The Effect of Ambiguity Aversion on Insurance and Self-protection*

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Abstract

In this paper, we derive a set of simple conditions such that ambiguity aversion always raises the demand for self-insurance and the insurance coverage, but decreases the demand for self-protection. We also characterize the optimal insurance design under ambiguity aversion, and exhibit a case in which the straight deductible contract is optimal as in the expected utility model.

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1 Introduction

Almost all models used in insurance economics up to now have been based on subjective expected utility theory. Therefore this literature has assumed that ambiguity over probabilities does not matter for decisions. This assumption is inconsistent with many experimental results (see, e.g., Camerer, 1995). The most famous observation illustrating the violation of the subjective expected utility theory is the Ellsberg (1961)'s paradox. The paradox can be explained by ambiguity aversion, which can be thought as an aversion to any mean-preserving spread in the space of probabilities. For example, ambiguity averse agents prefer the lottery that yields a specific gain with a probability 1/2 to another lottery in which the probability of earning the same gain is uncertain, but with a subjective mean of 1/2 (for example if the agent believes that the probability is either 1/4 or 3/4 with equal probability); see, e.g., Halevy (2007) for recent experimental evidence. This psychological trait differs from risk aversion, which is an aversion to any mean-preserving spread in the payoff of the lottery. Since the seminal work by Gilboa and Schmeidler (1989), decision theorists have proposed various decision models that exhibit a form of ambiguity aversion (for a literature survey, see, e.g., Etner, Jeleva and Tallon, 2009).

In this paper, we study the effect of ambiguity aversion on insurance and self-protection decisions using the theory of ambiguity axiomatized by Klibanoff, Marinacci and Mukerji (2005). This theory captures the idea that mean-preserving spreads in probabilities reduce the welfare of ambiguity averse agents. Also, this theory permits to separate the effect of ambiguity aversion from that of risk aversion. Therefore our results permit to examine whether the effect of ambiguity aversion differs from that of risk aversion.

There has been several papers studying the effect of risk aversion on insurance and self-protection decisions within the subjective expected utility model. A well-known result is that risk aversion increases the demand for insurance, i.e. it raises the coverage rate, and it reduces the straight deductible. Indeed, in the case of coinsurance, this result is a direct consequence of the well-known Pratt's result that risk aversion decreases the demand for the risky asset. Similarly, it can be shown that risk aversion always increases the demand for self-insurance (Ehrlich and Becker, 1972). In contrast, the effect of risk aversion on self-protection is not clear, as first shown by Dionne and

Eeckhoudt (1985).[1]

Intuitively, ambiguity aversion reinforces risk aversion. Under this intuitive view, ambiguity aversion should raise the demand for insurance, and it should have an ambiguous effect on self-protection. Following Gollier (2011), we show that ambiguity aversion is not equivalent to an increase in risk aversion. For example, Gollier shows that ambiguity aversion may raise the demand for an ambiguous risky asset in a simple one-risky-one-risk-free asset portfolio model. In general, the effect of ambiguity aversion on the optimal decision is complex, but strongly connected to the comparative statics analysis of the effect of risk.[2]

In this paper, we mostly consider specific ambiguous contexts where the ambiguity is concentrated on one state of nature. That is, we assume that conditional to the information that this state does not occur the distribution of final wealth is unambiguous. This restrictive structure of ambiguity allows us to get simple results regarding the effect of ambiguity aversion. Our results are driven by the observation that the behavioral effect of ambiguity aversion is as if an expected utility maximizer would use a more pessimistic distribution of his beliefs when he acts. And it happens that the comparative statics of a change in subjective probability is simpler than that of risk aversion within the subjective expected utility model. We in turn derive interpretable conditions so that ambiguity aversion increases self-insurance and the demand for insurance but decrease self-protection decisions.

We finally examine the robustness of the celebrated Arrow (1971) result to the introduction of ambiguity aversion. Within the subjective expected utility model, Arrow showed that the optimal insurance contract has a straight deductible when the insurance tariff is based on the actuarial value of the contract. We show that under our specific structure of ambiguity the straight deductible contract remains optimal under ambiguity aversion. The only effect of ambiguity aversion is to reduce the level of the optimal deductible.

[1] An activity of self-protection consists in investing to reduce the probability of an accident. In fact, no general result can be obtained to sign the effect of risk aversion on self-protection in the general case (Sweeney and Beard, 1992). To sign this effet, it is necessary to specify the value of the probability of loss and to make further assumptions on the utility function (Dachraoui et al., 2004; Eeckhoudt and Gollier, 2005).

[2] Gollier (1995) summarizes the main findings on this comparative statics analysis in the case of a portfolio choice problem, which is further examined in Abel (2002). Strong restrictions on the utility function are required to guarantee that ambiguity aversion reduces the optimal risk exposure.

The paper is organized as follows. In the next section, we introduce the model of ambiguity aversion with the basic full insurance problem. We then study the self-insurance problem in section 3, and the self-protection problem in section 4. In section 5, we examine the problem of the optimal insurance design, and we conclude in the last section.

2 Full insurance

We consider an agent who faces an uncertain final wealth that can take values $w_1 < w_2 < ... < w_n$. The wealth distribution is represented by a vector $(p_1(\theta), ..., p_n(\theta))$ that belongs to the simplex of R^n. It is ambiguous in the sense that it depends upon a parameter θ that can take values in Θ. The ambiguity takes the form of a probability distribution for θ. Let $F : \Theta \to [0, 1]$ denote the cumulative distribution function describing this parametric uncertainty. Let $E_\theta g(\theta) = \int_\Theta g(\theta) dF(\theta)$ denote the expectation operator with respect to the parametric uncertainty.

Suppose that the true value of θ is known, and let $U(\theta)$ denote the expected utility reached for that specific θ. It is defined as follows:

$$U(\theta) = \sum_{s=1}^{n} p_s(\theta) u(w_s), \tag{1}$$

u being the vNM utility function. We say that the agent is ambiguity neutral if he evaluates his welfare ex ante by the expected value $E_\theta U(\theta)$, i.e., if he uses expected utility with the mean state probabilities $\overline{p}_s = E_\theta p_s(\theta)$. This agent is indifferent to any mean-preserving spread in state probabilities. In accordance to the resolution of the Ellsberg paradox, let us assume alternatively that the agent dislikes mean-preserving spreads in state probabilities, i.e., that he is ambiguity averse. Following the work by Klibanoff, Marinacci and Mukerji (2005) (KMM hereafter), let us assume that the agent evaluates his welfare ex ante by the certainty equivalent of the random variable $U(\theta)$ computed with an increasing and concave valuation function ϕ. The concavity of ϕ expresses ambiguity aversion, i.e. an aversion to mean-preserving spreads in the random state probabilities $p_s(\theta)$. The ex ante welfare is equal to the certainty equivalent of $U(\theta)$ using the concave function ϕ :

$$\phi^{-1}[E_\theta \phi[U(\theta)]].$$

In this section, we study the willingness to pay (hereafter WTP) P for risk elimination under ambiguity aversion, defined by:

$$u(\overline{w} - P) = \phi^{-1}[E_\theta \phi[U(\theta)]],$$

where $\overline{w}$ is equal to $\sum_{s=1}^{n} \overline{p}_s w_s$.

The special case of a linear ϕ function corresponds to ambiguity neutrality. In that case, the WTP for the elimination of risk is denoted P_0, which is defined by $u(\overline{w} - P_0) = E_\theta U(\theta)$. Observe that under ϕ concave, namely under ambiguity aversion, we have

$$\begin{aligned} \phi[u(\overline{w} - P)] &= E_\theta \phi[U(\theta)] \\ &\leq \phi[E_\theta U(\theta)] = \phi[u(\overline{w} - P_0)]. \end{aligned}$$

Therefore P_0 is always less than P under ϕ concave. In other words, ambiguity aversion always raises the WTP for risk elimination. The intuition is that eliminating the risk also eliminates all the ambiguity associated with the risk. Therefore this is no surprise that ambiguity averse agents are willing to pay an extra premium for risk elimination.

3 Self-insurance

Self-insurance is a technique aimed at increasing wealth in a specific state i against a cost incurred in all states. We assume that the ambiguity is concentrated on that state i in the sense that the distribution of final wealth conditional to the information that "the state is not i" is unambiguous. This means that for all $\theta \in \Theta$ and all $s \neq i$,

$$p_s(\theta) = (1 - p_i(\theta))\pi_s, \tag{2}$$

where π_s is the probability of state s conditional to the state not being i, with $\Sigma_{s \neq i} \pi_s = 1$. This structure of ambiguity is without loss of generality when there are only two states of nature. It is restrictive when there are more than two states of nature. Without loss of generality, we assume in this paper that p_i is increasing in θ.

3.1 Willingness to pay for self-insurance

We first examine the willingness to pay for an infinitesimal insurance for state i. Let $P(\varepsilon)$ denote the WTP for an increase in wealth in state i:

$$E_\theta \phi \left[p_i(\theta) u(w_i + \varepsilon - P(\varepsilon)) + (1 - p_i(\theta)) \sum_{s \neq i} \pi_s u(w_s - P(\varepsilon)) \right] = E_\theta \phi[U(\theta)].$$

Fully differentiating this equation with respect to ε yields

$$P'(0) = \frac{u'(w_i) E_\theta p_i(\theta) \phi'[U(\theta)]}{E_\theta \phi'[U(\theta)] \left[p_i(\theta) u'(w_i) + (1 - p_i(\theta)) E u'(w_{-i}) \right]}, \tag{3}$$

where $Eu'(w_{-i}) = \Sigma_{s \neq i} \pi_s u'(w_s)$ is the expected marginal utility conditional to the state not being i. We are interested in determining the effect of ambiguity aversion on the willingness to pay for insurance. Using (3) with $\phi' \equiv 1$ implies that the marginal WTP for insurance under ambiguity neutrality equals

$$\frac{\overline{p}_i u'(w_i)}{\overline{p}_i u'(w_i) + (1 - \overline{p}_i) E u'(w_{-i})}. \tag{4}$$

Under ambiguity aversion, we can define a distorted probability of state i as follows:

$$\widehat{p}_i = \frac{E_\theta p_i(\theta) \phi'[U(\theta)]}{E_\theta \phi'[U(\theta)]}. \tag{5}$$

We can then rewrite (3) as

$$P'(0) = \frac{\widehat{p}_i u'(w_i)}{\widehat{p}_i u'(w_i) + (1 - \widehat{p}_i) E u'(w_{-i})}. \tag{6}$$

The effect of ambiguity aversion on the marginal WTP for insurance in favor of state i is equivalent to a change in the probability of state i from $\overline{p}_i$ to $\widehat{p}_i$. It is easy to check that an increase in the distorted probability $\widehat{p}_i$ always raises the marginal WTP. Thus, ambiguity aversion raises the marginal WTP to insure state i if and only if it raises the distorted probability of state i.

Let us define ψ_1 as the certainty equivalent of w_{-i}:

$$Eu(w_{-i}) = u(\psi_1).$$

It is easy to check that $\widehat{p}_i$ is larger than $\overline{p}_i$ if and only if w_i is smaller than ψ_1. Observe that $U(\theta) = p_i(\theta) u(w_i) + (1 - p_i(\theta)) u(\psi_1)$ is decreasing in θ in

that case. Because ϕ' is decreasing under ambiguity aversion, the covariance rule (Kimball, 1951) implies that

$$\widehat{p}_i = \frac{E_\theta p_i(\theta)\phi'[U(\theta)]}{E_\theta \phi'[U(\theta)]} \geq \frac{E_\theta p_i(\theta) E_\theta \phi'[U(\theta)]}{E_\theta \phi'[U(\theta)]} = E_\theta p_i(\theta) = \overline{p}_i.$$

Thus, ambiguity aversion has the effect to increase the distorted probability of the state to insure if the corresponding state wealth is smaller than the certainty equivalent wealth in other states, and to reduce the distorted probability of the state to insure otherwise. This concludes the proof of the following proposition:

Proposition 1 *Consider the decision to self-insure state i in which ambiguity is concentrated. Ambiguity aversion raises the marginal WTP to self-insure state i if wealth in state i is smaller than the certainty equivalent wealth ψ_1, and it reduces it otherwise.*

There is a simple intuition to this result. When state i has a wealth smaller than the certainty equivalent ψ_1, raising w_i reduces the dispersion of $U(\theta)$. Therefore, it is positively valued by all ambiguity averse agents. The opposite result holds when wealth in state i is larger than ψ_1. Note that in the two-state case where a loss L occurs in state i, we simply have $w_i = w - L < \psi_1 = w$, so that ambiguity aversion raises the marginal WTP to self-insure state i.

3.2 Optimal self-insurance

We now examine the impact of ambiguity aversion on the optimal insurance in favor of state i. It is convenient to consider first a general model in which the expected utility $U(e, \theta)$ is a function of a decision e and of an unknown parameter θ. Under ambiguity neutrality, the agent selects the level e that maximizes her unconditional expected utility $E_\theta U(e, \theta)$. It yields the following first-order condition (hereafter subscripts with decision variables denote derivatives):

$$E_\theta U_e(e^*, \theta) = 0. \tag{7}$$

From now on, e^* denotes the solution of this equation, and is interpreted as the optimal prevention effort under ambiguity neutrality. Under ambiguity aversion, the objective of the decision-maker is to maximize $\phi[W(e)] =$

$E_\theta \phi[U(e,\theta)]$ where $W(e)$ could be interpreted as the certainty equivalent utility on the same way than the certainty equivalent wealth in the expected utility theory. Assuming that U is concave in e, the optimal level of effort is increased by ambiguity aversion if $W'(e^*)$ is positive. We have that

$$\phi'[W(e^*)]W'(e^*) = E_\theta U_e(e^*,\theta)\phi'[U(e^*,\theta)]. \tag{8}$$

As a simple application of the covariance rule, we obtain the following preliminary result.

Lemma 1 *In decision problem $max_e E_\theta \phi[U(e,\theta)]$ with $U_{ee} \leq 0$, ambiguity aversion increases (resp. decreases) the optimal decision e if and only if $U(e^*,\theta)$ and $U_e(e^*,\theta)$ are anti-comonotonic (resp. comonotonic), where e^* is defined by (7).*

Proof: If $U(e^*,\theta)$ and $U_e(e^*,\theta)$ are anti-comonotonic, we have that

$$E_\theta U_e(e^*,\theta)\phi'[U(e^*,\theta)] \geq E_\theta U_e(e^*,\theta)E_\theta \phi'[U(e^*,\theta)] = 0.$$

The first inequality comes from the fact that ϕ' is decreasing, which implies that $U(e^*,\theta)$ and $\phi'[U_e(e^*,\theta)]$ are comonotonic. The second equality is due to (7). This shows that $W'(e^*) \geq 0$, so that the optimal prevention under ambiguity aversion is less than e^*, the optimal effort under ambiguity neutrality. This proves sufficiency. For necessity, suppose by contradiction that there exists $(\theta,\theta') \in \Theta^2$ so that $U(e^*,\theta') > U(e^*,\theta)$ and $U_e(e^*,\theta') > U_e(e^*,\theta)$. Let θ be distributed as $(\theta, q; \theta', 1-q)$. Under this assumption, we obtain that $E_\theta U_e(e^*,\theta)\phi'[U(e^*,\theta)] < 0$, a contradiction. ■

The optimal effort is increased by ambiguity aversion if the marginal value of effort (expressed by U_e) is larger for less favorable priors, i.e. for θ yielding a smaller expected utility U. Indeed, ambiguity averse agents put more weight on these unfavorable priors than ambiguity neutral agents.

We apply this result to the case of the optimal insurance of state i. More specifically, suppose that one can undertake ex ante an effort e that raises wealth in state i to $w_i(e) - e$, and that reduces wealth in all other state $s \neq i$ to $w_s - e$. The problem can be written as follows:

$$U(e,\theta) = p_i(\theta)u(w_i(e) - e) + (1 - p_i(\theta)) \sum_{s \neq i} \pi_s u(w_s - e).$$

To make the problem interesting, we assume that $w_i'(e)$ is larger than unity. The condition $U_{ee} \leq 0$ is satisfied under $w_i''(e) \leq 0$. If e^* is the optimal

demand for insurance of the ambiguity neutral agent as defined by equation (7), we can redefine the certainty equivalent wealth level ψ_1 conditional to $s \neq i$ in this self-insurance context as follows:

$$\sum_{s \neq i} \pi_s u(w_s - e^*) = u(\psi_1 - e^*). \tag{9}$$

Suppose first that $w_i(e^*)$ is smaller than ψ_1. It implies that $U(e^*, \theta)$ is decreasing in θ. Observe also that

$$U_e(e^*, \theta) = p_i(\theta) u'(w_i(e^*) - e^*)(w_i'(e^*) - 1) + (p_i(\theta) - 1) \sum_{s \neq i} \pi_s u'(w_s - e^*).$$

The two terms in the right-hand side of the above equation are increasing in θ, independent of the value of $w_i(e^*)$ relative to ψ_1. Thus, Lemma 1 implies that the optimal self-insurance effort is increased by ambiguity aversion when the wealth level in the insurable state is smaller than the certainty equivalent wealth level in the other states. Because U and U_e are comonotonic when $w_i(e^*)$ is larger than ψ_1, the opposite result holds in that case. This proves the following proposition.

Proposition 2 *Consider the decision to self-insure state i in which ambiguity is concentrated. Ambiguity aversion raises the optimal self-insurance effort if wealth in state i is smaller than the certainty equivalent wealth ψ_1 defined by (9), and it reduces it otherwise.*

This result generalizes a result in Snow (2011), who considers the special case with only two states of nature. As mentioned above in such a case, ψ_1 is equal to the initial wealth and the result holds true. Indeed the intuition of our result is the same as for Proposition 1. When the insurable state has a low wealth level, self-insurance reduces the dispersion of conditional expected utility levels. Ambiguity averse agents will therefore invest more in self-insurance.

A special case of this result is obtained in the case of the demand for insurance, where e is reinterpreted as the insurance premium. Suppose that there are two states of nature. In the no-loss state, initial wealth is w, whereas it is only $w-L$ in the loss state i. For each dollar of premium, the policyholder receives an indemnity k if and only if a loss occurs. If a premium e is paid

ex ante, final wealth will be equal to $w_i(e) - e$ and $w_s - e$ respectively in the loss state and in the no-loss state, with $w_i(e) = w - L + ke$ and $w_s = w$. Assuming that the indemnity k can never exceed the loss, we have that $w_i(e^*)$ is always smaller than the certainty equivalent wealth $\psi_1 = w$. This yields the following corollary.

Corollary 1 *Consider the standard coinsurance problem with two states of nature. Ambiguity aversion always raises the insurance coverage rate.*

We can infer from Gollier (2011) that this result does not hold under a general structure of ambiguity as considered in section 2.

4 Self-protection

Self-protection is a technique aimed at reducing the probability of a specific state i at a cost incurred in all states. As in the previous section, we assume that the ambiguity is concentrated on that state i in the sense of equation (2).

4.1 Willingness to pay for self-protection

We first examine the impact of ambiguity aversion on the willingness to pay for a marginal investment in self-protection. Suppose that one can reduce the probability of state i by ε in all possible scenarios θ. The distribution of wealth conditional to $s \neq i$ is supposed to be unaffected by this investment. These two ceteris paribus assumptions mean that the self-protection effort affects the risk but has no impact on the degree of ambiguity. The WTP for this action is denoted $P(\varepsilon)$, which is defined as follows:

$$E_\theta \phi \left[(p_i(\theta) - \varepsilon) u(w_i - P(\varepsilon)) + (1 - p_i(\theta) + \varepsilon) \sum_{s \neq i} \pi_s u(w_s - P(\varepsilon)) \right] = E_\theta \phi \left[U(\theta) \right].$$

Straightforward computations lead to

$$P'(0) = \frac{\left(\sum_{s \neq i} \pi_s u(w_s) - u(w_i) \right) E_\theta \phi' \left[U(\theta) \right]}{E_\theta \left(p_i(\theta) u'(w_i) + (1 - p_i(\theta)) \sum_{s \neq i} \pi_s u'(w_s) \right) \phi' \left[U(\theta) \right]}. \tag{10}$$

This can be rewritten as follows:

$$P'(0) = \frac{u(\psi_1) - u(w_i)}{\widehat{p}_i u'(w_i) + (1 - \widehat{p}_i)u'(\psi_2)}, \tag{11}$$

in which $\widehat{p}_i$ is the distorted probability defined by equation (5), and ψ_1 and ψ_2 are respectively the certainty equivalent and the precautionary equivalent wealth level conditional to $s \neq i$:

$$u(\psi_1) = \sum_{s \neq i} \pi_s u(w_s) \quad \text{and} \quad u'(\psi_2) = \sum_{s \neq i} \pi_s u'(w_s). \tag{12}$$

We hereafter assume that the wealth level in state i is smaller than the certainty equivalent ψ_1, so that the marginal WTP for self-protection is positive. We see from equation (11) that the marginal WTP is the ratio between the marginal utility benefit of self-protection and the marginal utility cost of the corresponding effort. The marginal utility benefit is measured by the expected utility difference when $s = i$ and when $s \neq i$, whereas the utility cost of self-protection is measured by the ambiguity-distorted expected marginal utility $\widehat{p}_i u'(w_i) + (1 - \widehat{p}_i)u'(\psi_2)$. Equation (11) corresponds to the marginal WTP of an ambiguity neutral agent with distorted beliefs $\widehat{p}_i$. Thus, as in the previous section, the effect of ambiguity aversion is to transform probability $\overline{p}_i = E_\theta p_i(\theta)$ into a distorted probability $\widehat{p}_i$. We have shown in the previous section that $\widehat{p}_i$ is larger than $\overline{p}_i$ if and only if w_i is smaller than ψ_1. On the other hand, it is immediate from (11) that the marginal WTP for self-protection of state i is decreasing in $\widehat{p}_i$ if and only if w_i is smaller than the precautionary equivalent wealth ψ_2.

Let us assume that the utility function of the agent exhibits decreasing absolute risk aversion (DARA). From Kimball (1990), we know that this means that the precautionary equivalent ψ_2 is smaller than the certainty equivalent ψ_1. Let us first consider a situation in which the wealth level w_i in state i is smaller than $\psi_2 \leq \psi_1$. In that case, $\widehat{p}_i$ is larger than $\overline{p}_i$ and the marginal WTP is decreasing in $\widehat{p}_i$. This implies that ambiguity aversion has a negative impact on the marginal WTP for self-protection. The intuition is that ambiguity aversion induces the agent to put more weight on scenarios θ with a large probability $p_i(\theta)$, since $U(\theta)$ is decreasing with θ when $w_i < \psi_1$. This raises the distorted probability $\widehat{p}_i$ above $\overline{p}_i$. But this has the effect to raise the marginal utility cost $\widehat{p}_i u'(w_i) + (1 - \widehat{p}_i)u'(\psi_2)$ of the ex ante effort, since $w_i < \psi_2$. The marginal utility benefit of self-protection expressed by

$u(\psi_1) - u(w_i)$ being unaffected by ambiguity aversion, we can conclude that ambiguity aversion reduces the marginal WTP of self-protecting states whose wealth level is below the precautionary equivalent wealth. The opposite result prevails when w_i is in between ψ_2 and ψ_1. This is because the increased distorted probability $\widehat{p}_i$ now implies a reduction in the utility cost of effort. These results are summarized in the following proposition.

Proposition 3 *Consider the decision to self-protect state i in which ambiguity is concentrated. Suppose that the agent is DARA, and that the wealth level w_i in state i is less than the certainty equivalent wealth level ψ_1. Ambiguity aversion reduces the marginal WTP to self-protect state i if w_i is smaller than the precautionary equivalent wealth ψ_2, and it raises it if $w_i \in [\psi_2, \psi_1]$.*

4.2 Optimal self-protection

We now consider a specific prevention model in which the effort e affects the probability $p_i(e, \theta)$ of state i in which ambiguity is concentrated. We define expected utility as follows

$$U(e, \theta) = p_i(e, \theta)u(w_i - e) + (1 - p_i(e, \theta)) \sum_{s \neq i} \pi_s u(w_s - e). \tag{13}$$

We assume that the probability of state i is differentiable with respect to e, with $p_{ie}(e, \theta) \leq 0$. We also assume that the second order condition is satisfied.[3] Let e^* denote the optimal effort of self-protection for the ambiguity neutral agent. Suppose without loss of generality that $p_i(e^*, \theta)$ is increasing in θ. Using Lemma 1, we need to determine whether U and U_e are comonotonic or anti-commonotonic to determine the impact the ambiguity aversion on the optimal effort. Redefining the certainty and the precautionary equivalents ψ_1 and ψ_2 as

$$u(\psi_1 - e^*) = \sum_{s \neq i} \pi_s u(w_s - e^*) \quad \text{and} \quad u'(\psi_2 - e^*) = \sum_{s \neq i} \pi_s u'(w_s - e^*),$$

we can rewrite the conditional expected utility as

$$U(e^*, \theta) = p_i(e^*, \theta)u(w_i - e^*) + (1 - p_i(e^*, \theta))u(\psi_1 - e^*), \tag{14}$$

[3] In this model, the condition $U_{ee} \leq 0$ requires some specific restrictions on functions p (Jullien, Salanié and Salanié, 1999).

and its derivative with respect to e as

$$\begin{aligned} U_e(e^*,\theta) &= p_{ie}(e^*,\theta)\left[u(w_i-e^*)-u(\psi_1-e^*)\right] \\ &\quad -\left[p_i(e^*,\theta)u'(w_i-e^*)+(1-p_i(e^*,\theta))u'(\psi_2-e^*)\right]. \end{aligned} \tag{15}$$

As earlier, suppose that w_i is smaller than the certainty equivalent ψ_1. This means that we focus on a state satisfying the natural property that a reduction of its probability raises expected utility. In other words, from (14), $w_i \leq \psi_1$ implies that the conditional expected utility is decreasing in θ. The sensitiveness of U_e to changes in θ is more difficult to evaluate at that degree of generality, as can be seen from equation (15). As earlier in this section, let us now assume that the degree of ambiguity is not affected by the effort. In other words, let us assume at this stage that p_{ie} is independent of θ. This implies that the first term in the right-hand side of equation (15) is independent of θ. What remains in the right-hand side of the equation is decreasing in θ if w_i is smaller than ψ_2. Using Lemma 1 yields the following proposition.

Proposition 4 *Consider the decision to self-protect state i in which ambiguity is concentrated. Suppose that the agent is DARA, and that the wealth level w_i in state i is less than the certainty equivalent wealth level ψ_1. If $p_{ie}(e^*,\theta)$ is independent of θ, ambiguity aversion reduces the optimal self-protection effort if w_i is smaller than the precautionary equivalent wealth ψ_2, and it raises it if w_i is larger than ψ_2.*

The intuition of this result is the same as for Proposition 3. When the wealth level in the state whose probability is reduced by the effort is smaller than the precautionary equivalent wealth, the increase in the distorted probability $\widehat{p}_i$ that is induced by ambiguity aversion raises the marginal utility cost of effort, thereby reducing the optimal effort.

Let us now relax the assumption that the effort level has no impact on the degree of ambiguity. Suppose alternatively that an increase in effort raises the degree of ambiguity, i.e., that $p_{ie}(e^*,\theta)$ is increasing in θ. Possible probabilities of state i become more dispersed when effort is increased. Intuitively, this should reinforce the negative impact of ambiguity aversion on effort. This can be checked by observing that the first term in the right-hand side in equation (15) is decreasing in θ, as is the second term when $w_i \leq \psi_2$.

Proposition 5 *Under the conditions of Proposition 4, if $p_{ie}(e^*, \theta)$ is increasing in θ, then ambiguity aversion reduces the optimal self-protection effort if w_i is smaller than the precautionary equivalent wealth ψ_2.*

This result is related to Snow (2011), who exhibits a case in which ambiguity aversion *raises* the optimal self-protection effort.[4] Using a two-state model, Snow restricts the functional form describing the effect of self-protection on the probability of accident. Using our notation, Snow assumes that, for all (e, θ), $p_i(e, \theta) = p_i(\theta)\rho(e)$ where $\rho(e)$ captures the effect of the self protective effort, with $\rho(e) > 0$ and $\rho'(e) < 0$. This is a special case in which the proportional effect of effort on the loss probability is the same for all θ. Under this condition, Snow shows that the self-protection effort is increased by ambiguity aversion. Because $p_i(e, \theta) = p_i(\theta)\rho(e)$ implies that p_{ie} is decreasing in θ, Snow's result is not inconsistent with our Proposition 5 in which we instead assume that p_{ie} is increasing in θ.[5]

5 Optimal insurance design

In this section, we explore the problem of the optimal insurance contract design. As first proved by Arrow (1971), any insurance contract is dominated by a contract with a straight deductible. As it is intuitive, the deductible contract optimizes the risk transfer to the insurer for any insurance budget level, since it provides indemnities where the marginal utility of wealth is the largest. We are interested here in determining whether this Arrow's result is robust to the introduction of ambiguity aversion.

As a particular case of the specific structure of ambiguity considered in sections 3 and 4, we limit the analysis to the case in which the ambiguity is concentrated in the probability of the no-loss state. We have $w_i = w$ and

[4] Berger (2011) also examines a model of self-protection and self-insurance under ambiguity aversion, but in a two-period model.

[5] Here is a sketch of the proof of Snow (2011)'s result. Let us assume $w_i = w - L$, $w_s = w$ and $p_i(e, \theta) = p_i(\theta)\rho(e)$. We need to show that $U_e(e^*, \theta)$ increases in θ in this specific model. It can easily be shown that this holds if and only if $f(e^*) = \rho'(e^*)(u(w - L - e^*) - u(w - e^*)) - \rho(e^*)(u'(w - L - e^*) - u'(w - e^*))$ is positive. Note that the first term of $f(e^*)$ is positive while the second term is negative, illustrating that the effect is in general indeterminate when p_{ie} is decreasing in θ. Nevertheless, observe that the first order condition of this problem simply reduces to $(E_\theta p(\theta))f(e^*) = u'(w - e^*)$, which implies $f(e^*) > 0$.

$w_s = w - L_s$ where L_s is the loss in state s. Let $p(\theta)$ denote the ambiguous probability that a loss occurs. Conditional to the occurrence of a loss, the level of loss is a random variable L which is independent of θ. That is, the distribution of the severity of the loss is unambiguous. An insurance contract stipulates an indemnity $I(L_s)$ for each possible loss level L_s. To any such indemnity schedule, there is an insurance premium which is proportional to the actuarial value of the policy:

$$P = \overline{p}(1+\tau)EI(L),$$

where as before $\overline{p} = E_\theta p(\theta)$ is the expected probability of accident and τ is the loading factor. Conditional to θ, the policyholder's expected utility is written as

$$U(\theta) = (1-p(\theta))u(w-P) + p(\theta)Eu(w-L+I(L)-P).$$

The ambiguity averse policyholder selects the indemnity schedule $I(L_s)$ that maximizes its ex ante welfare which is measured by $E_\theta\phi[U(\theta)]$. By the concavity of u and ϕ, this maximization is well-behaved, and its first-order conditions are necessary and sufficient. We first prove the following proposition.

Proposition 6 *Suppose that the policyholder is ambiguity averse and that the ambiguity is concentrated on the probability of the no-loss state. Under this specification, the optimal insurance contract contains a straight deductible d:* $I(L_s) = \max(0, L_s - d)$.

Proof: Suppose by contradiction that the contract I_0 that maximizes $E_\theta\phi[U(\theta)]$ is not a straight deductible contract. It yields an insurance premium $P_0 = \overline{p}(1+\tau)EI_0(L)$. Then, let d_0 denote the deductible which yields the same insurance premium than P_0 :

$$\overline{p}(1+\tau)E(\max(0, L-d_0)) = P_0.$$

But we know from Arrow (1971) that this alternative contract with a straight deductible d_0 dominates any other insurance schedule as I_0 :

$$Eu(w - \min(L, d_0) - P_0) \geq Eu(w - L + I_0(L) - P_0).$$

It implies that for all θ, we have

$$\begin{aligned} & (1-p(\theta))u(w-P_0) + p(\theta)Eu(w - \min(L, d_0) - P_0) \\ \geq\ & (1-p(\theta))u(w-P_0) + p(\theta)Eu(w - L + I_0(L) - P_0). \end{aligned}$$

Because the expected utility conditional to θ is larger with the straight deductible d_0 than with contract I_0 for all θ, the former necessary yields a larger ex ante welfare $E_\theta \phi[U(\theta)]$. This is a contradiction.■

We now examine the impact of ambiguity aversion on the optimal deductible. Corollary 1 answered this question in the special case of only one possible loss, since in that case a proportional coinsurance or a straight deductible are formally equivalent. With more than one possible loss level, the decision problem can thus be rewritten as follows:

$$\max_d E_\theta \phi[U(d,\theta)], \tag{16}$$

with

$$U(d,\theta) = (1-p(\theta))u(w-P(d)) + p(\theta)Eu(w-\min(L,d)-P(d))$$

and $P(d) = \overline{p}(1+\tau)E\max(0, L-d)$. We suppose that $\overline{p}(1+\tau)$ is smaller than unity, so that reducing the deductible makes the policyholder wealthier in high loss states. Let d^* denote the optimal deductible under ambiguity neutrality. By Lemma 1, and because U is concave in d,[6] ambiguity aversion reduces the optimal deductible if $U(d^*,\theta)$ and $U_d(d^*,\theta)$ are comonotonic. Suppose that p is increasing in θ, so that $U(d^*,\theta)$ is decreasing in θ. We have

$$\begin{aligned} U_d(d^*,\theta) &= -P'(d^*)\left[(1-p(\theta))u'(w-P^*) + p(\theta)Eu'(w-\min(L,d^*)-P^*)\right] \\ &\quad -p(\theta)(1-F(d^*))u'(w-d^*-P^*) \end{aligned} \tag{17}$$

where and $P^* = P(d^*)$ and F is the cumulative distribution function of L. This equality can be rewritten as follows:

$$\begin{aligned} U_d(d^*,\theta) = \quad & -P'(d^*)(1-p(\theta))u'(w-P^*) + p(\theta)(1-F(d^*)) \\ & \times \left[\overline{p}(1+\tau)Eu'(w-\min(L,d^*)-P^*) - u'(w-d^*-P^*)\right]. \end{aligned} \tag{18}$$

This is obviously decreasing in θ, since $P'(d^*) < 0$, $Eu'(w-\min(L,d^*)-P^*) \leq u'(w-d^*-P^*)$ and $\overline{p}(1+\tau) \leq 1$. This proves the following proposition.

Proposition 7 *Under the specification of Proposition 6, the optimal deductible is reduced by ambiguity aversion.*

[6] See Schlesinger (1981).

The intuition for this result can be obtained from the observation that the effect of ambiguity aversion is here equivalent to an increase in the implicit loss probability defined by

$$\widehat{p} = \frac{E_\theta \phi'[U(d^*, \theta)] p(\theta)}{E_\theta \phi'[U(d^*, \theta)]}.$$

Because $U(d^*, \theta)$ and $p(\theta)$ are negatively correlated and because ϕ' is decreasing, we have that $\widehat{p}$ is larger than $E_\theta p(\theta) = \overline{p}$, under ambiguity aversion.

We assumed in this section that the ambiguity is concentrated on the no-loss state. We finally show that the optimal insurance schedule is in general not a straight deductible for other structures of ambiguity. This is done with a numerical counter-example. Suppose that the agent has an initial wealth $w = 3$ and that there are three possible loss levels $L = (0, 1, 2)$. The ambiguity is structured as follows. There are two equally-likely probability distributions for L with support $(0, 1, 2)$: The good distribution is $p(\theta = 1) = (1/2, 1/2, 0)$, whereas the bad distribution is $p(\theta = 2) = (1/4, 1/4, 1/2)$. Observe that, under our terminology, the ambiguity is here concentrated on the large loss $L = 2$. The loading factor of the insurance premium is $\tau = 0.2$. The policyholder's utility function is $u(c) = -1/c$, and we calibrate ambiguity aversion with $\phi[u] = -e^{-100u}$. Using Mathematica, we solve numerically the following problem:

$$\max_{I_1 \in [0,1],\ I_2 \in [0,2],\ P} \frac{1}{2}\phi\left[\frac{1}{2}u(3-P) + \frac{1}{2}u(2+I_1-P)\right]$$
$$+\frac{1}{2}\phi\left[\frac{1}{4}u(3-P) + \frac{1}{4}u(2+I_1-P) + \frac{1}{2}u(1+I_2-P)\right]$$

subject to

$$P = 1.2\left(\frac{3}{8}I_1 + \frac{1}{4}I_2\right).$$

We obtain $I_1^* = 0.407$ and $I_2^* = 1.630$. Because $I_2^* - I_1^* > L_2 - L_1$, we see that the optimal contract is not a straight deductible, i.e., there is no d such that $I^*(L_s) = \max(0, L_s - d)$ for all L_s. Final wealth is larger in state with $L = 2$ than in state with $L = 1$. This is done to compensate for the relatively large ambiguity associated to the large loss. We conclude that the result expressed in Proposition 6 is not robust to the relaxation of the condition that ambiguity is concentrated on the no-loss state.

6 Conclusion

In this paper, we have shown that ambiguity aversion tends to raise the incentive to insure and to self-insure, but to decrease the incentive to self-protect. The intuition for our results is that ambiguity aversion has an effect which is behaviorally equivalent to the effect of more pessimism under subjective expected utility theory. Since the effect of a change in probability in expected utility models is usually different from that of a change in risk aversion, this explains why the effect of ambiguity aversion differs from that of risk aversion. We have also exhibited a case where the optimal insurance contractual form contains a straight deductible as in the expected utility model.

This paper has thus generalized the analysis of some standard problems in insurance economics to ambiguous risks. Some technical difficulties remain however. In particular, it could be interesting to consider more general ambiguous probability distributions, ambiguity averse insurers and other forms of ambiguity sensitive preferences. At a more conceptual level, we finally observe that if the distinction between (self-)insurance and self-protection has classically been done in the insurance economics literature, the case for maintaining such a distinction may become more problematic under conditions of ambiguity.

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