerson level in the posterior environment if i wins and everyone has been truthful. The induction hypothesis implies $\pi^*(t_i) = \pi_i(t_i, t_i)$ for every $t_i \in T_i$. Eq. (7) hence implies the desired claim

$$U_i(t_i) = \mathrm{E}_{\underline{b}_i} u_i(t_i, t_i|\underline{b}_i) = \mathrm{E}_{\underline{b}_i} U_i^*(t_i|\underline{b}_i).$$

Step 3 *If every bidder has been truthfully throughout stage one, following the strategy profile $\mathcal{S}^*$ thereafter constitutes a continuation equilibrium starting at stage two.*

Given truthful bidding, Proposition 3.1 implies that the bidder w who won in stage one knows that his type is higher than the type of any bidder indexed below him and hence wants to resell the good only to those indexed above him. The the amount of payment w made upon winning is independent on the past bids of the bidders $j > w$, the only news he learns is that $t_j \leq \alpha_{jw}(t_w)$ for all $j > w$. This is also the only news all such j learn, since the public announcement in the stage-one mechanism has made w's type t_w commonly known. Thus, by Proposition 3.1, the secondary owner w faces an environment that satisfies the condition of the induction hypothesis, which then implies that it is optimal for w to choose a resale mechanism according to $\mathcal{S}^*$ and that a continuation equilibrium of this resale mechanism is that everyone follows $\mathcal{S}^*$ thereafter.

Step 4 *If a bidder is truthful in $\mathcal{Z}(0, t_0, (F_i)_{i=1}^{N+1})$ when he is the leading bidder, then he is truthful when he is not the leading bidder.*

As a bidder gets zero payoff if the current mechanism skips him and designates a higher-indexed bidder to lead (and hence a higher-indexed bidder will win), under-reporting one's type cannot improve a non-leading bidder's payoff. Thus, the only kind of deviations to consider is over-reporting one's type. Such a deviation is pivotal only when the bidder wins in stage one by over-reporting while he should not have won in stage one had he been truthful. To win, bidder i must be able to lead and his bid $\hat{t}_i$ when he is leading must be at least his minimum bid $\rho_i(t_{<i})$ and must satisfy $\alpha_{ji}(\hat{t}_i) \geq t_j$ $(\forall j > i)$. Since within this step he is assumed to be truthful when he is leading, his type t_i must satisfy

$$(\forall j > i)\ \alpha_{ji}(t_i) \geq t_j \text{ and } t_i \geq \rho_i(t_{<i}) \tag{9}$$

in any pivotal case. Consequently, as long as his over-reporting is pivotal,

bidder i will be a winner sooner or later whether he over-reports or not. (10)

By Eq. (9), he will win now if he gets to lead via over-reporting. Suppose that he does not over-report and some bidder w indexed below him wins. With everyone else following the strategy profile $\mathcal{S}^*$, the Myerson allocation in w's posterior environment will be implemented (induction hypothesis). That allocation is determined by the ranking of the posterior virtual utility functions, which by Lemma 3.2a is equivalent to the ranking of the current virtual utility functions. Thus, the good will go to one of the bidders $j = i, \ldots, N+1$, since $t_i \geq \rho_i(t_{<i})$ implies that $V_i(t_i) \geq V_w(t_w)$ (Eq. (5)). But since whoever reselling the good to these bidders is expected to follow $\mathcal{S}^*$, bidder i must be the winner according the selection rule (3), due to the fact that $\alpha_{ji}(t_i) \geq t_j$ $(\forall j > i)$ and that the the winner-selection rule in that future stage will use the same α functions (Proposition 3.1). This proves (10).

It follows that bidder i's decision on whether to over-report is equivalent to whether to win now or to win later. I shall show that winning now cannot make him better-off. This is done by showing that the bidder's (a) expected profit from possible resale and (b) expected payment upon winning are unchanged by the bidder's decision. Let us handle item (a) first. Notice that it is unprofitable for bidder i to resell the good to bidders indexed below him. To see that, imagine that he is truthful and hence wins in a later stage. As the player reselling the good to him is expected follow $\mathcal{S}^*$ and hence to select i according to the rule (3), Lemma 3.1 implies the fact that i's type is higher than than the type of any bidder indexed below him. This fact remains true whether bidder i is truthful or not. It follows from this fact that item (a) is the maximum expected profit from reselling to bidders $i+1, \ldots, N+1$. (Even if a bidder $j < i$ is willing to pay more than j's true type because of an expectation of resale profit, bidder i can always obtain the same profit by being a reseller himself.) That takes care of item (a). To prove that item (b) is also invariant to bidder i's decision, notice from Eqs. (6) and (7) that i's payment $\wp_i(t_i|\underline{b}_i)$ upon winning remains the same conditional on the same minimum bid $\underline{b}_i$, whether he wins now or later. Hence it suffices to prove that his minimum bid is the same in both cases. If bidder i wins now, then his minimum bid is

$\rho_i(t_{<i})$ by Lemma 4.1. If he wins later from a reseller w $(w < i)$ whose type is t_w, then the same lemma implies that his minimum bid is

$$\underline{b}_i' := \max\{\inf T_i, V_{i|t_w}^{-1}(t_w), \max\{V_{i|t_w}^{-1}(V_{j|t_w}(t_j)) : j = w+1, \ldots, i-1\}\},$$

which is $\max\{\inf T_i, V_i^{-1}(V_w(t_w)), \max\{V_i^{-1}(V_j(t_j)) : j = w+1, \ldots, i-1\}\}$ by Lemma 1. Thus, $\underline{b}_i' \leq \rho_i(t_{<i})$. To show that $\underline{b}_i' \geq \rho_i(t_{<i})$, we need only $V_{i|t_w}^{-1}(t_w) \geq V_i^{-1}(V_j(t_j))$ for all $j < w$. Since $V_{i|t_w}^{-1}(t_w) = V_i^{-1}(V_w(t_w))$ for all $j < w$ (Lemma 1 and $V_{w|t_w}(t_w) = t_w$), it suffices to show that $V_w(t_w) \geq V_j(t_j)$ for all $j < w$, which follows from Lemma 3.1. Thus, $\underline{b}_i' = \rho_i(t_{<i})$, as desired.

Step 5 *A bidder i is truthful in $\mathcal{Z}(0, t_0, (F_i)_{i=1}^{N+1})$ when he is the leading bidder.*

Let $h_i^*(\hat{t}_i, t_i)$ denote the probability of the event "the leading bidder i wins at the current stage and resells the good to some bidder with his optimal resale mechanism" given that his type is t_i and he has bid $\hat{t}_i$ when he was leading. By the Envelope Theorem (Milgrom and Segal [16, Theorem 3]), Appendix A proves: for all $x, t, t' \in T_i$,

$$Q_i(x|\underline{b}_i)\left[\pi_i(x,t') - \pi_i(x,t)\right] = -\int_t^{t'} h_i^*(x,z)dz. \tag{11}$$

Let us observe that, for any $x, t, t' \in T_i$,

$$t < t' \Rightarrow h_i^*(x,t) \geq h_i^*(x,t'). \tag{12}$$

This follows from the definition of h_i^*: Given the same bid x, bidder i's probability of winning at the current stage and the posterior environment after his winning are each determined. Thus, the only difference between $h_i^*(x,t)$ and $h_i^*(x,t')$ is the probability of i reselling the good. To optimize for i, this probability is lower if i's type is lower.

We are ready to verify that truthful bidding is optimal for the leading bidder i. Denote $\bar{q}_i^*(t_i|\underline{b}_i) := \mathrm{E}_{t_{>i}} q_i^*(t_i, t_{>i}|\underline{b}_i)$. For any $t, t' \in T_i$,

$$\begin{aligned}
&u_i(t,t|\underline{b}_i) - u_i(t',t|\underline{b}_i)\\
&\quad= U_i^*(t|\underline{b}_i) - U_i^*(t'|\underline{b}_i) + (t'-t)Q_i(t'|\underline{b}_i) + [Q_i(t'|\underline{b}_i)\pi_i(t',t') - Q_i(t'|\underline{b}_i)\pi_i(t',t)]\\
&\quad= \int_{t'}^{t} \bar{q}_i^*(z|\underline{b}_i)dz - (t-t')Q_i(t'|\underline{b}_i) + \int_{t'}^{t} h_i^*(t',z)dz\\
&\quad\geq \int_{t'}^{t} \bar{q}_i^*(z|\underline{b}_i)dz - (t-t')Q_i(t'|\underline{b}_i) + \int_{t'}^{t} h_i^*(t',t')dz\\
&\quad= \int_{t'}^{t} \left(\bar{q}_i^*(z|\underline{b}_i) - \bar{q}_i^*(t'|\underline{b}_i)\right)dz\\
&\quad\geq 0.
\end{aligned}$$

Here the second equality follows from Eq. (11); the first inequality follows from the fact (12); the third equality comes from the fact that a leading bidder is the final owner of the good if and only if he wins now and cannot resell the good in the next stage; and the final inequality is due to the BNE-feasibility of the Myerson allocation (hence the monotonicity of $\bar{q}_i^*(\cdot|\underline{b}_i)$). Thus, truth-telling is optimal for a leading bidder.

All the steps of the proof have been completed. ■

The equilibrium constructed in the above proof is worth noticing, because it exhibits a tower of optimal auctions via optimal resale auctions, and during the equilibrium play, the environment inherits by a secondary seller is similar to the environment of his immediate predecessor. The next corollary summarizes these features.

Corollary 4.1 *Suppose Assumptions 1 through 5. Then the auction-resale game has an equilibrium where the initial owner uses the mechanism* $\mathcal{Z}(0, t_0, (F_i)_{i=1}^n)$ *to sell the good and every subsequent owner* $k \in \{1, \ldots, n-1\}$*, facing an environment that also satisfies Assumptions 1 through 5, uses the mechanism* $\mathcal{Z}(k, t_k, (F_i(\cdot|t_i \leq \alpha_{ik}(t_k)))_{i=k+1}^n)$ *to resell the good to bidders in* $\{k+1, \ldots, n\}$.

When the number n of bidders is less than three, Assumptions 3, 4, and 5 are not needed for our result. To see that, first notice that Assumptions 4 and 5 are vacuously true when $n \leq 2$. Further, Assumption 3 is needed for only two purposes. One is to ensure the existence of the inverse functions α_{ij}^{-1}, which in turn are needed only for Assumption 4 when $n > 2$. The other role of Assumption 3 (strict monotonicity of α_{ij}) is used by Lemma 2.4, which is vacuously true when $n \leq 2$. Thus, we have:

Corollary 4.2 *If the number of bidders is less than 3, then the conclusions of Theorem 1 and Corollary 4.1 are true if Assumptions 1 and 2 are satisfied.*

A Proofs

Lemma 2.2. With f_i differentiable (Assumption 1), the hypothesis of this lemma implies that

$$f_i(x)^2 + (1 - F_i(x))f_i'(x) \geq 0 \tag{13}$$

for every x interior to T_i and $(F_i(a) - F_i(x))/f_i(x)$ is differentiable for every x interior to $T_i \cap (\infty, a]$. It suffices to show that the derivative of $(F_i(a) - F_i(x))/f_i(x)$ with respect to x is nonpositive for all such x. As f_i is positive on T_i (Assumption 1), the sign of this derivative is the sign of $-[f_i(x)^2 + (F_i(a) - F_i(x))f_i'(x)]$. This is negative when $f_i'(x) \geq 0$. Thus, consider the case $f_i'(x) < 0$. Then

$$[F_i(a) - F_i(x)]f_i'(x) = -[F_i(a) - F_i(x)]|f_i'(x)| \geq -[1 - F_i(x)]|f_i'(x)| = [1 - F_i(x)]f_i'(x),$$

hence $-[f_i(x)^2 + (F_i(a) - F_i(x))f_i'(x)]$ is at most $-[f_i(x)^2 + (1 - F_i(x))f_i'(x)]$, which is nonnegative by Eq. (13), as desired. ∎

Lemma 2.3. Pick any bidders $i, j \in \{1, \ldots, n\}$ such that $i > j$. Let $x \in T_j$. For any $a \in T_i$ with $a \geq V_i^{-1}(V_j(x))$, define

$$\varphi_x(a) := V_i^{-1}(V_j(x)) - \frac{F_i(a) - F_i(V_i^{-1}(V_j(x)))}{f_i(V_i^{-1}(V_j(x)))}.$$

To prove the lemma, it suffices to show that there is a unique $\alpha_{ij}(x) \in T_i \cap [V_i^{-1}(V_j(x)), \infty)$ such that $\varphi_x(\alpha_{ij}(x)) = x$ (Eq. (2)). The uniqueness of $\alpha_{ij}(x)$ follows from the fact that the function φ_x is one-to-one (by its definition and the assumption that F_i is strictly increasing on T_i). To prove the existence of $\alpha_{ij}(x)$, we apply the Intermediate-Value Theorem to function φ_x: First, the function is continuous by Assumption 1. Second, by the definition of φ_x,

$$\varphi_x(V_i^{-1}(V_j(x))) = V_i^{-1}(V_j(x)) \geq x,$$

with the inequality due to $i > j$ (hence $V_i(x) \geq V_j(x)$ by Assumption 2). Third,

$$\lim_{a \to \sup T_i} \varphi_x(a) = V_i^{-1}(V_j(x)) - \frac{1 - F_i(V_i^{-1}(V_j(x)))}{f_i(V_i^{-1}(V_j(x)))} = V_i(V_i^{-1}(V_j(x))) = V_j(x) < x,$$

with the inequality due to Eq. (1). Thus, $\alpha_{ij}(x)$ exists. ∎

Lemma 2.4. With the notations of the lemma, we calculate:

$$t_i > \alpha_{ij}(t_j) \geq \alpha_{ij} \circ \alpha_{kj}^{-1}(t_k) \geq V_i^{-1} \circ V_k(t_k) \geq V_i^{-1} \circ V_i(t_k) = t_k.$$

Here the second inequality is due to Assumption 3, the third inequality is due to Assumption 4, and the fourth inequality is due to the fact $V_k(t_k) \geq V_i(t_k)$, which comes from $i > k$ and Assumption 2. Note that the third inequality implies $V_i(t_i) > V_k(t_k)$. ∎

Lemma 3.2. Pick any i, j, w, t_w specified by the hypothesis. By its definition, $V_{i|t_w}$ is strictly increasing (which also uses Lemma 2.2 and Assumption 1). Also notice from Eqs. (2) and (4) that

$$V_{i|t_w}\left(V_i^{-1}(V_w(t_w))\right) = t_w.$$

Thus, $V_{i|t_w}(t_i) >$ (resp. $=$) t_w is equivalent to $V_{i|t_w}(t_i) >$ (resp. $=$) $V_{i|t_w}(V_i^{-1}(V_w(t_w)))$, which, by the strict monotonicity of $V_{i|t_w}$, is equivalent to $t_i >$ (resp. $=$) $V_i^{-1}(V_w(t_w))$. This proves part (a). For part (b), let us calculate from Eq. (4):

$$\begin{aligned} V_{i|t_w}(t_i) - V_{j|t_w}(t_j) &= V_i(t_i) - V_j(t_j) + \frac{1 - F_i(\alpha_{iw}(t_w))}{f_i(t_i)} - \frac{1 - F_j(\alpha_{jw}(t_w))}{f_j(t_j)} \\ &= V_i(t_i) - V_j(t_j) + (t_w - V_w(t_w))\left(\frac{f_i(V_i^{-1}(V_w(t_w)))}{f_i(t_i)} - \frac{f_j(V_j^{-1}(V_w(t_w)))}{f_j(t_j)}\right), \end{aligned}$$

where the second equality follows from Eq. (2) and the definition of $V_w(V_i^{-1} \circ V_w(t_w))$. Notice that $t_w \geq V_w(t_w)$ by Eq. (1). Thus, the proved equality, coupled with Assumption 5, implies part (b). ■

Lemma 3.3. For any $t_j \in T_j$, let $x := V_{i|t_w}^{-1}(V_{j|t_w}(t_j))$. Then $V_{i|t_w}(x) = V_{j|t_w}(t_j)$, which implies $V_i(x) = V_j(t_j)$ by Lemma 3.2b. The strictly monotonicity of V_i gives $x = V_i^{-1}(V_j(t_j))$, hence $V_{i|t_w}^{-1}(V_{j|t_w}(t_j)) = V_i^{-1}(V_j(t_j))$. This fact, coupled with Eq. (2) that defines the α functions, implies that $\alpha_{ij|t_w} = \alpha_{ij}$. ■

Eq. (11). Denote i for a leading bidder, $b \in T_i$ for his bid while leading, $t \in T_i$ for his type. Denote $\mathcal{F}(b)$ for the set of equilibrium-feasible mechanisms for i to resell the good given that he has won by bidding b. (Note that $\mathcal{F}(b)$ is independent of t, since equilibrium-feasibility is relative to each player's posterior belief, which results from the actions, instead of types, of the players.) For any mechanism $M \in \mathcal{F}(b)$ and given the fact that he has bid b and his type is t, denote $h_i(M, b)$ for the probability of the event "bidder i wins in the current stage and resale occurs in his resale mechanism M," and denote $r_i(M, b)$ for i's expected revenue conditional on a resale in the mechanism M. (Note that both $h_i(M, b)$ and $r_i(M, b)$ are independent of i's type t.) With these notations, we have

$$Q_i(b|\underline{b}_i)\pi_i(b, t) = \max_{M \in \mathcal{F}(b)} h_i(M, b)[r_i(M, b) - t].$$

We shall apply the Envelope Theorem to this maximization, with t being the parameter. Denote $\phi_M(t) := h_i(M, b)[r_i(M, b) - t]$. By Milgrom and Segal [16, Theorem 3], the theorem is applicable if two conditions are satisfied: (a) $\frac{\phi_M(t') - \phi_M(t)}{t' - t}$ converges as $t' \to t$ uniformly across all $M \in \mathcal{F}(b)$; (b) for every $t \in T_i$, $\sup_{M \in \mathcal{F}(b)} |\phi_M(t)|$ is finite. Item (a) follows from the fact that $\frac{\phi_M(t') - \phi_M(t)}{t' - t} = -h_i(M, b)$. To prove Item (b), for every $t \in T_i$, we need upper and lower bounds of $\phi_M(t)$ for all M. Since $\phi_M(t) \leq r_i(M, b)$, one upper bound is the social surplus for the whole environment when all players are truthful. This surplus is finite because there are finitely many players and the expected value of each player's type is finite.

Since $\phi_M(t) \geq h_i(M,b)[\min_{j=1,\dots N+1} \inf T_j - t]$, with $h_i(M,b) \in [0,1]$ and $\inf T_j$ assumed to exist for each bidder j (Assumption 1), a lower bound of $\phi_M(t)$ exists for all M.

Thus, both items are satisfied. The Envelope Theorem then implies that $Q_i(b|\underline{b}_i)\pi_i(b,t) - Q_i(b|\underline{b}_i)\pi_i(b,t')$ is equal to the definite integral of the function $-h_i(M^*(b,\cdot),b)$ on the interval $[t',t]$, where $M^*(b,\tau)$ denotes an optimal resale mechanism given i's bid b and type τ. Recalling the notation h_i^*, we have $h_i(M^*(b,\cdot),b) = h_i^*(b,\cdot)$ and hence Eq. (11) is proved. ■

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