# Necessary and Sufficient Conditions for Prediction Market Accuracy[*]

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### Abstract

We consider a binary-event prediction market in which traders have heterogeneous prior beliefs. We derive conditions so that the prediction market is accurate in the sense that the equilibrium price equals the mean of traders' beliefs. We show that the prediction market is accurate i) for all distributions of beliefs if and only if the utility function is logarithmic, and ii) for all strictly concave utility functions if and only if the distribution of beliefs is symmetric about one half. We also exhibit a necessary and sufficient condition for the equilibrium price to be always above or below the mean beliefs for all symmetric beliefs distributions.

**Keywords**: Prediction market, heterogeneous beliefs, risk aversion, favorite-longshot bias, and asset prices.

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# 1 Introduction

Prediction markets are financial markets in which traders bet on the outcomes of uncertain events (e.g., political elections). Asset prices in prediction markets are often interpreted as probabilities. For instance, Arrow et al. (2008) introduce prediction markets as follows: *"Consider a contract that pays \$1 if Candidate X wins the presidential election in 2008. If the market price of an X contract is currently 53 cents, an interpretation is that the market 'believes' X has a 53% chance of winning"* (Arrow et al. 2008: 877). This interpretation that a price is a probability may be consistent with the empirical observation that the forecasts provided by prediction markets have been fairly accurate.[1] However, it is not clear what the "market believes" exactly means, and it has been argued that there is little theoretical support for this interpretation in general (Manski, 2006).

In this paper, we consider a simple binary-event prediction market in which otherwise identical traders have heterogeneous beliefs. We examine the conditions so that the prediction market is accurate in the sense that the equilibrium price equals the mean of traders' beliefs. Manski (2006) presented a first formal analysis of this question using a model with risk neutral traders and assuming limited investment budgets. He showed that the equilibrium price can largely differ from the mean beliefs of traders. Wolfers and Zitzewitz (2006) consider a more standard model with risk averse traders and show theoretically that the prediction market is accurate when the utility function of traders is logarithmic. Moreover Wolfers and Zitzewitz explore numerically how the equilibrium price is affected by beliefs heterogeneity for several utility functions and several beliefs distributions.[2]

Our main contribution in this paper is to derive the exact necessary and sufficient conditions for prediction market accuracy for general utility functions and for general distributions of beliefs. Specifically, we show that the prediction market is accurate i) for all distributions of beliefs if and only if the utility function is logarithmic, and ii) for all strictly concave utility functions if and only if the distribution of beliefs is symmetric about one half. More-

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[1] An early reference is Forsythe et al. (1992), showing that the Iowa electronic market performed extremely well. There is evidence that prediction markets have consistently performed better than any other forecasting institutions (see, e.g., Hanson 2006).

[2] See Gjerstad (2004) for theoretical results under constant relative risk aversion (CRRA) utility functions, and some numerical results. See Fountain and Harrison (2010) for further numerical results with wealth and beliefs heterogeneity.

over, we present several examples in which the joint distributions of traders'
beliefs, wealth and risk preferences lead to systematically violate prediction
market accuracy. Nevertheless, we provide indications about the direction of
the bias. Most significantly, we exhibit some conditions for the equilibrium
price to be always above or below the mean beliefs for all symmetric beliefs
distributions. This condition provides a rationale to the favorite-longshot
bias in a two-horse race model (Ali 1977; Manski 2006).

We organize the paper as follows. In the next section we introduce the
model and derive a sufficient condition for the equilibrium price to be unique.
In the next two sections we derive necessary and sufficient conditions for
prediction market accuracy. More precisely, section 3 derives a condition
on the utility function, and section 4 derives a condition on the probability
distribution representing the beliefs of traders. Then in section 5 we examine
the conditions leading to favorite-longshot bias in our simple model, and the
last section concludes.

## 2   The model

We consider a simple prediction market in which risk averse agents can buy
and sell a financial asset paying \$1 if a specific event occurs, and nothing
otherwise. The main assumption of the model is that the beliefs of the
agents about the occurrence of the specific event are heterogeneous. We
thus consider a model in which agents "agree to disagree", and therefore
have different prior beliefs. Namely, the heterogeneity in beliefs does not
come from asymmetric information but rather from intrinsic differences in
how agents view the world.[3] As a result, in our model agents do not update
their beliefs when they observe asset prices. In this section, we derive some
properties of the individual asset demand, and then of the equilibrium price
in this specific model.

### 2.1   Individual asset demand

In our model, each agent maximizes his expected utility based on his own
beliefs. Formally, when he decides how much to invest in the financial asset

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[3] Many models in finance have considered agents with heterogeneous prior beliefs. For a
justification and implications of these models, see for instance the survey papers by Varian
(1989) and Scheinkman and Xiong (2004).

paying $1 if the event occurs, he maximizes over $\alpha$ the following expected utility

$$pu(w + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi), \qquad (1)$$

in which $w$ is the agent's initial wealth, $p \in ]0, 1[$ his subjective probability that the event occurs (i.e., his belief), $\alpha$ his asset demand and $\pi$ the price of this asset. We assume the utility function $u(\cdot)$ to be strictly increasing, strictly concave and three times differentiable.

The first order condition of this optimization program is given by

$$p(1 - \pi)u'(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi u'(w - \alpha(p, \pi)\pi) = 0, \qquad (2)$$

in which $\alpha(p, \pi)$ is the unique solution. Differentiating with respect to $p$ the last equality, we obtain

$$
\begin{aligned}
0 &= (1 - \pi)u'(w + \alpha(p, \pi)(1 - \pi)) + \pi u'(w - \alpha(p, \pi)\pi) \qquad (3) \\
&\quad + \alpha_p(p, \pi)\{p(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) + (1 - p)\pi^2 u''(w - \alpha(p, \pi)\pi)\},
\end{aligned}
$$

and rearranging we have

$$\alpha_p(p, \pi) = \frac{(1 - \pi)u'(w + \alpha(p, \pi)(1 - \pi)) + \pi u'(w - \alpha(p, \pi)\pi)}{-p(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2 u''(w - \alpha(p, \pi)\pi)} > 0, \qquad (4)$$

that is, the asset demand increases with belief $p$. Since $\alpha(p, p) = 0$, we conclude that $\alpha(p, \pi) \geq 0$ if and only if $p \geq \pi$. Namely, the agent buys (respectively sells) the asset yielding $1 if the event occurs if and only if he assigns a probability for this event higher (respectively lower) than the asset price.

## 2.2   The equilibrium

Let $\widetilde{p}$ be the random variable representing the distribution of beliefs in the population of agents, and let $\pi^*$ be the equilibrium price. The equilibrium condition is

$$E\alpha(\widetilde{p}, \pi^*) = 0, \qquad (5)$$

in which $E$ denotes the expectation operator with respect to $\widetilde{p}$. Our main objective in the paper will be to compare $\pi^*$ to $E\widetilde{p}$. Notice that, when $\widetilde{p}$ is degenerate and equals $p$ with probability 1, then $\pi^* = p$ and there is no trade at the equilibrium. This is a trivial case always leading to prediction market

accuracy, $\pi^* = E\widetilde{p}$. We rule out this case, and consider nondegenerate $\widetilde{p}$ in the following.

It is easy to see that an equilibrium always exists in such a prediction market. Indeed, when $\pi$ tends to 0 (respectively tends to 1) $\alpha(p, \pi)$ is positive (respectively negative) for all $\widetilde{p}$, so its expectation is also positive (respectively negative) over $p$. Therefore when $\pi$ increases, the function $E\alpha(\widetilde{p}, \pi)$ must go from a positive to a negative region and thus must cross zero somewhere in between.

We now discuss the uniqueness of the equilibrium. We must show that $E\alpha(\widetilde{p}, \pi)$ only crosses the origin once. We know that $\alpha(p, \pi)$ has this single crossing property at $\pi = p$. But that does not guarantee that $E\alpha(\widetilde{p}, \pi)$ also has the single crossing property, as illustrated by the following example.

**Example 1** (Multiple equilibria): Consider agents with a quadratic utility function $u(w) = -(1 - w)^2$ and initial wealth $w = 1/2$. The optimal asset demand is equal to $\alpha(p, \pi) = \frac{p-\pi}{2(p-2p\pi+\pi^2)}$. In a prediction market with only two agents with respective beliefs denoted $p_1 = 0.1$ and $p_2 = 0.9$, the equilibrium condition is equivalent to $9 - 68\pi + 150\pi^2 - 100\pi^3 = 0$. Solving for this equation, it is found that there are three equilibrium prices in this prediction market: $\pi^* = (0.235, 0.5, 0.764)$.

A sufficient condition for the uniqueness of the equilibrium however is $\alpha_\pi(p, \pi) < 0$ everywhere. Indeed, this implies that the function $E\alpha(\widetilde{p}, \pi)$ is strictly decreasing in $\pi$, and therefore crosses zero at most once. Differentiating (2) with respect to $\pi$, we have

$$\alpha_\pi(p, \pi) \tag{6}$$
$$= \frac{-pu'(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)u'(w - \alpha(p, \pi)\pi)}{-p(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2 u''(w - \alpha(p, \pi)\pi)}$$
$$- \alpha(p, \pi)\frac{p(1 - \pi)u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi u''(w - \alpha(p, \pi)\pi)}{-p(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^2 u''(w - \alpha(p, \pi)\pi)}.$$

The first term is strictly negative but the second term is of ambiguous sign under risk aversion, so that the demand may increase when the price $\pi$ increases, as it is the case in Example 1. We now provide a sufficient condition for uniqueness by ensuring that the second term is also negative. We show that this is the case under nonincreasing absolute risk aversion.

**Proposition 1** *The equilibrium price $\pi^*$ is unique if $u$ has nonincreasing absolute risk aversion.*

*Proof*: We are done if we can show that the second term of the right hand side in (6) is negative. As this is simple to show, we only provide a sketch of the proof. Let denote $\widetilde{x} = (p, (1-\pi); 1-p, -\pi)$ so that this condition can be written more compactly $E[\widetilde{x}u'(w + \alpha\widetilde{x})] = 0$ implies that $-\alpha E[\widetilde{x}u''(w + \alpha\widetilde{x})] \leq 0$. Then the condition means that $-u'$ is more risk averse than $u$, which is equivalent to nonincreasing absolute risk aversion.∎

The intuition is the following. When the price of an asset increases, there are two effects captured by the two terms of the right hand side of equation (6). First, there is a substitution effect that leads to decrease its demand, but there is also a wealth effect that may potentially increase its demand. Intuitively, as the terminal wealth distribution deteriorates, the investor's attitude towards risk may change, and this wealth effect might prove sufficiently strong to increase the demand for the risky asset, as initially shown by Fishburn and Porter (1976) in the case of a first-order stochastic dominance (FSD) shift. Under decreasing absolute risk aversion (DARA) however, the negative wealth effect leads the agent to be more risk averse, and therefore this effect further decreases the demand for the risky asset. Under constant absolute risk aversion (CARA), there is no wealth effect, and only the first negative effect is at play. Finally, Example 1 featured multiple equilibria because the quadratic utility function has increasing absolute risk aversion.

When there is a unique equilibrium, one can make a simple comment on the effect of a change in the distribution of beliefs on the equilibrium price. Indeed, from the equilibrium condition $E\alpha(\widetilde{p}, \pi^*) = 0$ and since $\alpha_p(p, \pi) > 0$, any FSD improvement in the distribution of beliefs must increase the equilibrium price.

# 3   Which utility functions lead to prediction market accuracy?

Assuming a logarithmic utility function $u(w) = \log w$ (which displays DARA) we can obtain a closed-form solution of the first order condition (2):

$$\alpha(p, \pi) = w \frac{(p - \pi)}{\pi(1 - \pi)}.$$

Assuming identical wealth, this implies that the equilibrium condition (5) can simply be written

$$E\widetilde{p} = \pi^*.$$

This shows that the logarithmic utility function is sufficient for prediction market accuracy (Gjerstad 2004; Wolfers and Zitzewitz 2006). A natural question is whether the utility function must be logarithmic to guarantee prediction market accuracy or whether this is possible for other utility functions, i.e. whether $u(w) = \log w$ is also a necessary condition. We show in the following Proposition that this is indeed the case.

**Proposition 2** *For all* $\widetilde{p}$, $E\widetilde{p} = \pi^*$ *if and only if* $u(w) = \log w$

*Proof*: We just need to prove the necessity. Namely, let $\overline{p} = E\widetilde{p}$, we must show that $E\alpha(\widetilde{p}, \overline{p}) = 0$ for all $\widetilde{p}$ implies $u(w) = \log w$. We are done if we can show that this implication holds for a specific class of probability distribution in $\widetilde{p}$. We consider the class of "small" risks, that is we assume that $\widetilde{p}$ is close enough to $\overline{p}$ in the sense of a second-order approximation: $E\alpha(\widetilde{p}, \pi) = \alpha(\overline{p}, \pi) + 0.5 E(\widetilde{p} - \overline{p})^2 \alpha_{pp}(\overline{p}, \pi)$. Using this last equality, the necessary condition $E\alpha(\widetilde{p}, \overline{p}) = 0$ implies $\alpha(\overline{p}, \overline{p}) + 0.5 E(\widetilde{p} - \overline{p})^2 \alpha_{pp}(\overline{p}, \overline{p}) = 0$. Since for all $p$, we have $\alpha(p, p) = 0$, the necessary condition becomes $\alpha_{pp}(p, p) = 0$. Differentiating again (3) with respect to $p$ to compute $\alpha_{pp}(p, p)$ we obtain

$$
\begin{aligned}
0 = & \; 2\alpha_p(p, \pi)\{(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) - \pi^2 u''(w - \alpha(p, \pi)\pi)\} \\
& + \alpha_{pp}(p, \pi)\{p(1 - \pi)^2 u''(w + \alpha(p, \pi)(1 - \pi)) + (1 - p)\pi^2 u''(w - \alpha(p, \pi)\pi)\} \\
& + \alpha_p(p, \pi)^2\{p(1 - \pi)^3 u'''(w + \alpha(p, \pi)(1 - \pi)) - (1 - p)\pi^3 u'''(w - \alpha(p, \pi)\pi)\}.
\end{aligned}
$$

Taking $\pi = p$ in the last expression, we have $\alpha_p(p, p) = \frac{1}{p(1-p)} \times \frac{u'(w)}{-u''(w)}$ from (4), then rearranging yields

$$\alpha_{pp}(p, p) = \frac{(1 - 2p)}{p^2(1 - p)^2} \left[ \frac{u'(w)}{u''(w)} \right]^2 \left[ \frac{u'''(w)}{-u''(w)} - 2\frac{-u''(w)}{u'(w)} \right]. \qquad (7)$$

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Therefore a necessary condition is $\frac{u'''(w)}{-u''(w)} = 2\frac{-u''(w)}{u'(w)}$. Finally, integrating this differential equation gives $u(w) = \log w.\blacksquare$

We complement this result with two remarks.

**Remark 1** (Wealth heterogeneity): The result cannot be generalized to non-identical wealth, as possible correlation between wealth and beliefs would invalidate the result. Indeed, let $\widetilde{w}$ the random variable representing wealth heterogeneity. Assuming a logarithmic utility function, we can obtain

$$\pi^* = E\widetilde{p} + \frac{1}{E_{\widetilde{w}}\widetilde{w}}Cov(\widetilde{p}, \widetilde{w}). \tag{8}$$

Therefore there is no utility function that can always ensure prediction market accuracy when beliefs and wealth are correlated. Observe that, despite this impossibility result, the direction of the bias can be inferred if the analyst knows the sign of the correlation between beliefs and wealth. The intuition for equilibrium condition (8) is that richer individuals invest more, and therefore have more influence on the equilibrium price. Therefore, if wealth is positively (respectively negatively) correlated with beliefs, the equilibrium price will be higher (respectively lower).

**Remark 2** (Stakes): Suppose the agents have a (positive or negative) stake $\Delta$ in the event he predicts, so that they now maximize over $\alpha$ the following expected utility

$$pu(w + \Delta + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi).$$

Then it is easy to understand that the result is not guaranteed either. Indeed for the logarithmic utility function we have

$$\alpha(p, \pi) = w\frac{(p - \pi)}{\pi(1 - \pi)} - \Delta\frac{\pi(1 - p)}{\pi(1 - \pi)}$$

leading to the equilibrium condition

$$\pi^* = \frac{wE\widetilde{p}}{w + \Delta(1 - E\widetilde{p})}$$

The intuition is that when there is a positive (respectively negative) stake, the marginal utility decreases (respectively increases) if the event occurs. As a

result, the agents want to transfer wealth to the state in which the event does not occur (respectively occurs), and they typically use the prediction market as a hedging scheme to do this. The consequence is that the equilibrium is biased downward (respectively upward). Observe that if the stakes are individual-dependent but uncorrelated with beliefs, and if their mean across individuals is equal to zero, then we retrieve prediction market accuracy under a logarithmic utility function.

# 4 Which distributions of beliefs lead to prediction market accuracy?

The previous section has studied the conditions on the utility function so that there is market prediction accuracy for all $\widetilde{p}$. In this section, we want to study the dual problem: which conditions on $\widetilde{p}$ ensure prediction market accuracy for all $u$?[4] We show that the necessary and sufficient condition is that the probability distribution of beliefs is symmetric about one half.

**Proposition 3** *Assume that the equilibrium price $\pi^*$ is unique. For all $u$, $E\widetilde{p} = \pi^*$ if and only if $\widetilde{p}$ is symmetric about $1/2$.*

*Proof*: We first show that if $\widetilde{p}$ is symmetric about $1/2$ then $E\widetilde{p} = \pi^*$ for all $u$. Observe from the first order condition (2) that $\alpha(p, \pi) = -\alpha(1-p, 1-\pi)$. This implies that the equilibrium condition can be written $E\alpha(\widetilde{p}, \pi^*) = E\alpha(1-\widetilde{p}, 1-\pi^*) = 0$. Observe then that $\widetilde{p}$ symmetric about $1/2$ means that $\widetilde{p}$ is distributed as $1 - \widetilde{p}$. Consequently the equilibrium condition implies $E\alpha(\widetilde{p}, \pi^*) = E\alpha(\widetilde{p}, 1-\pi^*)$. Since the equilibrium is assumed to be unique, this last condition implies $\pi^* = 1 - \pi^*$, that is $\pi^* = 1/2 = E\widetilde{p}$.

We now demonstrate that if $E\widetilde{p} = \pi^*$ for all $u$ then it must be that the distribution is symmetric about $1/2$. This is proved by contradiction. Consider the following example. Let $u(w) = -e^{-rw}$ with CARA $r > 0$ and the probability density of beliefs $\tilde{p}$ is given by

$$f(p) = \begin{cases} \frac{2p}{b}, & \text{for } 0 < p \le b \\ \frac{2(1-p)}{1-b}, & \text{for } b < p < 1. \end{cases}$$

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[4] As made clear before, we consider all $u$ that are strictly increasing, strictly concave and three times differentiable.

Obviously, for $b \neq 1/2$, the belief distribution is not symmetric about $1/2$. It can be verified that $E\widetilde{p} > \pi^*$ for $0 < b < 1/2$ and $E\widetilde{p} < \pi^*$ for $1/2 < b < 1$. ∎

The intuition for the Proposition is simple. When $\widetilde{p}$ is symmetric about one half, the two states are formally indistinguishable, and therefore it cannot be that the price of an asset yielding one dollar in one state is different from that of an asset yielding one dollar in the other state, implying $\pi^* = 1/2$. We note, however, that if heterogeneity in individual utility functions is introduced, prediction market accuracy may not hold anymore even under $\widetilde{p}$ symmetric about $1/2$. The intuition is essentially the same as the one presented in Remark 1. This is illustrated by the following example which considers heterogeneity over (constant absolute) risk aversion.

**Example 2** (Heterogeneous CARA): Let $u_i(w) = -e^{-r_i w}$ in which $r_i > 0$ represents the CARA index of agents $i = 1, 2$ with respective beliefs $p_1 = 0.1$ and $p_2 = 0.9$. Under positive correlation between beliefs and risk aversion $(r_1, r_2) = (1, 3)$, we have $\pi^* = 1/4 < 1/2 = E\widetilde{p}$, while under negative correlation $(r_1, r_2) = (3, 1)$, we have $\pi^* = 3/4 > 1/2 = E\widetilde{p}$.

We have characterized the necessary and sufficient conditions for prediction market accuracy, for all $\widetilde{p}$ in Proposition 2, and then for all $u$ in Proposition 3. These conditions are rather stringent. However, one can relax these conditions in the sense that it is possible to find well-chosen pairs $(u, \widetilde{p})$ also yielding prediction market accuracy. This is shown in the following example which uses a specific constant relative risk aversion (CRRA) utility function and a specific nonsymmetric distribution of beliefs.

**Example 3** (Prediction market accuracy under CRRA). Consider agents with utility function $u(w) = -1/w$. Two groups of agents participate in the prediction market: one group has beliefs $p_1 = p$, and the other group has beliefs $p_2 = 1 - p$. Denoting $a$ the proportion of agents in the first group, we have $E\widetilde{p} = ap + (1-a)(1-p)$. One may then easily obtain that $E\alpha(\tilde{p}, \pi) = 0$ implies $\sqrt{\pi(1-\pi)}\{ap + (1-a)(1-p) - \pi\} = 0$ leading to $\pi^* = E\widetilde{p}$.

# 5 A necessary and sufficient condition for the favorite-longshot bias

In the previous analysis, we have examined under which conditions the prediction market is accurate in the sense that the equilibrium price $\pi^*$ is equal to mean beliefs $E\widetilde{p}$. In this section, we derive necessary and sufficient conditions for $\pi^*$ to be systematically above or below $E\widetilde{p}$.

The analysis developed in this section may provide a rationale for the favorite-longshot bias, namely for the empirical observation that longshots tend to be over-valued and that favorites tend to be under-valued (Ali 1977; Thaler and Ziemba 1988). To see why, consider a horse race with only two horses, and call the first horse the favorite (resp. longshot) if the mean beliefs that this horse wins are such that $E\widetilde{p} \geq 1/2$ (resp. $E\widetilde{p} \leq 1/2$). As we will see, the necessary and sufficient condition so that this horse is under-valued, i.e. $\pi^* \leq E\widetilde{p}$, critically depends on whether it is a favorite or a longshot. This result is presented in the following Proposition 4 in which $P(w) = -u'''(w)/u''(w)$ denotes the coefficient of absolute prudence (Kimball 1990) and $A(w) = -u''(w)/u'(w)$ denotes the coefficient of absolute risk aversion.

**Proposition 4** *Assume that the equilibrium price $\pi^*$ is unique. Then for all symmetric $\widetilde{p}$, $\pi^* \geq E\widetilde{p}$ if and only if $(1/2 - E\widetilde{p})(P(w) - 2A(w)) \geq 0$ for all $w$.*

*Proof*: See the appendix.

The difference between the mean beliefs and the equilibrium price therefore depends on whether the mean belief is less than $1/2$, and on whether absolute prudence is greater than twice absolute risk aversion. The condition $P \geq 2A$ is a familiar condition on utility functions derived from comparative statics analysis within expected utility models (Gollier 2001). Under CRRA utility functions, $P \geq 2A$ is equivalent to a parameter of constant relative risk aversion less than 1. Notice also that DARA is equivalent $P$ larger than $A$.

The result presented in Proposition 4 is illustrated in Figure 1. The horizontal axis represents the mean beliefs and the vertical axis represents the equilibrium price. The diagonal therefore represents prediction market accuracy, which holds everywhere if and only if $P = 2A$ (i.e., $u$ is logarithmic).

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The result therefore shows that there is a favorite-longshot bias if and only if the utility function displays $P > 2A$.

INSERT FIGURE 1 HERE

This result also generalizes the theoretical results of Gjerstad (2004) obtained for CRRA utility functions, and provide a theoretical foundation of the numerical simulations presented in Wolfers and Zitzewitz (2006) for various utility functions and probability distributions of beliefs. Moreover, this result is connected to an early result obtained by Varian (1985, 1989). Indeed Varian identifies a condition on the utility function so that the price of Arrow-Debreu securities in a complete market setting increases when beliefs are more dispersed in the sense of a mean-preserving spread of $\widetilde{p}$. Interestingly, Varian shows that the necessary and sufficient condition is $A' \geq -A^2$, which is in fact equivalent to $P \leq 2A$. Despite the obvious similarities, Varian's result is closely related, but not equivalent to our result however. Indeed, the nature of the comparison is not the same. As recognized by Varian, his result is not a comparative statics result, as he did not compare two equilibria, but compared the price of different assets within the *same* equilibrium.[5] This explains why our result critically depends on whether mean beliefs are lower or higher than $1/2$, while Varian's result does not.

One may wonder whether the condition $(1/2 - E\widetilde{p})(P(w) - 2A(w)) \geq 0$ is also necessary and sufficient for all distributions $\widetilde{p}$, not only symmetric ones. Note that $\pi^* \geq E\widetilde{p}$ is equivalent to $E\alpha(\widetilde{p}, \overline{p}) \geq 0$, and since $\alpha(\overline{p}, \overline{p}) = 0$, by Jensen's inequality the necessary and sufficient condition for all $\widetilde{p}$ is simply given by $\alpha_{pp}(p, \overline{p}) \geq 0$ for all $p$ and $\overline{p}$. The computation of $\alpha_{pp}(p, p)$ in (7) shows that the condition $(1/2 - E\widetilde{p})(P(w) - 2A(w)) \geq 0$ is indeed necessary for the favorite-longshot bias. However this condition is not sufficient, as the following example shows.

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[5]Let us illustrate this statement using the horse race parabola. Varian considers a race with $n > 2$ horses and compares the equilibrium price of two horses $s$ and $t$ within that race. He assumes that the mean beliefs that horses $s$ and $t$ win are identical, but that the beliefs over horse $s$ are more dispersed. In contrast, we consider a race with only two horses. Yet we allow the mean beliefs over the two horses to be different, so that there is a favorite (and thus a longshot) in that race. We then compare the equilibrium price of the favorite to the mean beliefs that this horse wins the race.

**Example 4** (Nonsymmetric beliefs): Consider agents with $u(w) = \sqrt{w}$ (i.e., $P > 2A$) and heterogeneous beliefs $p_1 = 0.1$ and $p_2 = 0.9$. When the proportion of agents with beliefs $p_1 = 0.1$ is 75% then $\pi^* = 0.272 < E\widetilde{p} = 0.3$ (i.e., the longshot is undervalued), and when the proportion of agents with beliefs $p_1 = 0.1$ is 25% then $\pi^* = 0.727 > E\widetilde{p} = 0.7$ (i.e., the favorite is overvalued).

# 6   Conclusion

This paper has derived generic conditions so that asset prices in prediction markets reflect the mean of the beliefs held by the traders. Unsurprisingly, these conditions are stringent. They require the utility function of all traders to be logarithmic or their beliefs to be distributed symmetrically. Moreover, no general conditions can be found when heterogeneity over traders' individual characteristics (wealth, risk aversion) is introduced.

Consistent with Manski (2006), the general message from this theoretical analysis is that we cannot realistically expect that asset prices in prediction markets only reflect the average of traders' beliefs. Typically, they should also reflect the dispersion of beliefs, as well as the individual characteristics of the traders. This suggests that the theoretical research on prediction markets should move beyond its initial focus on predictive accuracy.

# Appendix: Proof of Proposition 4

Recall that, when the equilibrium is unique, $\pi^* \geq \overline{p}$ if and only if $E\alpha(\widetilde{p}, \overline{p}) \geq 0$. For symmetric distributions, this holds true if and only if for all $\overline{p}$ (hereafter denoted $p$) we have

$$g(\delta) = \alpha(p + \delta, p) + \alpha(p - \delta, p) \geq 0, \tag{9}$$

in which $\alpha(p + \delta, p)$ is the unique solution of

$$(p+\delta)(1-p)u'(w+\alpha(p+\delta,p)(1-p))-(1-p-\delta)pu'(w-\alpha(p+\delta,p)p) = 0 \tag{10}$$

and $\alpha(p - \delta, p)$ is the unique solution of

$$(p-\delta)(1-p)u'(w+\alpha(p-\delta,p)(1-p))-(1-p+\delta)pu'(w-\alpha(p-\delta,p)p) = 0 \tag{11}$$

for $\delta \in [0, \min\{p, 1 - p\}]$.

Observe that $g(0) = 0$ and $g'(0) = 0$. Moreover, we have $g''(0) = 2\alpha_{pp}(p, p)$. Then, taking $\alpha_{pp}(p, p)$ from (7), we can see that $g''(0) \geq 0$ is equivalent to $(1/2 - p)(P(w) - 2A(w)) \geq 0$ for all $w$. This provides the necessity part of the Proposition.

We now prove the sufficiency. From (11), condition (9) is equivalent to

$$(p-\delta)(1-p)u'(w-\alpha(p+\delta,p)(1-p))-(1-p+\delta)pu'(w+\alpha(p+\delta,p)p) \geq 0. \tag{12}$$

Denoting $\phi(x) = 1/u'(x)$ and $\alpha = \alpha(p+\delta, p) \geq 0$, $\pi^* \geq p$ is therefore satisfied if

$$(p + \delta)(1 - p)\phi(w - \alpha p) - (1 - p - \delta)p\phi(w + \alpha(1 - p)) = 0 \tag{13}$$

which implies

$$(p - \delta)(1 - p)\phi(w + \alpha p) - (1 - p + \delta)p\phi(w - \alpha(1 - p)) \geq 0. \tag{14}$$

We now introduce two random variables:

$$\tilde{x} = \begin{cases} w + \alpha p, & \frac{p-\delta}{2p} \\ w - \alpha p, & \frac{p+\delta}{2p} \end{cases}, \qquad \tilde{y} = \begin{cases} w + \alpha(1 - p), & \frac{1-p-\delta}{2(1-p)} \\ w - \alpha(1 - p), & \frac{1-p+\delta}{2(1-p)} \end{cases}.$$

Then it can be verified that $E\tilde{x} = E\tilde{y} = w - \alpha\delta$ and $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$ if and only if $p \geq 1/2$. Note that $\phi''(x) \geq 0$ if and only if $P \leq 2A$. Therefore, when $p \geq 1/2$ and $P \leq 2A$, we have

$$E\phi(\tilde{x}) \geq E\phi(\tilde{y}), \tag{15}$$

14

which is equivalent to

$$\frac{1}{2p}\left[(p-\delta)\phi(w+\alpha p)+(p+\delta)\phi(w-\alpha p)\right]$$
$$\geq\frac{1}{2(1-p)}\left[(1-p-\delta)\phi(w+\alpha(1-p))+(1-p+\delta)\phi(w-\alpha(1-p))\right].$$

This last inequality then leads to

$$(1-p)(p-\delta)\phi(w+\alpha p)-p(1-p+\delta)\phi(w-\alpha(1-p))$$
$$\geq-\left[(1-p)(p+\delta)\phi(w-\alpha p)-p(1-p-\delta)\phi(w+\alpha(1-p))\right]=0,$$

where the last equality is given by (13). This shows that the condition (14) is satisfied. Hence $\pi^* \geq p$ when $p \geq 1/2$ and $P \leq 2A$. Moreover, when $p \leq 1/2$, $\tilde{y}$ is a mean-preserving spread of $\tilde{x}$, and $\phi''(x) \leq 0$ is equivalent to (15), leading to $\pi^* \geq p$. The case $\pi^* \leq p$ under $(1/2-p)(P-2A) \leq 0$ can be demonstrated in an analogous fashion. This concludes the proof. ∎
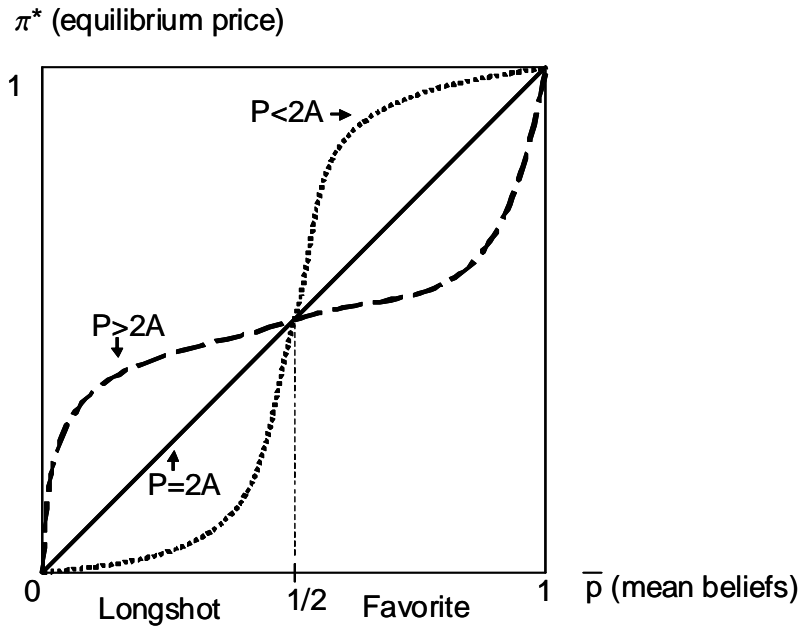
π* (equilibrium price)

1

P<2A →

P>2A
↓

P=2A

0

Longshot    1/2    Favorite    1    p̄ (mean beliefs)

Figure 1:

**Favorite-longshot bias**. This figure plots the equilibrium price $\pi^*$ as a function of mean beliefs $\overline{p}$. There is a favorite-longshot bias when $\pi^* > \overline{p}$ if and only if $\overline{p} < 1/2$. This is satisfied for the class of utility functions $u$ for which absolute prudence $P(w) = -u'''(w)/u''(w)$ is greater than twice absolute risk aversion $A(w) = -u''(w)/u'(w)$.

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