# From Bottom of the Barrel to Cream of the Crop: Sequential Screening with Positive Selection*

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**Abstract**

In a number of interesting environments, dynamic screening involves positive selection: in contrast with Coasian dynamics, only the most motivated remain over time. The paper provides conditions under which the principal's commitment solution is time consistent and uses this result to derive testable predictions under permanent or transient publicly observable shocks. By contrast, under common agency, in partnerships or when the principal's or the agent's types are shifting, time consistency does not hold despite positive selection, but simple equilibrium characterizations can still be obtained.

*Keywords*: screening, positive selection, time consistency, shifting preferences, exit games.

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## 1 Introduction

The poll tax on non-Muslims that was levied from the Islamic conquest of then-Copt Egypt in 640 through 1856 led to the (irreversible) conversion of poor and least religious Copts to Islam to avoid the tax and to the shrinking of Copts to a better-off minority. To the reader familiar with Coasian dynamics, the fact that most conversions occurred during the first two centuries raises the question of why Muslims did not raise the poll tax over time to reflect the increasing average wealth and religiosity of the remaining Copt population.[1]

This paper studies a new and simple class of dynamic screening games. In the standard intertemporal price discrimination (private values) model that has been the object of a voluminous literature, the monopolist moves *down* the demand curve: most eager customers buy or consume first, resulting in a right-truncated distribution of valuations or "negative selection". Similarly, in the dynamic version of Akerlof's lemons model (common values), the buyer first

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[1] A possible conjecture is that the Muslims wanted to preserve a tax base. However, they might also have wanted immediate income and further were also aiming at maximizing conversions to Islam. See Saleh (2013) for an analysis of the impact of the poll tax on the correlation of religious and socio-economic status over these twelve centuries.

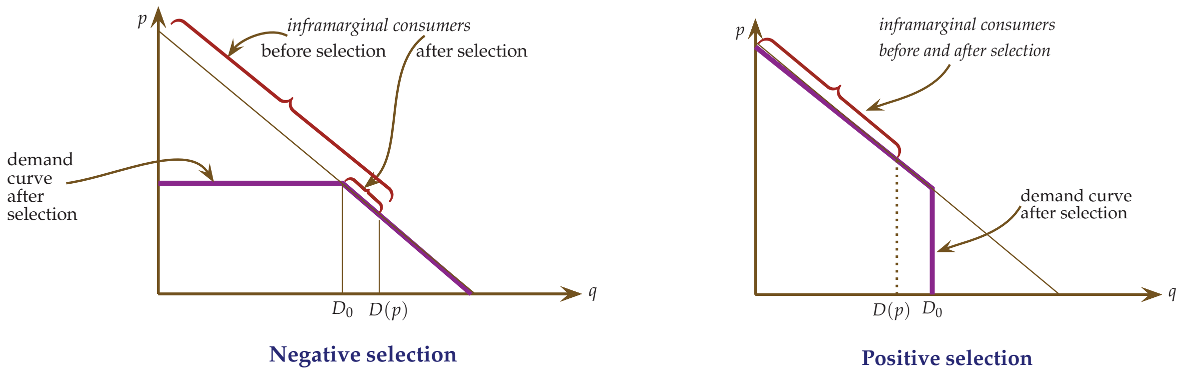


Figure 1

deals with the most-eager-to-trade seller types –the owners of lemons– and thereafter raises price to account for the information that the seller is less eager to trade than expected. Again there is negative selection and the price setter moves down the demand curve.

In the poll tax example by contrast, the monopolist (the state) moves *up* the demand curve: Copts who most value their religion or are richer and therefore less sensitive to the pool tax remain in the tax base, while converts and their descendants, under the threat of apostasy forever disappear from that tax base. As we will later note, a number of interesting economic contexts share with the conversion game "positive selection" and, at least approximately, "absorbing exit".

One might conjecture that the distinction between positive and negative selection is just a matter of sign convention, but this is not the case. To start building some intuition about why this is so, consider a demand curve $D(p) = 1 - F(p)$ obtained by aggregating demands from individual consumers with willingnesses to pay $\theta$ distributed according to some cumulative distribution $F(\theta)$. Suppose that the distribution for some reason has been truncated at $\theta_0$ and let $D_0 = 1 - F(\theta_0)$; let $\eta_R$ and $\eta_L$ denote the elasticities of the right- and left-truncated residual demands. Then $\eta_R = [-D'(p)p]/[D(p) - D_0]$ (for $D(p) > D_0$) and $\eta_L = [-D'(p)p]/[D(p)]$ (for $D(p) < D_0$). That is, right truncations increase the elasticity of demand, while left ones leave it unchanged. This difference is illustrated in Figure 1, which emphasizes the reduction (invariance) of the set of inframarginal consumers under negative (positive) selection.

This observation has a number of implications.[2] The most obvious is that the monopoly price is invariant to (moderate) left-, but not to right-truncations of the distribution of willingnesses

[2] For instance, auctions of incentive contracts, which amount to a left truncation of the winner's efficiency's probability distribution, deliver for the winner the same power of incentive scheme as if the winner were a monopolist; optimal auctions thus only reduce the fixed component of rewards (Laffont and Tirole 1987). Another application of this property of the hazard rate is Niedermayer and Shneyerov (2014), in which a platform matches buyers and sellers. The mechanism that maximizes platform profit can as usual be derived by maximizing total virtual surplus. An interesting result in that paper is that this platform monopoly profit is also attainable in a decentralized manner, where the platform charges membership fees to the participants and then buyers and sellers (independently of the platform) make take-it-or-leave-it offers to each other (random proposer model). Buyers with low values and sellers with high values do not join the platform because of the membership fees. For appropriate fees, the membership is the same as under centralization and decentralized trade is efficient: all buyers and sellers who pay the membership fees trade.

to pay, a property whose implications for dynamic screening we will investigate. Indeed, this paper focuses on the properties and time consistency of monopoly pricing under left truncations. The dynamics first studied by Coase (1972) have the monopolist move down the demand curve once the cream has been skimmed off and the remaining market is the bottom of the barrel; the intuition is provided by the increasing elasticity of demand under right truncations. The monopolist's incentive to reduce price over time has been shown to result in the time inconsistency of optimal monopoly pricing and therefore in an erosion of monopoly power. By contrast, the invariance of the elasticity under left truncations suggests that the monopolist will not be tempted to move up the demand curve and so optimal market segmentation is time-consistent.

Consider the following example: agents – the consumers – have unit demand in each of two periods, $t = 0, 1$, and willingnesses to pay (relative to the outside opportunity) $\theta$ uniformly distributed on $[0, 1]$; consumers can consume at date 1 only if they have consumed at date 0. The principal – the monopoly seller – produces costlessly. If the principal can commit at date 0 to a sequence of prices, the optimal price sequence is $p_0 = p_1 = 1/2 = \arg\max \{p(1 - p)\}$. Because the consumers are the same at the two dates, the absorbing-exit constraint is not binding. Suppose now that the principal lacks commitment ability. Let the principal charge $p_0 = 1/2$ at date 0, and suppose that consumers expect that the date-1 price will not be any lower. Consumers with type $\theta < 1/2$ then do not want to consume at date 0 and exit at that date. Consumers with type $\theta \geq 1/2$ benefit from consuming at $t = 0$ regardless of their expectation of $p_1$ and therefore consume at date 0. Consider then date 1. The monopolist faces uniform posterior distribution $2[\theta - 1/2]$ on $[1/2, 1]$ on its remaining goodwill and so picks price $p_1 = \arg\max \{p(1 - (2p - 1)\} = 1/2$, vindicating consumers' expectations. So the monopolist's ability to implement monopoly pricing is not hampered by the lack of commitment.

This paper investigates several questions raised by this toy example. First, how general is the "weak time consistency" result that there exists *one* equilibrium that delivers the commitment payoff for the principal? Second, do *all* equilibria under non-commitment deliver the commitment payoffs? That is, are the commitment payoffs "strongly time consistent"? Third, can one characterize exit dynamics when the environment is uncertain and evolves over time (in the example above, the seller's cost and the consumers' willingnesses to pay might follow stochastic processes), when the realization of the shocks is either publicly or privately observed? Fourth, does the analysis extend to inflows of new agents and to finite re-entry costs? Finally, how are exit dynamics affected by the multiplicity of principals (common agency) or agents (Myerson-Satterthwaite teams)?

Section 2 describes the baseline model. Ignoring the transfers between the two parties, the agent's flow payoff from the relationship (net of her outside option payoff) is increasing in a privately known type $\theta$ and may depend on peers' decisions as well as on the current state of nature $s_t$. Again ignoring transfers, the principal's flow payoff depends on the state of nature and the composition of the remaining installed base; the model thus accommodates private and common values. At date $t$, provided that the agent has remained in the installed base (the parties are still in a relationship), the principal offers a (positive or negative) price $p_t$ which the agent accepts or refuses, in which case the game is over.

The paper's first contribution is to show that provided that types (although not necessarily willingnesses to pay) are permanent, the optimal outcome is indeed time-consistent, and so no commitment ability is required to implement it (Section 3). More precisely, we first derive weak time consistency. We then prove strong time consistency under the assumption of no strongly-positive network externalities, together with, for the case of an infinite horizon, the added requirement that either the equilibrium be Markov perfect or that the principal weakly benefit from a greater clientele at the optimal, non-frontloaded price (we later derive a sufficient condition for this property to obtain).

While the optimal *outcome* (payoffs for the principal and the agents, agents' exit pattern) is time consistent, not every optimal *policy* is; for, there is in general a continuum of contingent price sequences that implement the commitment outcome. Prices can be arbitrarily frontloaded, with the impact of high initial prices being offset by the promise of low prices in the future; however, frontloaded policies in general are time inconsistent as the principal would want to renege on this promise.

Time consistency transforms the search for a perfect Bayesian equilibrium of the no-commitment game into a simple dynamic stochastic optimization problem. We use this fact in Section 4 to compute the equilibrium in simple cases. One prominent case has an unambiguous aggregate evolution: the consumption becomes either more or less attractive over time; alternatively, the principal over time becomes more or less eager to retain agents. In the borderline case of an invariant environment, equilibrium leads to all exits ("conversions" in the religious example) taking place early on. More generally, when consumption deterministically becomes more attractive, all exit occurs at the initial date. By contrast, when it becomes less attractive over time (perhaps in a stochastic fashion), exit is spread over time and both the principal and the agent in equilibrium behave myopically, i.e., as if this were the last period (Section 4.1).

Section 4.2 extends the model to finite re-entry costs in the context of monotone attractiveness and provides a lower bound on re-entry costs (equal to 0 in the case of decreasing/constant attractiveness) for re-entry to be irrelevant and therefore for the results obtained previously to apply.

The robustness of the insights to inflows of new agents is analyzed also for monotone-attractiveness environments in Section 4.3. Following the literature on negative selection, we look at whether the absence of price discrimination among identical cohorts impacts the outcome under commitment and non-commitment. Under negative selection, uniform pricing has the potential to restore some of the monopoly power that is eroded by the temptation to lower the price as the principal moves down the demand curve. One may wonder whether, conversely, the combination of the inflow of new cohorts and of uniform pricing might undermine the time consistency of optimal policies and thereby destroy principal value under positive selection.

For the class of monotone-attractiveness games, we obtain two interesting results. First, under commitment, uniform pricing does as well as discriminatory pricing. Second, time consistency obtains for decreasing/constant attractiveness, but not for strictly increasing attractiveness. In the case of decreasing/constant attractiveness, these results stem from the observation that the principal's optimal behavior under price discrimination is myopic and so identical across cohorts; therefore there is no cost for the principal of not being able to make use of cohort information. Neither is there any time consistency issue.

Under strictly increasing attractiveness, the optimal policy under price discrimination is described by invariant, but cohort-specific cut-offs; all exit within a cohort occurs in the cohort's first period of existence, but new cohorts have a lower cut-off, i.e., higher membership, than older ones. We show that under uniform pricing, a specific pattern of frontloaded payments is both necessary and sufficient to achieve the price-discrimination/cohort-specific cut-offs. This frontloading is what makes the optimal outcome time-inconsistent under uniform pricing (while it is time consistent under discriminatory pricing).

When the economy is subject to transient aggregate shocks (Section 4.4), which in a sense is the polar case of the permanent shocks studied in Sections 4.1 through 4.3, shocks have long-lasting effects, the exit volume decreases over time and this volume is serially negatively correlated. At each date $t$, the participation depends only on the worst shock so far and is given by a simple condition.

Sections 5 and 6 by contrast study environments in which time consistency does not hold. Solving for equilibria of principal-agent relationships in the absence of commitment is notoriously difficult as it no longer boils down to solving an optimization problem. Interestingly, though, the simple structure of games with positive selection allows us to provide equilibrium characterizations.

Section 5 looks at the possibility that either the agent's or the principal's type, and not only the publicly observable environment, changes over time. It is well-known that optimal policies are not time consistent in such environments. With a shifting principal type, both the principal and the agent must factor in the possibility that future principals be more or less eager to pursue the relationship than the current one. We prove two simple propositions. First, if the environment is invariant (except for the principal's type), the date-$t$ remaining installed base is determined by the optimal level for the least eager principal so far. Second, whether attractiveness is monotonically increasing or decreasing, the installed base at any date $t$ is smaller than what the current principal would induce if he and the agent anticipated that the principal's type would no longer change. We then allow the agent's type to shift over time. When the agent's type moves in an iid fashion, the equilibrium hazard rate for the termination of the relationship is constant and there is more exit than in the commitment solution.

Section 6 studies the cases of multiple principals and multiple agents. Under common agency, the agent's exit pattern is jointly determined by the principals. The analysis thus is a dynamic extension of the standard commons (or moral-hazard-in-team) problem. Despite positive selection, time consistency is invalidated by the principals' desire to influence each other's future policies. Nonetheless, equilibrium in the dynamic retention game can be characterized and shown to have the same qualitative properties as in the case of coordinated principals (i.e., the single-principal case).

Section 6 finally provides a preliminary analysis of Myerson and Satterthwaite (1983) partnerships, in which exit by one member implies the dissolution of the partnership. It demonstrates that despite positive selection, the time consistency property does not extend to partnerships whenever agents are informed at the end of the period only of the continuation of the relationship.

Section 7 concludes with suggestions for future research.

**Economic environments with positive selection**

Left truncations are closely related to the economics of incumbency; that is, they arise whenever an authority, firm or technology has an installed base of "customers" that forms a potential tax base, but may irremediably exit:

*Emigration*: In an illustration closely related to the conversion game, suppose that immigration is an irrevocable decision (or at least one that is costly to reverse). Then an economic,[3] ethnic or religious group may see some of its members leave as the government levies more taxes or enacts adverse "non-price" policies toward the group. Over time, only the most attached to the country or the least mobile members of the group remain in the country.

*Employee retention*: A firm or an academic department at any given point in time comprises the subset of legacy employees who are the most committed or immobile. Organization-beneficial policies asking for public service, personal sacrifices or wage moderation create a risk for the organization of losing valuable employees.[4]

*Technology disadoption/licensing*: A firm or group of intellectual property owners licensing key patents enabling the implementation of an incumbent technology may be concerned that users might defect for a new technological alternative. Again, the loyal tax base is composed of those with the highest benefits from the incumbent technology or highest switching costs.

*Entry of generics*: In a closely related example that has been the focus of much empirical literature, generic drugs that enter when a brand-name drug goes off-patent are typically far less expansive (by a factor of 5 or 10) than the brand-name version. Interestingly, the US retail prices of brand-name drugs tend to increase just prior to patent expiration and continue to increase (a bit less) post-patent expiration. This phenomenon, dubbed "price rigidity", is consistent with the idea of positive selection. The brand-name drug manufacturer is left with the most loyal consumers and has little incentive to lower the price despite a declining market.[5]

*Dyad game*: Two parties are involved in some relationship. One of the parties is uncertain about the other party (friend, spouse, co-worker)'s commitment to the relationship and therefore does not know how much effort is needed to keep the relationship going. Assuming that the dyad

[3]The economic group here refers to a well-to-do group. Emigration of course may rather concern the lower end of the income bracket.

[4]Following Burdett and Mortensen (1998)'s work on the impact of on-the-job search on labor markets, a literature has developed that studies firms' retention policies under the threat of incoming outside opportunities. This literature in particular looks at how a steeply rising wage contract can approximate optimal "sell-out contracts" that are enabled by entry (or quitting) fees (Burdett and Coles 2003, Stevens 2004). The focus is rather different from that in the paper since there is no asymmetric information at the contracting date and thus no dynamic screening issue; furthermore, commitment is assumed, while the present paper studies the time consistency of optimal commitment policies.

[5]In this example, "re-entry" (going back to the brand-name drug after switching to the generic) is relatively costless. However, as will be shown in Section 4.2, re-entry costs need not be large (and actually can be nil in the case of constant/decreasing attractiveness, a reasonable assumption for this application) for the results to hold.

once dissolved does not re-form, the dyad game is an illustration of our framework.[6]

**Related literature**

Following up on early work on dynamic "Myersonian" mechanism design (e.g., Baron and Besanko 1984, Courty and Li 2000), remarkable progress has recently been made to characterize optimal mechanisms under commitment. The literature has derived generalizations of the envelope characterization of the first-order condition and conditions under which attention can be focused on single, rather than compound deviations (Esö and Szentes 2007, Boleslavsky and Said 2013, and Pavan et al 2014). It has obtained necessary and sufficient conditions for the attainment of the optimal allocation either asymptotically (Battaglini 2005) or overall (Bergemann and Välimäki 2010, Athey and Segal 2013, Skrzypacz and Toikka 2013). Kruse-Strack (2014) studies the optimal commitment policy when the agent must choose a stopping time, the analog of our absorbing exit condition. The agent's type evolves over time (making the optimum time inconsistent). Under a dynamic single crossing assumption, the paper provides an elegant characterization of optimal stopping rules and their implementation.

The work cited in the previous paragraph presumes "double commitment": commitment to a long-term mechanism if the initial offer is accepted, and commitment by the principal not to make further offers if the initial offer is rejected. However, commitment is often not to be taken for granted. Public policies generally lack commitment, and so do a number of policies in private-sector environments, especially when future policies are hard to describe, let alone contract upon, in advance. More precisely, long-term commitment may be infeasible ("no commitment"); or it may be feasible but renegotiable (i.e., it is renegotiated if the parties involved in the long-term contract all find it advantageous to do so; this is the paradigm of "commitment and renegotiation" studied e.g., in Dewatripont 1989 or Laffont and Tirole 1990); finally, further offers cannot be precluded, as in Coase's durable good model.

Much less is known for environments in which either form of commitment is violated. In specific environments (usually two permanent types), both the commitment-and-renegotiation model and the no-commitment model (without agent anonymity) have been shown to exhibit Coasian dynamics (Hart and Tirole 1988, Maestri 2013, Strulovici 2013).[7] Similar Coasian dynamics have been obtained in the common value counterpart of Coase's model, the dynamic version of Akerlof's lemons model (e.g. Daley and Green (2012), Fuchs and Skrzypacz (2013), Gerardi and Maestri (2013), Gerardi et al (2014)).

Little is known also regarding general properties of sequential screening in such environments. Bester and Strausz (2001) demonstrate that the cardinality of messages can be confined to that of types for a finite type space. Skreta (2006) looks at the standard risk-neutral seller-buyer game, in which, say, the buyer's invariant valuation $\theta$ is drawn from some distribution with support the interval $[0, 1]$, and the seller's cost is 0. As long as the buyer is not served, the seller keeps

[6]For this to be the case, it is important that the informational asymmetry be one-sided. The case of two-sided asymmetric information will be studied in Section 6.2.

[7]Dynamic screening with left truncations shares features with both the "rental game" (in which the price is set each period for that specific period) and the "sale game" (except that, as already noted, individuals with low willingnesses to pay are progressively removed out of the market, while durable-goods buyers with high willingness to pay are removed from the market in Coase's model).

making offers at $t = 0, \cdots, T$. These offers need not be prices $p_t$. Rather they can be full-fledged mechanisms, resulting in date-$t$ probabilities of trade and expected transfers. The central result is that an optimal mechanism is to simply post a price in each period, generalizing the Riley and Zeckhauser (1983) classic result to sequential mechanism design and thereby simplifying the search for equilibria in this class of games.

The literature so far has been concerned with a principal selling goods in a market. The principal's objective is then to attract consumers without lowering price too much. By contrast, we consider a principal who somehow has already established a customer base and who is trying to retain it while incurring a low cost or charging a high price. In this sense, this model is a mirror image of existing screening models, with the less motivated jumping off ship instead of the most motivating getting on board.

Board and Pycia (2014) study pricing by a durable-good monopolist when the consumers can enjoy an alternative, outside option with positive value. In this framework it is easy to show that the monopoly price (the price $p^m$ that maximizes $[\theta - c][1 - F(\theta)]$, where $F(\theta)$ is the distribution of willingnesses to pay and $c$ is the marginal cost) is time consistent: the monopolist charges at date 0 the monopoly price, those consumers with $\theta \geq p^m$ purchase immediately and the others opt for the outside option. The principal's (out of equilibrium) beliefs from date 1 on are that remaining consumers have types above $p^m$. Board and Pycia's striking result is that this is the only equilibrium.[8] Our framework differs from theirs in two important and related aspects. First, Board and Pycia's principal can commit to a long-term contract as is implicit in the durable good framework (selling can be viewed as a commitment to long-term rental). Second, their model exhibits both left- and right-truncations (the more motivated buy the durable good and the least motivated buy the outside option), whereas the model studied in this paper has only left truncations and Coase's traditional model has only right truncations.

## 2 Model

Time is discrete: $t = 0, 1, \cdots, T$, where $T$ is finite or infinite. At the beginning of each period, a state of nature $s_t$ is realized in some set $S_t$. Let $s^t \equiv (s_0, \cdots, s_t) \in S^t \equiv S_0 \times \cdots \times S_t$ follow a stochastic process with conditional distribution $G(s^\tau | s^t)$ for $\tau > t$. We will say that $s^\tau \succ s^t$ for $\tau > t$ if there exists $(s_{t+1}, \cdots, s_\tau)$ such that $s^\tau = (s^t, s_{t+1}, \cdots, s_\tau)$.

The players are a price-setting principal and an agent (or continuum of agents with mass 1) with a unit demand in each period. All have identical discount factor $\delta \in (0, 1)$.

The agent is characterized by a privately-known type $\theta \in [\underline{\theta}, \bar{\theta}]$, distributed according to smooth c.d.f. $F(\theta)$ with density $f(\theta)$. Each period $t$, the agent consumes ($x_t = 1$) or does not consume ($x_t = 0$); her consumption decision however is relevant only if she has kept consuming in the past. Let $X^t \equiv \Pi_{z=0}^{z=t} x_z$.

*Information, timing and strategies.* At the beginning of each period $t$, the state $s_t$ is realized and

[8] In Fudenberg et al (1987), the possibility for the seller to consume the good himself or to switch to bargaining with another buyer can restore commitment power but there are multiple equilibria. In Board-Pycia, the outside option is on the buyer side, commitment is fully restored and the equilibrium is unique.

publicly observed. In (the no-commitment version of) the game, the principal sets a price $p_t$ for date-$t$ membership/consumption, and previously loyal consumers (those for whom $X^{t-1} = 1$) decide whether to consume. Strategies $\{\sigma^P_\cdot, \sigma^A_\cdot\}$ are price choices, $p_t = \sigma^P_t(p^{t-1}, s^t) \in \mathbb{R}$ for the principal and consumption choices $x_t = \sigma^A_t(p^t, s^t, \theta) \in \{0, 1\}$ for the agent, where $p^t \equiv (p_0, \cdots, p_t)$. We focus on pure strategies and the equilibrium concept is perfect Bayesian equilibrium.

*Agents' preferences.* Relative to the payoff obtained by not consuming, the agent's net surplus from date-$t$ consumption is linear in the date-$t$ transfer $p_t \in \mathbb{R}$ to the principal (a price, or more generally the conditions demanded by the principal for belonging to the consuming group[9]); his gross surplus from consumption depends on his type $\theta$, on the date-$t$ payoff relevant state $s_t$ and on the set $\Theta_t \subseteq [\underline{\theta}, \overline{\theta}]$ of types who consume at date $t$.

The dependence of preferences on $\Theta_t$ allows for social image/self views and (in the case of a continuum of agents) network externalities to affect consumption decisions. For example, the agent's utility may depend, positively or negatively, on the mass $\mu(\Theta_t)$ of agents in the consumption group; more generally, network externalities may also depend on the identity of members of that group. Allowing for externalities adds an argument in the surplus function, but given their relevance in a number of applications, it is worthwhile to show that the results hold when their are present.

*Skimming property.* We assume that higher types have a strictly higher gross surplus for all $(\Theta_t, s_t)$. It is then straightforward[10] to show that, conditionally on having consumed up to date $t$, if type $\theta$ consumes at date $t$ for history $(p^t, s^t)$ (i.e., $x_t(p^t, s^t, \theta) = 1$ ), then so does type $\theta' > \theta$ (i.e., $x_t(p^t, s^t, \theta') = 1$). Intuitively, this results from the fact that type $\theta'$ obtains a strictly higher utility from consumption at date $t$ and that the agent's continuation valuation at date $(t+1)$ is weakly increasing in type (as type $\theta'$ can always mimic type $\theta$'s behavior from date $(t+1)$ on). Thus incentive compatibility implies the existence of a unique cut-off $\theta^*_t(p^t, s^t)$ such that $\Theta_t = [\theta^*_t(p^t, s^t), \overline{\theta}]$ and $\mu(\Theta_t) = 1 - F\left(\theta^*_t(p^t, s^t)\right)$. Absorbing exit then implies cutoff monotonicity: $\theta^*_t(p^t, s^t) \geq \theta^*_{t-1}(p^{t-1}, s^{t-1})$.

We can therefore write the agent's payoff function as a function of the cutoff:

$$\phi(\theta, \theta^*_t, s_t) - p_t.$$

$\phi$ is assumed to be strictly increasing in its first argument and differentiable in its first two arguments. The intertemporal utility of a type-$\theta$ agent is

$$E_{s^T}\left[\Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t)\left[\phi(\theta, \theta^*_t, s_t) - p_t\right]\right].$$

[9]The quasi-linearity of preferences is assumed solely for expositional simplicity. Similarly, transfers more generally can involve deadweight losses. The key assumption is positive selection.

[10]The proof follows the standard lines (see, e.g., Fudenberg et al 1985).

*Principal's preferences.* The principal also has quasi-linear preferences, with flow payoff

$$\int_{\theta_t^*}^{\overline{\theta}} \psi\big(\theta, \theta_t^*, s_t\big) f(\theta) d\theta + p_t\big[1 - F(\theta_t^*)\big],$$

and intertemporal utility

$$E_{s^T}\left[\Sigma_{t=0}^{t=T} \delta^\tau \Big[\int_{\theta_t^*}^{\overline{\theta}} \psi\big(\theta, \theta_t^*, s_t\big) f(\theta) d\theta + p_t\big[1 - F(\theta_t^*)\big]\Big]\right].$$

The principal's objective function deserves some comment as well. Often, the economic model defines the $\psi$ function directly. For example, $\psi\big(\theta, \theta_t^*, s_t\big)$ could stand for the marginal productivity of worker $\theta$ enjoying (positive or negative) production externalities depending on the set of coworkers $\big[\theta_t^*, \overline{\theta}\big]$ in state $s_t$. Our formalism thus allows for common values: the principal may care about the agent's type. For example, loyal employees may be loyal because they are enthusiastic about their job and then are highly productive; or they may stay because they are unable to find another job and then are likely to have a low productivity for the firm. Similarly, the principal in the dyad game may exhibit reciprocal altruism and then experience a welfare that depends on (his perception of) $\theta$. Both cases are illustrations of common values.

Sometimes, the economic model gives instead the principal's gross surplus directly; one must then define the $\psi$ function accordingly. For example, the cost function of serving a number $1 - F(\theta_t^*)$ of agents might be $C\big(1 - F(\theta_t^*), s_t\big)$. One can then define $\psi\big(\theta, \theta_t^*, s_t\big) = -C_1\big(1 - F(\theta), s_t\big)$ for all $\theta$, where $C_1$ is the derivative with respect to the first argument, and so $\int_{\theta_t^*}^{\overline{\theta}} \psi\big(\theta, \theta_t^*, s_t\big) f(\theta) d\theta \equiv C(0, s_t) - C\big(1 - F(\theta_t^*), s_t\big)$. This latter example illustrates the possibility of "network externalities" arising on the principal/cost side. The notion of "externality augmented virtual surplus" introduced below is therefore relevant under non-constant returns even when there are no direct externalities among agents.

*Examples.* The model embodies the premises of dynamic screening with left truncations: the absorbing-exit condition and non-commitment. Let us provide a few examples.

In the basic *conversion game*, $\phi(\theta, \theta_t^*, s_t) = \theta$ and $\psi(\theta, \theta_t^*, s_t) = -c$. The agent has preferences $X^t[\theta - p_t]$ where $\theta$ is the ratio of the agent's religiosity over her marginal utility of income; needless to say, we could enrich this basic set up with in- and out-religious group externalities. The principal is the Muslim rulers, with overall date-$t$ instantaneous payoff $c[1 - \mu(\Theta_t)] + p_t\mu(\Theta_t)$; the parameter $c$ reflects the rulers' intrinsic preference for conversion to Islam, leading to a "markup" $p_t - c$ on poll-tax-paying Copts.

The same payoffs can be used to describe the *dyad game*, where $p_t$ represents (minus) the effort exerted to keep the uncommitted party on board.

With only very slight modifications, the model also accommodates persecutions such as those brought about by the inquisition (against the Albigensian heresy in the 13th century by the Dominicans on behalf of the Pope or against Spanish Jews and Moslems in the late 15th century Spain by Queen Isabella and the Tribunal of the Holy Office of the Inquisition). The screening instrument employed is then purely wasteful, except perhaps for the confiscations, but the results in this paper do not rely on the tax being a pure transfer between the principal and the agent.[11]

In the *technology-disadoption game*, $\phi(\theta, \theta_t^*, s_t) = \theta + \alpha[1 - F(\theta_t^*)] + s_t$ and $\psi(\theta, \theta_t^*, s_t) = -c$, where $\alpha$ is a network-externality coefficient and $s_t$ might stand for shifts in the attractiveness of the challenging technology. The agent has flow preferences $X^t[\theta + \alpha\mu(\Theta_t) + s_t - p_t]$. The principal's flow profit is the $(p_t - c)\mu(\Theta_t)$.

# 3 Time consistency

## 3.1 Commitment

Suppose, first, that the principal can commit to an incentive compatible mechanism that specifies for each $\theta$ a (present-discounted) payment $P(\theta)$ and a state-contingent consumption pattern $\{x_t(\theta, s^t)\}_{\substack{t \in \{0,\cdots,T\} \\ s^t \in S^t}}$ (such that $x_t(\theta, s^t) = 0 \Rightarrow x_{t+1}(\theta, s^{t+1}) = 0$ if $s^{t+1} \succ s^t$). Letting

$$U(\theta) \equiv \max_{\{\widehat{\theta} \in [\underline{\theta}, \bar{\theta}]\}} \left\{ E_{s^T} \left[ \Sigma_{t=0}^{t=T} \delta^t X^t(\widehat{\theta}, s^t) \phi(\theta, \theta_t^*, s_t) \right] - P(\widehat{\theta}) \right\},$$

the participation and incentive constraints require that

$$U(\underline{\theta}) \geq 0$$

and

$$\frac{dU}{d\theta} = E_{s^T} \left[ \Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \frac{\partial \phi}{\partial \theta}(\theta, \theta_t^*, s_t) \right].$$

Consider an optimal policy under commitment. Let $U(\theta) \geq 0$ denote the ex-ante rent of type $\theta$, and $V$ denote the principal's ex-ante payoff. Using the standard decomposition between efficiency and rent, the principal's payoff can be written as:

$$V^* = E_\theta E_{s^T} \left[ \Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \left[ \phi(\theta, \theta_t^*(s^t), s_t) + \psi(\theta, \theta_t^*(s^t), s_t) \right] - U(\theta) \right],$$

[11] Let the Catholic rulers' objective function at date $t$ be

$$-c\big[1 - F(\theta_t^*)\big] - K(i_t)$$

where $c$ is their disutility of non-conversion, $i_t$ is the intensity of inquisition (the probability of detecting non-converts) and $K$ is an increasing and convex cost function. Let the utility of a non-convert with religiosity $\theta$ be $\theta - i_t d$ where $d$ is the relative cost of being caught and tried. The remaining installed based at date $t$ is

$$Y^{t-1}\big[1 - F(\theta_{t-1}^*)\big]$$

where $Y^{t-1} = (1 - i_0) \cdots (1 - i_{t-1})$. Thus the elasticity at any $\theta$ remains the same under left truncations, and the analysis carries over. In particular, varying the rulers' religiosity ($c$) over time or the impact of the environment (for example, through $K$), one can as in Sections 4.1 and 5.1 derive dynamics under endogenous changes in inquisition intensity.

where (using $U(\underline{\theta}) = 0$)

$$E_\theta[U(\theta)] = \int_{\underline{\theta}}^{\overline{\theta}} U(\theta) dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \frac{dU(\theta)}{d\theta}[1 - F(\theta)]d\theta$$

$$= E_{s^T} \int_{\underline{\theta}}^{\overline{\theta}} \left[\Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \frac{\partial \phi}{\partial \theta}(\theta, \theta_t^*(s^t), s_t)\right][1 - F(\theta)]d\theta.$$

And so the principal's payoff can be rewritten in the standard, expected virtual surplus fashion:

$$V^* = \int_{\underline{\theta}}^{\overline{\theta}} E_{s^T} \Bigg[\Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \Big[ \left[\phi(\theta, \theta_t^*(s^t), s_t) + \psi(\theta, \theta_t^*(s^t), s_t)\right] f(\theta)$$

$$- \frac{\partial \phi}{\partial \theta}(\theta, \theta_t^*(s^t), s_t)[1 - F(\theta)]\Big] d\theta\Bigg]$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} E_{s^T} \left[\Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \Gamma(\theta, \theta_t^*(s^t), s_t)\right] f(\theta) d\theta$$

where

$$\Gamma(\theta, \theta^*, s) \equiv \phi(\theta, \theta^*, s) + \psi(\theta, \theta^*, s) - \frac{\partial \phi}{\partial \theta}(\theta, \theta^*, s) \frac{1 - F(\theta)}{f(\theta)}$$

denotes the virtual surplus.

The optimization must respect the feasibility constraints $(F)$[12]: For all $(t, s^t)$,

$$X^t(\theta, s^t) = 1 \iff \theta \geq \theta_t^*(s^t),$$

or, equivalently, *cutoff monotonicity*:

$$\theta_t^*(s^t) \geq \theta_{t-1}^*(s^{t-1}) \quad \text{if } s^t \succ s^{t-1}. \tag{F}$$

As usual, the policy must be optimal for any subform; that is, for all $(t, s^t)$, $\{X^\tau(\cdot, \cdot), \theta_\tau^*(\cdot)\}_{\tau \geq t}$ must also maximize :

$$V_t(s^t) = \int_{\theta_{t-1}^*(s^{t-1})}^{\overline{\theta}} E_{s^T} \left[\Sigma_{\tau=t}^{\tau=T} \delta^{\tau-t} X^\tau(\theta, s^\tau) \Gamma(\theta, \theta_\tau^*(s^\tau), s_\tau)\right] \left[\frac{f(\theta)}{1 - F(\theta_{t-1}^*(s^{t-1}))}\right] d\theta$$

subject to the relevant set of feasibility constraints for all $(\tau, s^\tau)$ such that $\tau \geq t$ and $s^\tau \succsim s^t$.

Next, we show that there is no loss of generality in considering commitments to a sequence of state-contingent prices that the agent accepts or turns down. That is, it does not matter

[12] Because the cutoffs are weakly increasing, this condition need only be checked for the last cutoff. Note furthermore that the condition "if $s^t \succ s^{t-1}$" in condition (F) can be dispensed with (it does not matter what the cutoff is if $s^t$ is unfeasible given $s^{t-1}$).

whether the agent is able to commit or not. To show this, consider a contingent price sequence

$$\boldsymbol{p} \equiv \{p_t(s^t)\}_{\substack{t \in \{0,\cdots,T\} \\ s^t \in S^t}}$$

so as to implement a sequence of contingent, weakly increasing cutoffs $\boldsymbol{\theta}^* \equiv \{\theta_t^*(s^t)\}_{\substack{t \in \{0,\cdots,T\} \\ s^t \in S^t}}$.
Cutoffs must satisfy sequential incentive compatibility. Introducing the agent's value function:

$$U_t(\theta, s^t; \boldsymbol{p}, \boldsymbol{\theta}^*) \equiv \max\Big\{0, \phi(\theta, \theta_t^*(s^t), s_t) - p_t(s^t) + \delta E\big[U_{t+1}(\theta, s^{t+1}; \boldsymbol{p}, \boldsymbol{\theta}^*)\big]\Big\},$$

then

$$x_t(s^t, \theta; \boldsymbol{p}, \boldsymbol{\theta}^*) = 1 \text{ if and only if } \phi(\theta, \theta_t^*(s^t), s_t) - p_t(s^t) + \delta E\big[U_{t+1}(\theta, s^{t+1}; \boldsymbol{p}, \boldsymbol{\theta}^*)\big] \geq 0. \quad \text{(IC)}$$

The principal's maximization program[13] is

$$V^* \equiv \max_{\{\boldsymbol{p}, \boldsymbol{\theta}^* \text{ satisfying } (IC)\}} E_{s^T}\Big[\Sigma_{t=0}^{t=T}\delta^t\big[\int_{\theta_t^*(s^t)}^{\bar{\theta}} \psi(\theta, \theta_t^*(s^t), s_t) f(\theta) d\theta + p_t[1 - F(\theta_t^*)]\big]\Big].$$

Note that (for the sake of the definition of $V^*$ only), we let the principal maximize not only over prices, but also over cutoffs. Indeed, there is no guarantee that a price strategy $\boldsymbol{p}$ leads to a unique sequence of cutoffs $\boldsymbol{\theta}^*$. Our allowing for network externalities, implies a possible multiplicity of static equilibria if externalities are positive and strong; under Assumption 1 below, though, the principal can guarantee himself $V^*$ by choosing $\boldsymbol{p}$ only.

**Lemma 1 (irrelevance of agent commitment).** *Consider an optimal commitment allocation $\{P(\cdot), x_{\cdot}(\cdot,\cdot)\}$ with associated cutoffs $\theta_{\cdot}^*(\cdot)$. Then there exists a sequence of short-term prices $\boldsymbol{p} \equiv \{p_t(s^t)\}_{\substack{t \in \{0,\cdots,T\} \\ s^t \in S^t}}$ such that, if the principal commits to the sequence of state-contingent prices $\boldsymbol{p}$, the outcome delivers the same payoff for the principal and (for all $\theta$) the agent, and satisfies the "cutoff myopia" property:*

$$\forall s^t: \quad p_t(s^t) = \phi(\theta_t^*(s^t), \theta_t^*(s^t), s_t).$$

*Thus, there is no need for commitment by the agent.*

[13] We will assume but not investigate the existence of an optimal policy $(\boldsymbol{p}, \boldsymbol{\theta}^*)$. As will be shown in the proof of Proposition 1 below, the optimization boils down to a search for a plan specifying state-contingent cutoffs $\{\theta_t^*(s^t)\}$ so as to solve

$$\sup_{\{\theta_{\cdot}^*(\cdot)\}} \Big\{E_{s^T}\Big[\Sigma_{t=0}^{t=T}\delta^t W\big(\theta_t^*(s^t), s_t\big)\Big]\Big\}$$

subject to

$$\theta_t^*(s^t) \geq \theta_{t-1}^*(s^{t-1})$$

where

$$W\big(\theta_t^*(s^t), s_t\big) \equiv \int_{\theta_t^*(s^t)}^{\overline{\theta}} \Gamma\big(\theta, \theta_t^*(s^t), s_t\big) f(\theta) d\theta,$$

and $\Gamma$ is the virtual surplus. The state of the system at date $t$ is therefore $(s^t, \theta_{t-1}^*(s^{t-1}))$. One can then apply standard results in dynamic programming as stated, say, in Lucas et al (1989).

*Proof.* Consider the price sequence $\boldsymbol{p}$ defined in the lemma. Note first that this sequence leaves no rent to the lowest type: $U(\underline{\theta}) = 0$. It thus remains to show that the cutoff delivered by the sequence of short-term prices are exactly the cutoffs $\boldsymbol{\theta^*}$ that obtain under commitment. Note first that for types $\theta > \theta_t^*(s^t)$, not exiting at date $t$ in state $s^t$ is a dominant strategy as they enjoy a strictly positive instantaneous surplus and can always exit later on. Now consider a type $\theta < \theta_t^*(s^t)$. Because the cutoff sequence is necessarily monotonic, $p_\tau > \phi(\theta, \theta_\tau^*(s^\tau), s_\tau)$ for all $\tau \geq t$ and so not exiting delivers a strictly negative payoff. ■

### 3.2 Non-commitment and time consistency

In practice principals may find it difficult to commit to a long-term, state-contingent policy. The absence of commitment is particularly natural either when the principal is a government or when specifying "non-price" dimensions of the future relationship in a contract is complex. This raises the time-consistency issue.

**Definition** (**time consistency**). *Weak* time consistency holds if there exists a perfect Bayesian equilibrium of the non-commitment game that delivers expected payoff $V^*$ for the principal. *Strong* time consistency holds if all perfect Bayesian equilibria of the non-commitment game deliver payoff $V^*$ for the principal.

To obtain results on strong time consistency in the following proposition, we will use the following assumption:

**Assumption 1** (**no strongly positive network externalities**). *For all $s$, the function $\phi(\theta^*, \theta^*, s)$ is strictly increasing in $\theta^*$ (i.e., $\phi_1(\theta^*, \theta^*, s) + \phi_2(\theta^*, \theta^*, s) > 0$).*

Assumption 1 is satisfied whenever network externalities are negative or non-existent ($\phi_2 \geq 0$). It is also satisfied for positive network externalities ($\phi_2 < 0$) provided they are not too large. For example, the technology-disadoption game ($\phi(\theta^*, \theta^*, s) = \theta^* + \alpha[1 - F(\theta^*)] + s$) satisfies Assumption 1 provided that $1 - \alpha \sup\{f(\theta)\} > 0$. Assumption 1 prevents multiple equilibria in the static game; in its absence, a "wrong coordination" of agents by itself might induce a payoff for the principal that lies below $V^*$.

For part (iii) of the proposition, we will further need either to focus on Markov perfect equilibria or to make the following assumption:

**Assumption 2** (**static benefits of a greater clientele**). *For all $(t, s^t)$,*

$$\frac{\partial}{\partial \theta^*}\Big(\int_{\theta^*}^{\overline{\theta}} \psi\big(\theta, \theta^*, s_t\big) f(\theta) d\theta + p_t[1 - F(\theta^*)]\Big) \leq 0$$

*at any $\theta^* \leq \theta_t^*(s^t)$ and for $p_t = \phi\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big)$, where $\big\{\theta_t^*(s^t)\big\}$ denotes the sequence of cutoffs along some optimal commitment policy.*

Assumption 2 says that the principal does not mind having a greater clientele provided that the price is set at the valuation of the current cutoff's surplus at the optimal program. In

Section 4, we will provide a sufficient condition for Assumption 2 to be satisfied. For instance, it is satisfied by the conversion and technology disadoption games, provided that attractiveness is constant or decreasing as defined in the next section.[14]

A *Markov perfect equilibrium* is an equilibrium in which the players' strategies at date $t$ depends only on the previous cutoff $\theta_{t-1}^*$ and the part of the state that is a sufficient statistic for the Markov process (i.e., $s_t$ if $s$ follows a first-order Markov process), and, for the agent, on $p_t$ as well.

**Proposition 1 (time consistency).**

*(i) Weak time consistency always obtains.*

*(ii) If $T < +\infty$ and Assumption 1 holds, strong time consistency obtains.*

*(iii) If $T = +\infty$ and Assumption 1 holds, strong time consistency obtains if either Assumption 2 holds or one focuses on Markov perfect equilibria.*

*Proof.* Suppose that the principal cannot commit and rather sets a price $p_t(s^t)$ in each period.

(i) To prove weak time consistency, consider the following strategies on the equilibrium path:

- The principal sets price $p_t(s^t) = \phi(\theta_t^*(s^t), \theta_t^*(s^t), s_t)$ for all $(t, s^t)$, where $\theta_t^*(s^t)$ corresponds to an optimal allocation cutoff.
- The agent consumes at date $t$ in state $s^t$ if and only if $\theta \geq \theta_t^*(s^t)$.

Myopic agent behavior is indeed optimal given the principal's strategy (the current cutoff has zero continuation utility and so do a fortiori all types below the cutoff; higher types strictly benefit from consuming during the period and so do not want to exit). Furthermore, the principal obtains his highest feasible payoff $V_t(s^t)$ starting at any $(t, s^t)$. And so the principal cannot benefit from deviating from his strategy in any subform $(t, s^t)$.

Because the principal's strategy is a function of his beliefs, whenever his beliefs are well-defined so is his strategy, which, as verified above, is optimal for any subform. Therefore it remains to consider the subforms in which the principal's beliefs are not uniquely pinned down. There are two possible deviations by the agent that would lead to non-uniquely specified beliefs for the principal. Suppose that $F(\theta_{t-1}^*(s^{t-1})) = 0$, but the agent has failed to consume at $t-1$ (or earlier). Then beliefs are irrelevant because the game is over due to the no-reentry constraint. Second, it could be that $F(\theta_{t-1}^*(s^{t-1})) = 1$, but the agent has always consumed up to $t-1$ (included). Then, specify that the principal puts all weight on type $\overline{\theta}$ for the rest of the game and sets $p_\tau(s^\tau) = \phi(\overline{\theta}, \overline{\theta}, s_\tau)$ for all $(\tau, s^\tau)$ with $\tau \geq t$ and $s^\tau \succsim s^t$. The agent therefore cannot obtain a strictly positive continuation utility by deviating. The specified strategies therefore form a perfect Bayesian equilibrium.

[14]Under increasing attractiveness, the principal may want to temporarily "price below marginal cost" so as to keep the clientele. And so a lower cutoff may be (at least temporarily) costly to the principal.

(ii) To prove strong time consistency when $T$ is finite, consider an arbitrary perfect Bayesian equilibrium and let $\delta\, U_{t+1}(\theta, h^t)$ denote the expected continuation payoff of a type $\theta$ that has not yet exited at the end of date $t$ given the entire public history $h^t$ (which includes the realization of $s_t$ and the price $p_t$). Let $\theta_t^*(h^t) < \overline{\theta}$ denote the equilibrium cutoff given history $h^t$ (if $\theta_t^*(h^t) = \overline{\theta}$, the game is over anyway). Let us show by backward induction that cutoff myopia prevails: $U_{t+1}\big(\theta_t^*(h^t), h^t\big) = 0$. Consider date $T$, with previous cutoff $\theta_{T-1}^* = \theta_{T-1}^*(h^{T-1})$. Suppose that the cutoff enjoys a rent:

$$\phi\big(\theta_{T-1}^*, \theta_{T-1}^*, s_T\big) > p_T$$

for some $s_T$ and some optimal price $p_T$ for the principal given $\theta_{T-1}^*$ and $s_T$. The principal's date-$T$ payoff is:

$$\int_{\theta_{T-1}^*}^{\overline{\theta}} \psi\big(\theta, \theta_{T-1}^*, s_T\big) f(\theta) d\theta + p_T\big[1 - F(\theta_{T-1}^*)\big]$$

as all remaining types ($\theta \geq \theta_{T-1}^*$) accept offer $p_T$. But let the principal offer instead

$$p_T' = \phi\big(\theta_{T-1}^*, \theta_{T-1}^*, s_T\big).$$

Then, $p_T' > p_T$ and, from Assumption 1, all types $\theta \geq \theta_{T-1}^*$ accept; and so the principal has increased his date-$T$ payoff by

$$\big(p_T' - p_T\big)\big[1 - F(\theta_{T-1}^*)\big] > 0,$$

a contradiction.

Now consider date $T-1$. The previous cutoff is $\theta_{t-2}^*(h^{T-2})$, the state is $s_{T-1}$ and the principal sets price $p_{T-1} \in$ support $\big(\sigma_{T-1}^P\big(p^{T-2}, s^{T-1}\big)\big)$. Either $\theta_{T-1}^* > \theta_{T-2}^*$, and then

$$\phi\big(\theta_{T-2}^*, \theta_{T-1}^*, s_{T-1}\big) - p_{T-1} + \delta\, U_T\big(\theta_{T-2}^*, h^{T-1}\big)$$
$$< \ \phi\big(\theta_{T-1}^*, \theta_{T-1}^*, s_{T-1}\big) - p_{T-1} + \delta\, U_T\big(\theta_{T-1}^*, h^{T-1}\big) \leq 0$$

where the weak inequality (which is an equality if $\theta_{T-1}^* < \overline{\theta}$) results from the fact that $\theta_{T-1}^*$ is the cutoff; and so $\theta_{T-2}^*$ exits at date $T-1$.

Or $\theta_{T-1}^* = \theta_{T-2}^*$. From the induction hypothesis again, type $\theta_{T-2}^*$ has no continuation value and net utility from $T-1$ on therefore equal to:

$$\phi\big(\theta_{T-2}^*, \theta_{T-2}^*, s_{T-1}\big) - p_{T-1}.$$

Were this utility to be strictly positive (it cannot be strictly negative, otherwise $\theta_{T-2}^*$ and nearby types would exit), the principal would raise price $p_{T-1}$ to $p_{T-1}' = \phi\big(\theta_{T-2}^*, \theta_{T-2}^*, s_{T-1}\big)$, still inducing no exit and raising payoff.

The same reasoning shows by backward induction that the cutoff type never has a strictly positive continuation utility.

Finally, suppose that at date 0 the principal offers $p_0 = \phi\big(\theta_0^*(s^0), \theta_0^*(s^0), s_0\big)$. The cutoff $\theta_0^\dagger$

must necessarily satisfy $\theta_0^\dagger \leq \theta_0^*(s^0)$; for, if $\theta_0^\dagger > \theta_0^*(s^0)$, $\phi(\theta_0^\dagger, \theta_0^\dagger, s_0) > p_0$ and so types $\theta_0^\dagger$ and just below should accept $p_0$. But if $\theta_0^\dagger < \theta_0^*(s^0)$, type $\theta_0^\dagger$ has negative date-0 payoff and has zero continuation utility, a contradiction. Hence $\theta_0^\dagger = \theta_0^*(s^0)$. By the same reasoning, the principal by setting $p_1(s^1) = \phi\big(\theta_1^*(s^1), \theta_1^*(s^1), s_1\big)$ uniquely induces cutoff $\theta_1^\dagger = \theta_1^*(s^1)$, and so forth by induction.

(iii) Allow now $T = +\infty$ and make Assumption 2 as well. Suppose that the principal stubbornly sets $p_t^*(s^t) = \phi\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big)$ for all $t$, $s^t$. There is no commitment to the sequence, nor is this sequence optimal for the principal in every subform. This is simply a strategy choice that reacts to the state of nature, but not to the observed amount of exit.

At date 0, the cutoff type satisfies $\theta_0^\dagger \leq \theta_0^*(s^0)$ from Assumption 1 and so types $\theta > \theta_0^*(s^0)$ optimally accept offer $p_0^*(s^0)$ regardless of the expectation concerning the continuation behavior of the principal. Assumption 2 then implies that the principal's date-0 payoff weakly exceeds

$$\int_{\theta_0^*(s^0)}^{\overline{\theta}} \psi\big(\theta, \theta_0^*(s^0), s_0\big) f(\theta) d\theta + p_0^*(s^0)\big[1 - F(\theta_0^*(s^0))\big].$$

Similarly, at date 1, the cutoff in state $s^1$ will be some $\theta_1^\dagger \leq \theta_1^*(s^1)$ and yield a weakly higher payoff than $\int_{\theta_1^*(s^1)}^{\overline{\theta}} \psi\big(\theta, \theta_1^*(s^1), s_1\big) f(\theta) d\theta + p_1^*(s^1)\big[1 - F(\theta_1^*(s^1)\big]$, and so forth. Thus the principal can guarantee himself the commitment payoff.

Alternatively, we can focus on Markov perfect equilibria and mimic the proof of part (ii). Suppose that for some history $p_t < \phi\big(\theta_{t-1}^*, \theta_{t-1}^*, s_t\big)$. Then $\theta_t^* = \theta_{t-1}^*$. But then the principal could charge $p_t' = \phi\big(\theta_{t-1}^*, \theta_{t-1}^*, s_t\big)$ and still keep all $\theta \geq \theta_{t-1}^*$ on board; and so the cutoff would remain $\theta_{t-1}^*$. The (random) payoff-relevant state in all future periods would be unchanged and so would continuation payoffs in a Markov perfect equilibrium. The principal's payoff would therefore strictly increase, a contradiction. We thus conclude that $p_t \geq \phi\left(\theta_{t-1}^*, \theta_{t-1}^*, s_t\right)$, which, together with cutoff monotonicity, implies that the cutoff in a given period never enjoys a strictly positive continuation valuation. In turn, agent myopic behavior implies that stubbornly setting $p_t^*(s^t) = \phi\left(\theta_t^*(s^t), \theta_t^*(s^t), s_t\right)$ delivers the commitment payoff. ■

As we noted in the introduction, while the optimal commitment *outcome* is time consistent, not every optimal commitment *policy* is. Prices can be arbitrarily frontloaded,[15] with the impact of high initial prices being offset by the promise of low prices in the future; however, frontloaded policies in general are time inconsistent as the principal would want to renege on this promise.

From now on, we will assume that either strong time consistency obtains, or that the equilibrium of the no-commitment game delivering the commitment outcome is selected.

[15] A commitment policy $(\boldsymbol{p}, \boldsymbol{\theta^*})$ is frontloaded if there exists $(t, s^t)$ such that

$$p_t(s^t) > \phi\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big).$$

That is, the cutoff type at date $t$ in state $s^t$ must expect some strictly positive surplus (i.e. a price below his gross surplus) at some future date $\tau$ in some state $s^\tau$.

### 3.3 Unequal time preference

Suppose, just for the sake of this section, that the principal and the agent have different discount factors ($\delta_A$ for the agent and, for comparison purposes, $\delta$ for the principal). We assume that the agent cannot commit. Lemma 1 shows that this assumption is innocuous when the two parties are equally patient; by contrast, it is not innocuous under heterogeneous discounting: when the agent is impatient, the agent's contributions are optimally backloaded, but, as we will see, the agent's inability to commit prevents this backloading. The non-commitment assumption is best motivated by thinking of the per-period transfer $p_t$ as being a non-monetary contribution, and appealing to a no-slavery rule.

**Proposition 2 (unequal discount factors).** *Suppose that $\delta_A$ and $\delta$ differ and that the agent cannot commit. Make Assumptions 1 and 2 and assume a finite horizon. Keeping the principal's discount factor $\delta$ fixed and varying the agent's discount factor $\delta_A$, let $V^{nc}(\delta_A)$ denote the principal's non-commitment payoff that prevails when the principal cannot commit either.*

*(i) The principal's payoff does not depend on the agent's discount factor: $V^{nc}(\delta_A) = V^*$.*

*(ii) The outcome is time consistent if and only if $\delta_A \leq \delta$ (impatient or equally patient agent).*

Intuitively, when the agent is impatient ($\delta_A < \delta$), the principal would like to backload the agent's contributions. The agent's lack of commitment however makes this impossible; and so the best policy under commitment satisfies the cutoff myopia property (see Lemma 1). The agent's discount factor is then irrelevant since the current cutoff never has a positive continuation utility.

When the agent is patient ($\delta_A > \delta$) and under commitment, the principal would like to frontload the agent's contributions. This policy however is infeasible when the principal cannot commit; by backward induction, the principal never leaves any surplus to the previous cutoff type and again cutoff myopia prevails, making the agent's discount factor irrelevant. Proposition 2 is proved in Appendix A.

## 4 Characterization of sequential screening outcomes

Proposition 1 transforms the search for a perfect Bayesian equilibrium of the no-commitment game into a simple optimization problem. This section characterizes the sequential screening outcome in two "polar" cases. In the first, the state improves or deteriorates monotonically over time; thus shocks are highly persistent. The second case is that of transient shocks.

The section also provides several robustness results by allowing inflows of new agents and finite re-entry costs.

### 4.1 Monotone attractiveness

Let us first consider the case in which the consumption offered by the principal becomes (stochastically) more or less attractive over time.

**Definition 1.** Let $s_t \in \mathbb{R}$. Define the externality-augmented virtual surplus as

$$\Lambda(\theta^*, s) \equiv \Gamma(\theta^*, \theta^*, s) - \frac{\int_{\theta^*}^{\overline{\theta}} \frac{\partial \Gamma\left[\theta, \theta^*, s\right]}{\partial \theta^*} f(\theta) d\theta}{f(\theta^*)}.$$

$\Lambda(\theta^*, s) f(\theta^*) d\theta^*$ is the loss of aggregate virtual surplus if we exclude the marginal types $[\theta^*, \theta^* + d\theta^*]$ from consumption.[16]

**Assumption 3 (externality augmented virtual surplus).**

*(i) For all* $\theta^*$ *and* $s$, $\dfrac{\partial \Lambda(\theta^*, s)}{\partial \theta^*} > 0$.

*(ii) Furthermore, states are ranked:* $s \in \mathbb{R}$, *and for all* $\theta^*$ *and* $s$, $\dfrac{\partial \Lambda(\theta^*, s)}{\partial s} > 0$.

Part (ii) of Assumption 3 simply says that higher states are defined as better ones. Part (i) is the generalization of the standard assumption of monotonicity of the virtual surplus to allow for externalities. It is satisfied in the conversion and disadoption games provided that the virtual type $\theta - [1 - F(\theta)]/f(\theta)$ is increasing (for this, it suffices that the Mills ratio $(1 - F)/f$ be decreasing) and network externalities are not too strong if they are positive.

When there are no network externalities and $\dfrac{\partial \psi}{\partial \theta}(\theta, \theta^*, s) \geq 0$ (high types are valued weakly more by the principal), a sufficient condition for part (i) to be satisfied is:

$$\frac{1 - \left(\dfrac{1 - F(\theta^*)}{f(\theta^*)}\right)'}{\dfrac{1 - F(\theta^*)}{f(\theta^*)}} \geq \left(\frac{\dfrac{\partial^2 \phi}{\partial \theta^2}}{\dfrac{\partial \phi}{\partial \theta}}\right)(\theta^*, s)$$

for all $(\theta^*, s)$, where the left-hand side is positive for a log-concave distribution.

**Definition 2 (monotone attractiveness).** Suppose that states are ranked as in Assumption 3. Increasing (resp. decreasing) attractiveness holds when $s_{t+1} \geq s_t$ (resp. $s_{t+1} \leq s_t$) for all $s_{t+1}$ in the support of $G$ conditionally on $s^t$.

Increasing attractiveness for example captures habit formation on the demand side and learning by doing on the supply side. By contrast, decreasing attractiveness may result from a

[16]For example, for the examples given above

$$\Lambda(\theta^*, s) = \theta^* - c - \frac{1 - F(\theta^*)}{f(\theta^*)}$$

in the conversion game, and

$$\Lambda(\theta^*, s) = \theta^* + 2\alpha\left[1 - F(\theta^*)\right] + s - c - \frac{1 - F(\theta^*)}{f(\theta^*)}$$

in the technology disadoption game.

decreasing interest in the incumbent consumption or gradual improvements in the alternative option.[17] Obviously, increasing and decreasing attractiveness include as a special case the case of a constant demand.

**Proposition 3 (monotone attractiveness).** *Let $S_t \equiv S \subseteq \mathbb{R}$ for all $t$.*

*(i) Under deterministic increasing attractiveness ($s^T$ is a singleton), exit occurs only in the initial period: there exists $\theta^*$ such that at the equilibrium outcome $\theta^*_t(s^t) = \theta^*$ for all $t$.*

*(ii) Under decreasing attractiveness, for any $(t, s^t, s^{t+1})$ such that $s^{t+1} \succ s^t$, then $\theta^*_{t+1}(s^{t+1}) > \theta^*_t(s^t)$ when $s_{t+1} < s_t$. The cutoffs are then given by myopic principal optimization: $\Lambda\big(\theta^*_t(s^t), s_t\big) = 0$ for all $(t, s^t)$.*

*Proof.* For a given allocation $(\boldsymbol{p}, \boldsymbol{\theta}^*)$, the tree (with generic element $(z, s^z)$) can be decomposed into the union $\mathcal{T}$ of disjoint subtrees $\mathcal{S}$ of complete subpaths over which the cutoff is constant:

$$\begin{aligned}\mathcal{S} \in \mathcal{T} \iff\ & \exists\, (t, s^t) \,\text{such that} \\ & (i) \quad U_t(\theta^*_t(s^t), s^t; \boldsymbol{p}, \boldsymbol{\theta}^*) = 0 \\ & (ii) \quad U_{t-1}(\theta^*_t(s^t), s^{t-1}; \boldsymbol{p}, \boldsymbol{\theta}^*) > 0 \ (\text{where}\, s^t \succ s^{t-1}) \\ & (iii) \quad \forall (z, s^z) \,\text{such that}\, s^z \succ s^t, \theta^*_z(s^z) = \theta^*_t(s^t) \ \text{if and only if}\, (z, s^z) \in \mathcal{S}.\end{aligned}$$

The principal maximizes

$$V = \int_{\underline{\theta}}^{\overline{\theta}} E_{s^T}\Big[\Sigma_{t=0}^{t=T} \delta^t X^t(\theta, s^t) \Gamma\big(\theta, \theta^*_t(s^t), s_t\big)\Big] f(\theta) d\theta,$$

subject to the feasibility constraint $(F)$ yielding first-order condition:[18]

- either the cutoff is constrained by the previous one: $\theta^*_t(s^t) = \theta^*_{t-1}(s^{t-1})$,
- or the expected discounted virtual surplus along a constant-cutoff sub-tree is equal to 0:

$$\widetilde{\Lambda}\big(\theta^*_t(s^t), s_t\big) \equiv \Lambda\big(\theta^*_t(s^t), s_t\big) + E\Big[\Sigma_{\tau=0}^{\tau=T-t} \delta^\tau \mathbb{I}_{\{\theta^*_{t+\tau}(s^{t+\tau}) = \theta^*_t(s^t)\}} \Lambda\big(\theta^*_{t+\tau}(s^{t+\tau}), s_{t+\tau}\big) | s^t\Big] = 0,$$

- or $\theta^*_t(s^t) = \underline{\theta}$ and $\widetilde{\Lambda}\big(\underline{\theta}, s_t\big) \geq 0$,
- or $\theta^*_t(s^t) = \overline{\theta}$ and $\widetilde{\Lambda}\big(\overline{\theta}, s_t\big) \leq 0$.

When the optimal cutoff is not at one of the boundaries of the support of types, either the cutoff monotonicity constraint binds or we have an interior optimum. We will show that

[17] One may here have in mind a temporary recession or lack of attractiveness of employer (bad management, scandal).

[18] To obtain this first-order condition, maximize $V$ over $\{x_\cdot(\cdot, \cdot), \theta^*_\cdot(\cdot)\}$ subject to the constraints:

$$\int_{\underline{\theta}}^{\overline{\theta}} x_t(\theta, s^t) f(\theta) d\theta \leq 1 - F(\theta^*_t(s^t))$$

and $\theta^*_{t+1}(s^{t+1}) \geq \theta^*_t(s^t)$ if $s^{t+1} \succ s^t$.

the constraint binds at all periods under deterministic increasing attractiveness, and that the optimum is interior in all periods under decreasing attractiveness.

(i) With a deterministic state, we can subsume the dependence of variables on the state through a time index: $\Lambda_t(\theta_t^*)$. A constant-cutoff sub-tree $\mathcal{S}$ is formed by a set of periods $\{t, \cdots, z\}$ such that $\theta_t^* = \cdots = \theta_z^*$. Suppose that there are at least two such subtrees and so the cutoff is not constant over time. Consider the first two, from 0 to $t-1$ and from $t$ to $z$ say. One has $\theta_0^* < \theta_t^*$ and from the previous characterization:

$$\Sigma_{\tau=0}^{t-1}\, \delta^{\tau} \Lambda_\tau(\theta_0^*) \geq 0 \geq \Sigma_{\tau=t}^{\tau=z}\, \delta^{\tau-t} \Lambda_\tau(\theta_t^*).$$

Because $\Lambda_\tau(\theta^*)$ is weakly increasing in $\tau$, one necessarily has

$$\Lambda_{t-1}(\theta_0^*) \geq 0 \geq \Lambda_t(\theta_t^*),$$

which is inconsistent with $\theta_0^* < \theta_t^*$ and $\Lambda_\cdot(\cdot)$ being weakly increasing in time and strictly increasing in the cutoff. Hence, the cutoff must be constant over time.

(ii) Under decreasing attractiveness, the principal-myopic optimum given by (uniquely so from Assumption 3)

$$\forall (t, s^t): \quad \Lambda(\theta_t^*(s^t), s_t) = 0,$$

satisfies the feasibility constraints as $\theta_{t+1}^*(s^{t+1}) \geq \theta_t^*(s^t)$ for $s^{t+1} \succ s^t$. It can be implemented through prices

$$p_t(s^t) = \phi\big(\theta_t^*(s^t), s_t\big). \qquad \blacksquare$$

*Remark.* We need the assumption that the state's evolution is deterministic in part (i) of Proposition 3. That is, it does not suffice that $s_t$ be stochastically increasing with time. To see this, suppose there are only two periods and two states at date 1: $s_0 = s_1^L < s_1^H$. In general, the principal will want to keep the participation high ($\Lambda_0 < 0$) so as to keep an option value of setting $\theta_1^*(s_1^H) = \theta_0^*$ low in state $s_1^H$. If "disappointing" news ($s_1^L$) accrue, then the principal raises the cutoff to $\theta_1^*(s_1^L) > \theta_0^*$. Despite increasing attractiveness, exit is not clustered at date 0.

**Corollary 1 (sufficient condition for Assumption 2 to hold).** *Assumption 2 (the principal benefits from a greater clientele at price $p_t = \phi\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big)$) holds provided that:*

*(a)* $\displaystyle\int_{\theta^*}^{\overline{\theta}} \psi\big(\theta, \theta^*, s\big) f(\theta) d\theta = -C(1 - F(\theta^*), s)$ *with* $C_{11} \leq 0$ *(constant or increasing returns),*

*(b)* $\phi$ *is separable in* $\theta$ *and* $\theta^*$*:* $\phi\big(\theta, \theta^*, s\big) = \xi(\theta, s) + \nu(\theta^*, s)$ *and Assumption 1 holds, and*

*(c) decreasing (including constant) attractiveness holds.*

*Proof.* Suppose that $\psi\big(\theta,\theta^*,s\big) = -C_1(1-F(\theta),s)$ and so: $\int_{\theta^*}^{\overline{\theta}} \psi\big(\theta,\theta^*,s\big) f(\theta)d\theta = -C(1-F(\theta^*),s) + C(0,s)$. Then Assumption 2 is equivalent to

$$\phi\big(\theta_t^*(s^t),\theta_t^*(s^t),s_t\big) \geq C_1\big(1-F(\theta^*),s_t\big)$$

for all $\theta^* \leq \theta_t^*(s^t)$. Now if $C_{11} \leq 0$, this is satisfied provided that

$$\phi\big(\theta_t^*(s^t),\theta_t^*(s^t),s_t\big) \geq C_1\big(1-F(\theta_t^*(s^t)),s_t\big).$$

When $\phi$ is separable ($\phi\big(\theta,\theta^*,s\big) \equiv \xi(\theta,s) + \nu(\theta^*,s)$), the condition $\Lambda\big(\theta_t^*(s^t),\theta_t^*(s^t),s_t\big) = 0$ (see part (ii) of Proposition 2) takes the following form at $\theta^* = \theta_t^*(s^t)$ and $s = s_t$:

$$\begin{aligned}\phi\big(\theta^*,\theta^*,s\big) - C_1\big(1-F(\theta^*),s\big) &= \frac{1-F(\theta^*)}{f(\theta^*)}\,\xi_1(\theta^*,s) + \frac{\int_{\theta^*}^{\overline{\theta}} \nu_1(\theta^*,s) f(\theta)d\theta}{f(\theta^*)} \\ &= \frac{1-F(\theta^*)}{f(\theta^*)}\,\big[\xi_1(\theta^*,s) + \nu_1(\theta^*,s)\big] \geq 0\end{aligned}$$

under (a) and (b). ■

### 4.2 Finite re-entry costs

We have assumed so far that re-entry costs were infinite. Intuitively, our results should carry over for large re-entry costs. We may then wonder, how large is "large"? This section sheds some light on this question. We generalize the model by considering an unchanged flow payoff for the agent while the principal's flow payoff is reduced by the expected re-entry cost:

$$\int_{\underline{\theta}}^{\overline{\theta}} \max\big\{0,(x_t(\theta)-x_{t-1}(\theta))r\big\} f(\theta)d\theta,$$

where $r \geq 0$ is the re-entry cost incurred at date $t$ whenever $x_t = 1$ and $x_{t-1} = 0$. For expositional simplicity, we assume that $r$ represents a cost that is borne by the principal (this assumption also allows us to abstract from the possibility that the principal lowers the price substantially in order to attract re-entrants, who might then fake re-entry – i.e., not spend $r$– and thereby "take the money/surplus and run"). We assume that there is a continuum of agents, and that the principal cannot price discriminate (i.e., he charges a uniform price $p_t$ in each period).[19]

**Proposition 4 (re-entry).** *Suppose that exit is not necessarily definitive: re-entry involves cost* $r \geq 0$.

*(i) Under decreasing/constant attractiveness, payoff* $V^*$ *without re-entry is still an equilibrium*

[19]With a single agent and, say, $r$ small, a ratchet effect would arise: an early rejection by the agent would lead to a price cut relative to an early acceptance. Similarly, ratcheting might occur with a continuum of agents and price discrimination between the installed base and re-entrants. Note that this distinction between the single-agent and continuum-of-agents models arises only in this subsection.

*payoff with re-entry, for all $r \geq 0$.*

*(ii) Under deterministic, strictly increasing attractiveness, equilibrium payoff $V^*$ without re-entry is still an equilibrium payoff with re-entry, provided that $r \geq \underline{r}$ for some $\underline{r}$.*

*Proof.* Under decreasing/constant attractiveness, when the principal charges $p_t = \phi\big(\theta_t^*(s_t), \theta_t^*(s_t), s_t\big)$ for all $(t, s_t)$ with $\Lambda(\theta_t^*(s_t), s_t) = 0$ ($\theta_t^*$ here depends on $s^t$ only through $s_t$) and so $\theta_t^*(s_t) \geq \theta_{t-1}^*(s_{t-1})$, the agent has no incentive to exit to later re-enter. The principal cannot obtain more than $V^*$, because even for $r = 0$ the commitment outcome is $\{\theta_t^*(s_t)\}_{t,s_t}$.

By contrast, with increasing attractiveness, for $r$ small, the principal might want the agent to exit and re-enter later on. The solution takes the following form (we write it for simplicity in the case of no network externalities). Let $T(\theta)$ be defined by (if not equal to 0 or $+\infty$):

$$\Gamma\big(\theta, s_{T(\theta)}\big) \geq (1-\delta)r > \Gamma\big(\theta, s_{T(\theta)-1}\big).$$

In words, $T(\theta)$ is type $\theta$'s optimal re-entry date, if any. Re-entry however can be strictly optimal only if two conditions hold. First re-entry must be profitable relative to exit at date 0 and no re-entry:

$$\Sigma_{t=T(\theta)}^{t=T} \delta^t \, \Gamma(\theta, s_t) > \delta^{T(\theta)} r.$$

Second, it must be profitable relative to no exit at date 0:

$$-\Sigma_{t=0}^{t=T(\theta)-1} \delta^t \, \Gamma(\theta, s_t) > e^{T(\theta)} r.$$

So if for all $\theta$

$$r \geq \min \left\{ \Sigma_{t=T(\theta)}^{t=T} \delta^{t-T(\theta)} \Gamma(\theta, s_t) \; , \; -\Sigma_{t=0}^{t=T(\theta)-1} \delta^{t-T(\theta)} \Gamma(\theta, s_t) \right\} \equiv \underline{r},$$

the optimum involves no re-entry. The first term in the min is increasing in $\theta$ and the second term decreasing.[20] Thus $\underline{r} \equiv \Sigma_{t=T(\widehat{\theta})}^{t=T} \delta^{t-T(\widehat{\theta})} \Gamma(\widehat{\theta}, s_t)$ where $\widehat{\theta}$ is uniquely defined by $\Sigma_{t=0}^{t=T} \delta^t \Gamma(\widehat{\theta}, s_t) = 0$. Note finally that even if $r > \underline{r}$, the commitment solution still resembles that of the absorbing exit paradigm: Types $\theta \geq \theta^*$ never exit (for some $\theta^*$); types $\theta < \theta^*$ exit at date 0, and may re-enter, with higher types re-entering earlier. ■

### 4.3 Inflow of new agents

We have so far assumed that all agents are present at date 0. Suppose by contrast that at each date $t$, a new cohort of agents enters, that has the same type distribution $F(\theta)$ as previous ones.[21] Newcomers, who live from date $t$ through date $T$, have a chance to interact with the principal at date $t$. Non-membership at the entry date is, like exit, an absorbing state.

[20] For the first term and using the envelope theorem, we know that $\Sigma_{t=T(\theta)}^{t=T} \delta^{t-T(\theta)} \Gamma(\theta, s_t) - r\delta^{T(\theta)}$ is increasing in $\theta$ and that $T(\theta)$ is decreasing in $\theta$. Similarly, for the second term, $T(\theta)$ can be seen as the time minimizing $\Sigma_{t=0}^{t=T(\theta)-1} \delta^{t-T(\theta)} \Gamma(\theta, s_t) + r\delta^{T(\theta)}$.

[21] We want to abstract from the standard issues associated with the impact of third-degree price discrimination under heterogeneous submarkets (see, e.g., Aguirre et al 2010 for a recent entry on this topic).

To abstract from direct interactions among cohorts, assume that there are no cross-cohorts network externalities (there can be within-cohort network externalities) and that returns to scale are constant; otherwise an agent's virtual surplus would depend on the number of retained agents in the other cohorts. To capture these requirements, we assume that the function $\Lambda$ is invariant to the presence of other cohorts, and so the only interaction among cohorts is through pricing.[22] If the principal is able to price discriminate among cohorts, each cohort is then treated in isolation and the previous analysis, including that of time consistency, applies. Rather, we ask whether time consistency still holds when the principal is constrained to practise uniform pricing.[23]

**Proposition 5 (inflow of new agents).** *Suppose that each period $t = 0, 1, \cdots, T$, a new cohort of arbitrary mass and type distribution $F(\theta)$ arrives. Suppose that $\Lambda$ is invariant to the presence of other cohorts (no cross-cohorts network externalities and no returns to scale). For the class of monotone-attractiveness games considered in Proposition 3:*

*(i) Under commitment, uniform pricing does as well for the principal as discriminatory pricing.*

*(ii) Time consistency obtains for decreasing/constant attractiveness, but not for deterministic, strictly increasing attractiveness.*

*Proof.* It will be convenient to consider the cases of decreasing/constant attractiveness and of strictly increasing attractiveness sequentially.

Under decreasing/constant attractiveness, the optimal policy for cohort $t$ when price discrimination is feasible (see Proposition 3) is given by myopic optimization:

$$\Lambda\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big) = 0 \quad \text{for all } (t, s^t).$$

Thus the cutoff $\theta_t^*(s^t)$ is independent of the cohort and can be implemented by cohort independent price

$$p_t(s^t) = \phi\big(\theta_t^*(s^t), \theta_t^*(s^t), s_t\big).$$

Furthermore, the function $\Lambda$ is left invariant by left truncations,[24] and so the price path just defined is time consistent.

Next, we consider the case of deterministic, strictly increasing attractiveness. We first want to show that uniform pricing does as well as discriminatory pricing. Given that attractiveness

[22] More precisely, we assume that the principal's intertemporal payoff $V$ is separable across cohorts. Let $c \in \{0, 1, \cdots, T\}$ denote a cohort, with mass $\alpha_c$. Let $X_c^t(\theta, s^t)$ denote the probability that type $\theta$ of cohort $c$ has not exited yet at $t \geq c$ in state $s^t$. Similarly, $\theta_{t,c}^*(s^t)$ is the date-$t$ cutoff for cohort $c$. Then

$$V = \Sigma_{c=0}^{c=T} \alpha_c \int_{\underline{\theta}}^{\overline{\theta}} E_{s^T} \left[ \Sigma_{t=c}^{t=T} \delta^t X_c^t(\theta, s^t) \Gamma\big(\theta, \theta_{t,c}^*(s^t), s_t\big) \right] f(\theta) d\theta,$$

[23] There has been substantial interest in the literature on negative selection regarding the impact of the arrival of new cohorts under uniform pricing (Conlisk et al 1984 and Sobel 1991 are classic references here).

[24] For example, when cohorts have equal sizes, at date $t$, the posterior cumulative over the $(t+1)$ existing cohorts is:
$F_t(\theta) = \dfrac{F(\theta)}{1 + t\big[1 - F\big(\theta_{t-1}^*\big)\big]}$ for $\theta \leq \theta_{t-1}^*$ and $F_t(\theta) = 1 - \dfrac{(t+1)[1 - F(\theta)]}{1 + t\big[1 - F\big(\theta_{t-1}^*\big)\big]}$ for $\theta \geq \theta_{t-1}^*$.

increases over time, the generation $t$ cutoff will enjoy future rents under uniform pricing. To cancel these rents, the principal ought to frontload the payment pattern. More precisely, under price discrimination, the optimal policy for each cohort $t$ consists in a constant cutoff $\theta^*(t)$ defined by (using the notation $\Lambda_\tau$ of the proof of Proposition 3):[25]

$$\Sigma_{\tau=t}^{\tau=T} \delta^{\tau-t} \Lambda_\tau\big(\theta^*(t)\big) = 0.$$

Because $s_t$ is strictly increasing, $\theta^*(t)$ is strictly decreasing.[26]

Let

$$P_t \equiv \Sigma_{\tau=t}^{\tau=T} \delta^{\tau-t} \phi\big(\theta^*(t), \theta^*(t), s_\tau\big)$$

denote the present discounted value of the cohort-$t$ marginal type's surplus. $P_t$ also represents what cohort $t$ will have to pay for membership from $t$ through $T$. Let $p_t$ be defined by $P_t = p_t + \delta P_{t+1}$. Then

$$p_t \equiv \phi\big(\theta^*(t), \theta^*(t), s_t\big) + \Sigma_{\tau=t+1}^{\tau=T} \delta^{\tau-t}\Big[\phi\big(\theta^*(t), \theta^*(t), s_\tau\big) - \phi\big(\theta^*(t+1), \theta^*(t+1), s_\tau\big)\Big].$$

The second term on the right-hand side of this expression of $p_t$ is the present discounted rent of the cohort-$t$ marginal type and is strictly positive. The difference $p_t - \phi\big(\theta^*(t), \theta^*(t), s_t\big)$ thus measures the required frontloading of the payment that delivers cutoff $\theta^*(t)$ for generation $t$. The price sequence $\{p_t\}$ generates cutoff sequence $\{\theta^*(t)\}$.[27]

Finally, suppose that there is no commitment and that the principal charges uniform prices. Is the sequence $\{p_t\}$ defined above an equilibrium of the non-commitment game? To see that this is not the case, consider the two-period version of the model: $t = 0, 1$ and assume for notational simplicity that there are no network externalities, even within a cohort. Necessarily, for time consistency to obtain,

$$p_1 = \phi\big(\theta^*(1), s_1\big)$$

and

$$p_0 + \delta p_1 = \phi\big(\theta^*(0), s_0\big) + \delta\phi\big(\theta^*(0), s_1\big).$$

Furthermore (recalling the assumption of no network externalities and constant returns to scale, so $\Lambda = \Gamma$)

$$\phi\big(\theta^*(1), s_1\big) + \psi\big(\theta^*(1), s_1\big) - \frac{\partial \phi}{\partial \theta}\big(\theta^*(1), s_1\big)\,\frac{1 - F(\theta^*(1))}{f(\theta^*(1))} = 0$$

and

$$\Sigma_{\tau=0}^{\tau=1} \delta^\tau \Big[\phi\big(\theta^*(0), s_\tau\big) + \psi\big(\theta^*(0), s_\tau\big) - \frac{\partial \phi}{\partial \theta}\big(\theta^*(0), s_\tau\big)\,\frac{1 - F(\theta^*(0))}{f(\theta^*(0))}\Big] = 0.$$

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[25] For conciseness, we assume interior solutions ($\underline{\theta} < \theta^*(t) < \overline{\theta}$). The result however does not hinge on this assumption.

[26] $\Sigma_{\tau=t}^{\tau=T} \delta^{\tau-t} \Lambda_\tau\big(\theta^*(t)\big) = 0 \quad \Longrightarrow \quad \Sigma_{\tau=t+1}^{\tau=T} \delta^{\tau-(t+1)} \Lambda_\tau\big(\theta^*(t)\big) > 0$. Hence if $\theta^*(t+1) \geq \theta^*(t)$,

$$\Sigma_{t=t+1}^{\tau=T} \delta^{\tau-(t+1)} \Lambda_\tau\big(\theta^*(t+1)\big) > 0,$$

a contradiction.

[27] It is unique if $T$ is finite. We conjecture that it is also unique if $T$ is infinite.

Because $\theta^*(0) > \theta^*(1)$, the principal's date-1 payoff in the neighborhood of $p_1 = \phi\big(\theta^*(1), s_1\big)$ is, letting $\alpha_t$ denote the weight of cohort $t$ and $\theta_1^*(p_1)$ be defined by $p_1 = \phi\big(\theta_1^*(p_1), s_1\big)$:

$$p_1\Big[\alpha_1\left[1 - F(\theta_1^*(p_1))\right] + \alpha_0\left[1 - F(\theta^*(0))\right]\Big]$$

$$+ \alpha_1 \int_{\theta_1^*(p_1)}^{\overline{\theta}} \psi(\theta, s_1) f(\theta) d\theta + \alpha_0 \int_{\theta^*(0)}^{\overline{\theta}} \psi(\theta, s_1) f(\theta) d\theta.$$

The derivative at $p_1 = \phi\big(\theta^*(1), s_1\big)$ is strictly positive, reflecting the fact that the demand of cohort 0 is locally inelastic. Thus the principal cannot obtain the commitment payoff. ■

### 4.4 Transient shocks

Suppose now that the realizations of the shocks $s_t$ are identically and independently distributed over time. Let $g(s_t)$ and $G(s_t)$ denote the density and the cumulative distribution of the shock. We assume that the horizon is infinite, so as to provide a simpler characterization of the equilibrium outcome.[28] The principal's payoff can then be rewritten as:

$$V = \int_{\underline{\theta}}^{\overline{\theta}} \left[E_{s^\infty} \Sigma_{t=0}^{\infty} \delta^t x_0(\theta, s^0) \cdots x_t(\theta, s^t) \Lambda\big(\theta, s_t\big)\right] f(\theta) d\theta.$$

**Proposition 6 (transient shocks).** *Suppose that shocks are identically and independently distributed with density $g(\cdot)$ and that $T = +\infty$. The equilibrium outcome is characterized by: for all $(t, s^t)$*

$$\theta_t^*(s^t) = \theta^u\big(\min_{\tau \le t} s_\tau\big),$$

*where $\theta^u(s)$, a decreasing function of $s$, is uniquely defined by $H(\theta^u(s)) = 0$ where*

$$H(\theta^*, s) \equiv \Lambda(\theta^*, s) + \frac{\delta}{1 - \delta[1 - G(s)]} \int_s^\infty \Lambda(\theta^*, \tilde{s}) g(\tilde{s}) d\tilde{s},$$

*if $H(\underline{\theta}, s) \le 0 \le H(\overline{\theta}, s)$, and $\theta^u(s) = \underline{\theta}$ if $H(\underline{\theta}, s) \ge 0$ and $\theta^u(s) = \overline{\theta}$ if $H(\overline{\theta}, s) \le 0$.*

*Proof.* Maximizing $V$ with respect to $x_t(\theta, s^t)$ yields the first-order condition

$$x_t(\theta, s^t) = 1 \iff \Lambda(\theta, s_t) + E_{s_{t+1}^\infty}\left[\Sigma_{\tau=t+1}^{\infty} \delta^{\tau - t} X_{t+1}^\tau(\theta, s^\tau) \Lambda(\theta, s_\tau)\right] \ge 0$$

where $X_{t+1}^\tau(\theta, s^\tau) \equiv x_{t+1}(\theta, s^{t+1}) \cdots x_\tau(\theta, s^\tau)$. Note that if $X^{t-1}(\theta, s^{t-1}) = 0$, then the first-order condition is irrelevant, but one can still impose this date-$t$ first-order condition without loss of generality. Because $\Lambda$ is strictly increasing in $\theta$, there is a unique threshold in $[\underline{\theta}, \overline{\theta}]$ for each $s^t$ such that this condition is satisfied if and only if $\theta \ge \theta_t^*(s^t)$. The stationarity of the problem then suggests looking for a strictly decreasing cutoff $\theta^u(s)$ (where "$u$" stands for "unconstrained

[28] A similar characterization is available by backward induction for a finite $T$, but the strategy then depends on the length of the remaining horizon.

by the previous exit pattern"). Noting that $X^{\tau}_{t+1}(\theta^u(s_t), s^{\tau}) = 0$ if $\min\{s_{t+1}, \cdots, s_{\tau}\} < s_t$, such a cutoff then satisfies (if interior, i.e., in $(\underline{\theta}, \overline{\theta})$):

$$\Lambda(\theta^u(s), s) + \delta\left[\int_s^{\infty} \Lambda(\theta^u(s), \tilde{s})g(\tilde{s})d\tilde{s} + [1 - G(s)]\delta\left[\int_s^{\infty} \Lambda\big(\theta^u(s), \tilde{s}\big)g(\tilde{s})d\tilde{s} + \cdots\right]\right] = 0$$

or

$$\Lambda(\theta^u(s), s) + \frac{\delta}{1 - \delta[1 - G(s)]}\int_s^{\infty} \Lambda(\theta^u(s), \tilde{s})g(\tilde{s})d\tilde{s} = 0.$$

Differentiating this condition and using it to eliminate two terms, one obtains:

$$\frac{\partial \Lambda}{\partial s} + \left[\frac{\partial \Lambda}{\partial \theta} + \frac{\delta}{1 - \delta[1 - G(s)]}\int_s^{\infty} \frac{\partial \Lambda}{\partial \theta} g(\tilde{s})d\tilde{s}\right]\frac{d\theta^u}{ds} = 0$$

and so

$$\frac{d\theta^u}{ds} < 0.$$

The tentative solution

$$\theta^*_t(s^t) = \max_{\tau \le t}\{\theta^u(s_{\tau})\} = \theta^u\big(\min_{\tau \le t} s_{\tau}\big)$$

indeed satisfies the first-order condition above. ■

**Corollary 2 (testable predictions for transient shocks).** *With transient shocks and an infinite horizon, let for all $t$, $s^{t-1}$*

$$\mathcal{V}_t = F\Big(\theta^u\big(\min_{\tau \le t} s_{\tau}\big)\Big) - F\Big(\theta^u\Big(\min_{\tau \le t-1} s_{\tau}\Big)\Big)$$

*denote the volume of exit at date $t$. Then the following properties hold for all $\tau > t$:*

*(i) Decreasing exit: $E\big[\mathcal{V}_t | s^{t-1}\big] \ge E\big[\mathcal{V}_{\tau} | s^{t-1}\big]$.*

*(ii) Negative serial correlation: $\dfrac{\partial}{\partial \mathcal{V}_t} E_t\big[\mathcal{V}_{\tau} | s^{t-1}, \mathcal{V}_t\big] < 0$.*

*Proof.* Let $\hat{s} \equiv \min_{\tau \le t-1} s_{\tau}$; and let $g_n(s) \equiv ng(s)[1 - G(s)]^{n-1}$ denote the density of the distribution of the minimum realization over $n$ periods.

(i) Then

$$E\Big[\mathcal{V}_{\tau} | s^{t-1}\Big] = \int_{\hat{s}}^{\infty}\Big[\int_{-\infty}^{\hat{s}}\Big[F\big(\theta^u(\tilde{s})\big) - F\big(\theta^u(\hat{s})\big)\Big]g(\tilde{s})d\tilde{s}\Big]g_{\tau-t}(s)ds$$

$$+ \int_{-\infty}^{\hat{s}}\Big[\int_{-\infty}^{s}\Big[F\big(\theta^u(\tilde{s})\big) - F\big(\theta^u(s)\big)\Big]g(\tilde{s})d\tilde{s}\Big]g_{\tau-t}(s)ds$$

$$\le \int_{-\infty}^{\hat{s}}\Big[F\big(\theta^u(\tilde{s})\big) - F\big(\theta^u(\hat{s})\big)\Big]g(\tilde{s})d\tilde{s} = E\big[\mathcal{V}_t | s^{t-1}\big].$$

(ii) Note that $E\left[\mathcal{V}_\tau|s^{t-1}\right]$ depends only on, and is increasing with $\hat{s}$; and that $E\left[\mathcal{V}_\tau|s^{t-1},\mathcal{V}_t\right] = E\left[\mathcal{V}_\tau|\min\{\hat{s},s_t\}\right]$. Because $\mathcal{V}_t$ is (weakly) decreasing in $s_t$, then $E\left[\mathcal{V}_\tau|s^{t-1},\mathcal{V}_t\right]$ is weakly decreasing in $\mathcal{V}_t$, and strictly so when $\mathcal{V}_t > 0$. ■

# 5 Time inconsistency: shifting types

The next two sections are devoted to the analysis of dynamic screening with positive selection in environments that do not satisfy the conditions for time consistency.

For the remainder of the paper, we will alleviate notation by making

**Assumption 4 (no network externalities).** $\frac{\partial \Gamma}{\partial \theta^*}(\theta,\theta^*,s) = 0$ *for all* $(\theta,s)$.

The absence of network externalities plays no major role in the results to come. By an abuse of notation, we omit the variable $\theta^*$ as an argument of $\phi$ and $\psi$. Note also that $\Gamma = \Lambda$ under Assumption 4.

## 5.1 Shifting principal type

Sometimes the principal's preferences may change over time. Indeed, in the conversion game, Muslim rulers exhibited varying degrees of piousness, altering the trade-off between tax receipts and adherence to the Muslim faith. Let $\gamma_t \in \mathbb{R}$ denote the date-$t$ principal's type, which is assumed to affect only the principal's objective function $\psi$ and not the agent's utility $\phi$. We assume that $\psi$ is increasing in $\gamma$ and so a high $\gamma$ principal prefers a lower cutoff compared with a low $\gamma$ one.

We assume that the realizations of $s_t$ and $\gamma_t$ are public information at the beginning of date $t$; otherwise the principal's price might signal his type. The parameters $s_t$ and $\gamma_t$ follow independent stochastic processes and differ in that the date-$t$ principal's payoff from date-$\tau$ agent participation for $\tau > t$ is (under Assumption 4) $\psi(\theta,s_\tau,\gamma_t)$ as opposed to $\psi(\theta,s_\tau,\gamma_\tau)$ for the date-$\tau$ principal. That is, $\gamma_t$ represents the change in the principal's preferences over time and will be the source of conflict among principals; by contrast, $s_t$ is the mere evolution of the part of the rest of the state. Thus, the date-$t$ principal's objective function is:

$$V_t(s^t,\gamma^t) = \int_{\theta^*_{t-1}(s^{t-1},\gamma^{t-1})}^{\overline{\theta}} E_{s^T,\gamma^T}\left[\Sigma_{\tau=t}^{T}\delta^{\tau-t}X^\tau(\theta,s^\tau,\gamma^\tau)\Gamma(\theta,s_\tau,\gamma_t)\left[\frac{f(\theta)}{1-F\left(\theta^*_{t-1}(s^{t-1},\gamma^{t-1})\right)}\right]d\theta\right]$$

Obviously, the principal's commitment policy in general will not be time consistent. Nonetheless simple equilibrium solutions again are available if we specialize the model somewhat.

Let us assume that the state $(s,\gamma) \in \mathbb{R}^2$, follows a first-order Markov process $G^s(s_{t+1}|s_t) \times G^\gamma(\gamma_{t+1}|\gamma_t)$ with full support. Suppose that $\partial\psi/\partial s > 0$ and that the virtual surplus

$$\Gamma(\theta,s,\gamma) = \phi(\theta,s) + \psi(\theta,s,\gamma) - \frac{\partial \phi(\theta,s)}{\partial \theta}\frac{1-F(\theta)}{f(\theta)}$$

is strictly increasing in $\theta$.

Suppose, first, that the "time-consistent" part of the state, $s_t$, is constant (only the principal's type varies over time). Intuitively, when inducing a cut-off at date $t$, type $\gamma_t$ constrains, but is not affected by future choices of types $\gamma' > \gamma_t$. By contrast, he is affected by future choices of types $\gamma' < \gamma_t$, but cannot do anything about it (altering these choices would require making future cutoffs even higher, while they are already too high).

**Proposition 7 (shifting principal type, invariant environment).** *Suppose that only the principal's type changes over time: $s_t = s$ for all $t$, that the virtual surplus $\Gamma$ is strictly increasing in $\theta$, and make Assumption 4.*

*Let $\theta^*_\gamma$ be defined by $\theta^*_\gamma = \underline{\theta}$ if $\Gamma(\underline{\theta}, s, \gamma) \geq 0$, $\theta^*_\gamma = \overline{\theta}$ if $\Gamma\left(\overline{\theta}, s, \gamma\right) \leq 0$ and $\Gamma\left(\theta^*_\gamma, s, \gamma\right) = 0$ otherwise. That is, $\theta^*_\gamma$ is the optimal cutoff for principal type $\gamma$. There exists a Markov perfect equilibrium of the game such that on the equilibrium path the cutoff is at each point of time the optimal cutoff for the least eager principal so far:*

$$\theta^*_t = \theta^*_{\underset{\tau \leq t}{min}\{\gamma_\tau\}}$$

*Proof.* Note that

$$M(\theta^*, \gamma) \equiv \int_{\theta^*}^{\overline{\theta}} \Gamma\left(\theta, s, \gamma\right) f(\theta) d\theta$$

is strictly quasi-concave with maximum at $\theta^*_\gamma$. Furthermore $\theta^*_\gamma$ is weakly decreasing in $\gamma$.

Consider an arbitrary date $\tau$ and history $h^{\tau-1} \equiv (\gamma_0, \cdots, \gamma_{\tau-1}, p_0, \cdots, p_{\tau-1})$ at that date. Let $\hat{\theta}(p_0, \cdots, p_{\tau-1})$ be defined by the solution to $\phi(\theta, s) = \max\{p_0, \cdots, p_{\tau-1}\}$ (if interior; otherwise $\hat{\theta} = \underline{\theta}$ if $\phi(\underline{\theta}, s) \geq \max\{p_0, \cdots, p_{\tau-1}\}$ and $\hat{\theta} = \overline{\theta}$ if $\phi(\overline{\theta}, s) \leq \max\{p_0, \cdots, p_{\tau-1}\}$) Suppose that at date $\tau$, principal $\gamma_\tau$ sets

$$p_\tau = \phi\left(\max\left\{\theta^*_{\gamma_\tau}, \hat{\theta}\big(p_0, \cdots, p_{\tau-1}\big)\right\}, s\right),$$

and that the agent behaves myopically ($x_\tau(\theta, h^\tau) = 1$ if and only if $\phi(\theta, s) \geq p_\tau$). Consider the date-$t$ principal, with type $\gamma_t$. Then

(i) either $\theta^*_{\gamma_t} \geq \hat{\theta}\big(p_0, \cdots, p_{t-1}\big)$ and then for all $\tau$ such that $\min\{\gamma_{t+1}, \cdots, \gamma_\tau\} \geq \gamma_t$, $M\left(\theta^*_{\gamma_t}, \gamma_t\right) \geq M\left(\theta^*_z, \gamma_t\right)$ for $z \in \{t+1, \cdots, \tau\}$ and so $\theta^*_{\gamma_t}$ provides a higher utility for such realizations. By contrast, consider $(\tau > t, \gamma_\tau)$ such that $\gamma_\tau < \gamma_t$. Then any alternative cutoff $\theta^*_t \geq \theta^*_{\gamma_\tau}$ would have no impact on the date-$\tau$ cutoff. And if $\theta^*_t < \theta^*_{\min\{\gamma_{t+1}, \cdots, \gamma_z\}}$ raising $\theta^*_t$ at the margin improves type $\gamma_t$'s welfare from the quasi-concavity of $M$.

(ii) or $\theta^*_{\gamma_t} < \hat{\theta}\big(p_0, \cdots, p_{t-1}\big)$. If $\hat{\theta}\big(p_0, \cdots, p_t, p_{t+1}, \cdots, p_\tau\big) = \hat{\theta}\big(p_0, \cdots, p_t\big) < \theta^*_{\gamma_\tau}$ the quasi-concavity of $M$ and the fact that $\hat{\theta}$ is weakly increasing in $p_t$ implies that any increase in $p_t$ above $\hat{\theta}\big(p_0, \cdots, p_t\big)$ would reduce profit not only at date $t$ but also at dates $t+1, \cdots, \tau$. And if $\hat{\theta}\big(p_0, \cdots, p_t, p_{t+1}, \cdots, p_\tau\big) = \theta^*_{\min\{\gamma_{t+1}, \cdots, \gamma_z\}}$, we are back to case (i). We thus conclude that the proposed strategies indeed form an equilibrium. ■

The next result allows the non-principal-related part of the state, $s_t$, to evolve over time, making the relationship either increasingly attractive or increasingly unattractive.

Let $\hat{\theta}_{t,\gamma_t}$ denote the optimal date-$t$ cutoff for principal $\gamma_t$ as characterized in Proposition 3. That is, the cutoff is that which would prevail in the thought experiment in which (a) the principal's type remains $\gamma_t$ for the rest of the game and (b) this principal is unconstrained by previous truncations of the distribution ($\theta^*_{t-1} = \underline{\theta}$, say): $\hat{\theta}_{t,\gamma_t}$ is the cutoff that would prevail in a different game in which both the principal and the agent both believed that $\gamma_\tau = \gamma_t$ for all $\tau > t$.

**Proposition 8 (shifting principal type, deterministic monotone attractiveness).** *Suppose that $s_t \in \mathbb{R}$, $\partial\Gamma/\partial s > 0$, $\partial\Gamma/\partial\theta > 0$ and Assumption 4 holds. Then there exists an equilibrium and a sequence $\theta^*_{t,\gamma_t}$ such that the cutoff $\theta^*_t$ induced by principal $\gamma_t$ at date $t$ is max $\left\{\theta^*_{t-1}, \theta^*_{t,\gamma_t}\right\}$ where*

$$\theta^*_{t,\gamma_t} \geq \hat{\theta}_{t,\gamma_t} \qquad \textit{under} \quad \begin{array}{l} \textit{either increasing attractiveness } (s_t \textit{ increasing}) \\ \textit{or decreasing attractiveness } (s_t \textit{ decreasing}), \end{array}$$

*where $\hat{\theta}_{t,\gamma_t}$ is the cutoff that would be selected by a date-$t$ principal with type $\gamma_t$, were the future principals also to have type $\gamma_t$.*

Proposition 8 says that there is too little retention going forward from the point of view of all successive principals. The reason why this is so is not the same for increasing and decreasing demand. Under increasing attractiveness the principal exerts cutoff moderation when having a constant type as he expects that he will prefer wider participation in the future. Cutoff moderation is like an investment, but with changing type, the investment has a lower value as the cutoff in future periods may be raised by less eager (lower $\gamma$) types. Under decreasing attractiveness, the date-$t$ principal would raise the cutoff over time if he were permanent (Proposition 3). Increasing the cutoff a bit above the myopic optimum is beneficial as this commits future, more eager (higher $\gamma$) types.

*Proof (sketch).*
(i) *Decreasing attractiveness.* Recall that the time-consistent cutoffs under decreasing attractiveness are given by myopic optimization, i.e., $\hat{\theta}_{t,\gamma_t}$ is given by

$$\Gamma\big(\hat{\theta}_{t,\gamma_t}\,, s_t\,, \gamma_t\big) = 0;$$

and that the sequence $\hat{\theta}_{t,\gamma_t}$ is monotonically increasing in $t$ for a given $\gamma_t$.

Let the agent behave myopically: $x_t = 1$ iff $p_t \leq \phi(\theta, s_t)$. Then setting prices is equivalent to setting cutoffs (subject to the cutoff being no smaller than the previous one). We look for an equilibrium in which for all $(t, \gamma_t)$

$$\theta^*_{t,\gamma_t} \geq \max\left\{\theta^*_{t-1}\,, \hat{\theta}_{t,\gamma_t}\right\}.$$

Given this, type $\gamma_t$ setting cutoff $\theta^*_t < \hat{\theta}_{t,\gamma_t}$ at date $t$ (assuming this is allowed by previous cutoffs) reduces the principal's date-$t$ payoff from the strict concavity of $M$ (the fact that $\Gamma$ is increasing in $\theta$). At a future date $\tau > t$, either $\theta_t^*$ is locally irrelevant $\left(\theta_t^* < \theta_\tau^*\right)$ or $\theta_t^* = \theta_\tau^*$. Because $\hat{\theta}_{\tau,\gamma_t} \geq \hat{\theta}_{t,\gamma_t}$ and by strict quasi-concavity, raising $\theta_t^*$ slightly would also raise type $\gamma_t$'s payoff at date $\tau$ in such events.

The existence of cutoff $\{\theta^*_{\cdot,\cdot}\}$ is obtained through a fixed-point argument.

(ii) *Increasing attractiveness.* The strategy of proof is identical to that of case (i). Again, let the agent behave myopically, and the principal set a cutoff

$$\theta_{t,\gamma_t}^* \geq \max\left\{\theta_{t-1}^*, \hat{\theta}_{t,\gamma_t}\right\}$$

where the time-consistent cutoff $\hat{\theta}_{t,\gamma_t}$ is no longer given by a myopic optimization (see Proposition 3). The strategy of proof again consists in using the strict monotonicity of the $\Gamma$ function to show that setting a cutoff $\theta_t^* < \hat{\theta}_{t,\gamma_t}$ is strictly suboptimal for type $\gamma_t$ at date $t$. ■

## 5.2 Shifting agent type

The framework so far rules out the case of a "shifting type" for the agent (for which asymmetries of information may be reduced over time), on which many of the recent advances on dynamic mechanisms design have focused. Time consistency of the optimal commitment policy is then not to be expected. Intuitively, the principal might want to promise low (and efficient) future prices in exchange of a higher price today. Again, price frontloading is not conducive to time consistency.

For example, in the Baron and Besanko (1984) two-period model of commitment with a Markovian type for the agent, the agent's date-0 virtual valuation is, as in the static case, determined by the date-0 inverse hazard rate (Mills ratio) and is equal to $\theta_0 - \frac{1-F_0(\theta_0)}{f_0(\theta_0)}$ where $\theta_0$ is his date-0 valuation ($\phi(\theta, s) = \theta$), and $F_0(\theta_0)$ is the date-0 distribution. By contrast, the agent's date-1 virtual valuation[29] is

$$\theta_1 - \frac{\dfrac{\partial\big(1 - F_1(\theta_1|\theta_0)\big)}{\partial\theta_0}}{f_1(\theta_1|\theta_0)} \, \frac{1 - F_0(\theta_0)}{f_0(\theta_0)}.$$

Thus, the commitment date-1 allocation depends not only on the agent's date-1 type, $\theta_1$, but also on his date-0 type $\theta_0$.

More generally, and as Pavan et al (2014) stress, "because of the serial correlation of types, it is optimal to distort allocations not only in the initial period, but at every history at which the agent's type is responsive to his initial type, as measured by the impulse response function." This memorization of past, now-payoff-irrelevant types in the optimal commitment allocation makes the commitment solution time-inconsistent. One therefore can no longer rely on solving an optimal control problem to obtain a perfect Bayesian equilibrium of the no-commitment environment. Nonetheless, explicit derivations are available in simple cases, as we now illustrate.

[29] See Pavan et al (2014) for the general expression of impulse response function with arbitrary horizon and payoff functions.

**Proposition 9** (**transient agent types**). *Assume that* $\phi(\theta, s) = \theta$ *and* $\psi(\theta, s) = 0$. *Suppose that at* $t = 0, 1, \cdots, \infty$, *the agent's type is drawn in an i.i.d. manner from density* $f(\theta)$ *and c.d.f.* $F(\theta)$ *on* $[\underline{\theta}, \overline{\theta}]$ *where* $\underline{\theta} \geq 0$ *and the virtual surplus* $\theta - \left[(1 - F(\theta))/f(\theta)\right]$ *is strictly increasing. Any Markov Perfect Equilibrium is characterized by a (uniquely defined and increasing in* $\delta$*) threshold* $\theta^*$ *given by the following generalized virtual surplus:*

$$J(\theta^*) \equiv \theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} + \frac{\delta \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta)}{1 - \delta[1 - F(\theta^*)]} = 0 \tag{1}$$

(if interior; $\theta^* = \underline{\theta}$ if $J(\underline{\theta}) \geq 0$ and $\theta^* = \overline{\theta}$ if $J(\overline{\theta}) \leq 0$)

The principal in each period sets price $p_t = \theta^* + \delta \int_{\theta^*}^{\overline{\theta}} (\theta - \theta^*) dF(\theta)$ conditional on the agent not having exited yet. Letting $\theta^m$ denote the monopoly price (i.e., $\theta^m = [1 - F(\theta^m)]/f(\theta^m)$), then $\theta^* \in [0, \theta^m)$.

*Proof.* Let $U$, $V$ and $W$ denote the continuation payoffs ($U$ for the agent, $V$ for the principal and $W \equiv U + V$). These are constant in a Markov Perfect Equilibrium, since the only payoff-relevant state variable is that the agent has not exited yet.

We treat only the case of an interior solution (the treatment of the corner solutions $\theta^* = \underline{\theta}$ or $\overline{\theta}$ is analogous). Price $p^*$ at date $t$ induces a cutoff $\theta^*$ given by

$$\theta^* - p^* + \delta U = 0.$$

The principal solves

$$\max_{\theta^*} \left\{[1 - F(\theta^*)]\left[(\theta^* + \delta U) + \delta V\right]\right\}$$

which yields

$$\theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} + \delta W = 0$$

with the continuation welfare given by

$$W = \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta) + \left[1 - F(\theta^*)\right]\delta W.$$

Simple computations show that $J'(\theta^*) > 0$ whenever $J(\theta^*) = 0$. Hence the solution $\theta^*$ is unique. The price $p^*$ is given by

$$p^* = \theta^* + \delta U$$

where

$$U = \int_{\theta^*}^{\overline{\theta}} \left[\theta - p^* + \delta U\right] f(\theta) d\theta = \int_{\theta^*}^{\overline{\theta}} (\theta - \theta^*) f(\theta) d\theta.$$

■

The commitment solution for the environment described in Proposition 9 can be implemented by a commitment to a sequence of prices:

$$p_t^c = 0 \quad \text{for all } t \geq 1 \quad \text{and} \quad p_0^c = \theta^c + \frac{\delta E[\theta]}{1-\delta},$$

where either

$$\theta^c - \frac{1-F(\theta^c)}{f(\theta^c)} + \frac{\delta E[\theta]}{1-\delta} = 0 \quad \text{or} \quad \theta^c = \underline{\theta} \quad \text{and} \quad \frac{\delta E[\theta]}{1-\delta} \geq \frac{1}{f(\underline{\theta})} - \underline{\theta}.$$

Note that

$$\theta^c \leq \theta^*$$

(with equality only when $\theta^* = \underline{\theta}$) and that for each $t \geq 1$

$$p_t^c \leq p^* \leq p_0^c$$

with strict inequalities whenever $\theta^* > \underline{\theta}$.

Relatedly, it is important for the analysis in Sections 3 and 4 that, in each period $t$, the state of nature be public knowledge prior to price setting (or, if not, that the principal be able to offer a state-contingent price $p_t$). Suppose by contrast that at each date $t$ the principal first learns the realization of $s_{t-1}$ (either directly or through the date-$(t-1)$ realized demand); the principal sets a price $p_t$; the agents then observe $s_t$ and decide whether to consume. The date-$t$ shock then plays a role similar to that of a transient shock to the agent's type, in that it confers an informational advantage to the agent for exactly one period.[30]

# 6 Common agency and teams

## 6.1 Multiple principals: retention as a dynamically provided public good

The agent's decision to exit often depends on the behavior of multiple principals rather than a single one. Retention in a work, volunteering, sports or religious community relies on the joint efforts by its members to make staying a comfortable option for the member. Immigration deci-

[30] To make the basic point in the simplest manner, suppose that $t = 0, 1$; that $s_t \in \mathbb{R}$ ; that $\phi = s_t$ (homogeneous preferences and no network externality); that $\psi = 0$ (costless production); and that $s_t$ is i.i.d. with distribution $G$.

At date $t = 1$, the principal sets $p^m = \arg\max p[1 - G(p)] \equiv \pi(p)$. Let $S(p) \equiv \int_p^\infty (s-p)dG(s)$.

In the absence of commitment, the principal chooses $p_0$ so as to solve:

$$\max_{\{p_0\}} \Big\{ [1 - G(p_0 - \delta S(p^m))] \, [p_0 + \delta\pi(p^m)] \Big\} = \max_{\{s_0^*\}} \Big\{ [1 - G(s_0^*)] \, [s_0^* + \delta \, [\pi(p^m) + S(p^m)]] \Big\}.$$

By contrast, the commitment outcome corresponds to the solution of

$$\max_{\{s_0^*\}} \Big\{ [1 - G(s_0^*)] \, [s_0^* + \delta S(0)] \Big\}$$

since $S(0) = \max_{\{p\}} [\pi(p) + S(p)]$. Under commitment, the principal charges a higher date-0 price and has a larger date-0 clientele.

sions similarly may be guided by a mixture of policies enacted by local and national authorities, workplace atmosphere, overall society openness, and so forth.

This section studies environments in which n principals each set a "price" every period for that period, and the agent's continuation decision is guided by the sum of those prices. Such environments are not conducive to time consistency since under commitment each principal might want to commit to relatively high prices in order to force other principals to bear the brunt of the retention effort in the future.[31]

Suppose that there are $n$ symmetrical principals with surplus $\psi(\theta, s_t)/n$ each. At date $t$, the principals simultaneously set prices $p_t^i$; principal $i$'s flow payoff given resulting cutoff $\theta_t^i$ is then

$$p_t^i\left[1 - F(\theta_t^*)\right] + \int_{\theta_t^*}^{\overline{\theta}} \frac{\psi(\theta, s_t)}{n} f(\theta) d\theta.$$

Provided that he does not exit, the agent's flow payoff is

$$\phi(\theta, s_t) - \Sigma_{i=1}^n p_t^i.$$

We will be focusing on *symmetric Markov perfect equilibria in which the agent behaves myopically*:

$$\begin{aligned} \theta_t^* &= \theta_{t-1}^* \quad \text{if} \quad \phi(\theta_{t-1}^*, s_t) \geq \Sigma_{i=1}^n p_t^i \\ \theta_t^* &= \overline{\theta} \qquad \text{if} \quad \phi(\overline{\theta}, s_t) \leq \Sigma_{i=1}^n p_t^i \end{aligned}$$

or, if the solution is interior:

$$\phi(\theta_t^*, s_t) = \Sigma_{i=1}^n p_t^i.$$

Markov behavior means that the vector of prices charged at date $t$, $\{p_t^i\}_{t=1}^n$ depends only on the previous cutoff $\theta_{t-1}^*$ and on the current state $s_t$ (provided that the state follows a first-order Markov process, or more generally on a statistics for the history of states that is a sufficient statistics for describing current and future payoffs). Furthermore, $p_t^i = p_t$ for all $i$ and all histories of the game.

The following assumption is the counterpart of Assumption 3 in the common agency context:

**Assumption 5.** *For all* $(\theta, s)$

$$\frac{\partial}{\partial \theta}\Big(\phi(\theta, s) - n\frac{1 - F(\theta)}{f(\theta)} \frac{\partial \phi}{\partial \theta}(\theta, s) + \psi(\theta, s)\Big) > 0$$

*We furthermore assume that states are ordered:* $s \in \mathbb{R}$ *and*

$$\frac{\partial}{\partial s}\Big(\phi(\theta, s) - n\frac{1 - F(\theta)}{f(\theta)} \frac{\partial \phi}{\partial \theta}(\theta, s) + \psi(\theta, s)\Big) > 0.$$

[31] This environment is different from that studied by Admati and Perry (1991) and the literature they initiated. Admati and Perry consider a cumulative-contribution game in which $n$ players make sequential commitments toward assembling a fixed amount needed to implement a project. There is no strategic agent involved, and a fortiori no screening of the agents' information.

**Proposition 10 (common agency).** *Under Assumptions 4 and 5, a symmetric Markov perfect equilibrium with myopic agent behavior exists and has the following properties:*

*(i) Under deterministic increasing attractiveness, exit occurs only in the initial period: there exists $\theta^*$ such that $\theta_t^*(s^t) = \theta^*$ for all $t$. Furthermore*

$$\Sigma_{t=0}^{t=T} \delta^t \left[\phi(\theta^*, s_t) - n\frac{1 - F(\theta^*)}{f(\theta^*)} \frac{\partial \phi}{\partial \theta}(\theta^*, s_t) + \psi(\theta^*, s_t)\right] = 0.$$

*(ii) Under (possibly stochastic) decreasing attractiveness, the cutoffs $\theta_t^* = \theta_t^*(s_t)$ are increasing over time and satisfy for all $(t, s_t)$:*

$$\phi(\theta_t^*, s_t) + \psi(\theta_t^*, s_t) = n\frac{1 - F(\theta_t^*)}{f(\theta_t^*)} \frac{\partial \phi}{\partial \theta}(\theta_t^*, s_t).$$

Proposition 10 can be viewed as a generalization of Cournot $n^{th}$ marginalization to dynamic games of exit/retention. When demand is increasing, all exit occurs in the first period, like in the single-principal case; retention is a collective investment and free riding implies that there is less retention than if the principals coordinated their price choices. When demand is decreasing by contrast, exit occurs progressively and the remaining installed base is determined by the static Cournot $n^{th}$ marginalization condition. It again involves insufficient retention.

*Proof.* Let us first consider the "unconstrained optimization" at date $t$; that is, one considers the thought experiment in which no exit has yet occurred at date $t$ ($\theta_{t-1}^* = \underline{\theta}$). Of course, cutoff monotonicity is imposed from date $t$ on. Let $\widehat{\theta}_t$ be defined like in Proposition 10, but for the game starting at $t$ with no exit prior to date $t$; $\widehat{\theta}_t$ satisfies:

$$\Sigma_{\tau=t}^{\tau=T} \delta^\tau \left[\phi(\widehat{\theta}_t, s_\tau) - n\frac{1 - F(\widehat{\theta}_t)}{f(\widehat{\theta}_t)} \frac{\partial \phi}{\partial \theta}(\widehat{\theta}_t, s_\tau) + \psi(\widehat{\theta}_t, s_\tau)\right] = 0,$$

under deterministic increasing attractiveness, and $\widehat{\theta}_t = \theta_t^*$ (as defined in part (ii) of the proposition) under decreasing attractiveness.

Consider the following Markov strategies. Principals all charge price $p_t^i = \phi(\theta_t, s_t)/n$ where $\theta_t = \max\left\{\theta_{t-1}^*, \widehat{\theta}_t\right\}$. And the agent behaves myopically, as described just prior to the statement of the Proposition. Assumption 5 guarantees that provided that other principals charge $\phi(\theta_t, s_t)$, each principal's flow payoff is strictly quasi-concave in the cutoff.

Consider first decreasing demand. The strict quasi-concavity of the instantaneous payoff in the cutoff implies that a deviation from the presumed price reduces the principal's current payoff; the deviation has no impact at date $\tau > t$ in state $s_\tau$ provided that the induced cutoff $\theta_t^*$ satisfies $\theta_t^* \leq \theta_\tau(s_\tau)$. Furthermore if $\theta_\tau(s_\tau) < \theta_t^*$, then the date-$t$ deviation also reduces the date-$\tau$ payoff. The proof for decreasing demand follows the same steps. ■

## 6.2 Partnerships

Consider now a partnership (whether professional or in private life) composed of $n \geq 2$ agents. This partnership is dissolved if any of its members exits. Let $\phi_i(\theta_i, s_t)$ denote agent $i$'s date-$t$

surplus, where $\theta_i$ is his type, distributed according to c.d.f. $F_i(\theta_i)$ and density $f_i(\theta_i)$ , with support $[\underline{\theta}_i\,,\overline{\theta}_i]$ and $s_t$ is as earlier the date-$t$ state. The distributions $F_i$ are independent. The principal, as we explain below, is a coordinating and taxing entity, and so we assume that $\psi = 0$.

In their celebrated contribution, Myerson and Satterthwaite (1983) look at the problem of efficiently *forming* a partnership. They derive the mechanism that delivers the highest expected social surplus subject to individual rationality, incentive compatibility and budget balance. Assuming that virtual valuations $\phi_i(\theta_i\,,s) - \mu\Big[\dfrac{1-F_i(\theta_i)}{f_i(\theta_i)}\Big]\dfrac{\partial\phi_i}{\partial\theta_i}$ are strictly increasing in $\theta_i$ for $\mu \in [0,1]$ and for simplicity that the environment stationary ($s_t = s$ for all $t$), they show that a partnership is optimally formed if and only if for some $\mu$ in $(0,1)$ reflecting the intensity of the budget-balance constraint, the sum of the virtual valuations is positive:

$$\Sigma_i\left[\phi_i(\theta_i\,,s) - \mu\left[\frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right]\frac{\partial\phi_i}{\partial\theta_i}\right] \geq 0 \tag{2}$$

Myerson and Satterthwaite (1983)'s efficient bargaining corresponds to the case of a *benevolent social planner* eager to maximize expected total surplus but unable to put money on the table. Alternatively, one can look at a *profit-maximizing multi-sided platform*, in which a platform enables agents to interact and thereby enjoy partnership surplus. This latter case admits the same characterization (2), except that the coefficient $\mu$ is now equal to 1, and thus is larger than for a profit-maximizing platform than for a social planner. Finally, a social planner with a positive shadow cost of public funds would also deliver condition (2), again with a $\mu$ in $(0,1)$. We will therefore call the allocation defined by condition (2), which defines the contours of the partnership, the "Myerson-Satterthwaite allocation", regardless of the identity of the principal (social planner with or without cost of public funds, for-profit platforms).

This section by contrast looks at the possibility of *dissolving*[32] a partnership, and does so in a dynamic rather than static context. For notational simplicity only, we assume a stationary state ($s_t = s$ for all $t$) and a finite horizon $T$.

Let us assume that a mechanism is designed, in which the agents truthfully reveal their type; the allocation specified by the mechanism (partnership/no partnership, transfers) is then implemented. Under commitment, $X^t = 1$ (for all $t$) if and only if (2) is satisfied.

Suppose by contrast that in each period t and conditional on $X^{t-1} = 1$ (the partnership has not been dissolved), the principal designs a mechanism for that period. A mechanism defines an allocation $x_t$ for period $t$, as well as payments $p_{it}$ conditional on the agents' reports.

Suppose that the date-0 outcome delivers the Myerson-Satterthwaite outcome. Because from (2), $X^{t-1} = 1$ implies that $\Sigma_i\phi_i(\theta_i\,,s) > 0$, and so ex post there are always gains from trade. If the principal is a social planner preoccupied solely by the efficiency of trading, the principal

[32]The material presented here differs from Cramton et al (1987) in two essential aspects (see Segal and Whinston 2014 and the references therein for more recent contributions concerning the impact of status-quo outcomes on the efficiency of bargaining). First, we study dynamics and time consistency while Gibbons et al, like Myerson-Satterthwaite, focus on a one-shot trade. Second, Gibbons et al study a situation in which the agents in the partnerships have initial shares $r_i$ (with $\Sigma_i r_i = 1$) in the partnership and therefore status quo utility (in our notation) $r_i\phi_i(\theta_i\,,s)$. The goal is to reshuffle ownership rights toward the agent with the highest surplus $\max_i\{\phi_i(\theta_i\,,s)\}$. Their striking result is that there exists an efficient mechanism provided that the initial shares are "not too different". Here each agent can destroy the other agents' status-quo utility by quitting the relationship.

is indifferent as to the vector of transfers and so the utilities from date-1 on (which condition truthtelling at date 0) are indeterminate. To avoid this indifference, we rather study the case of a profit-maximizing principal or that of a social planner that puts at least a bit of weight on his budget (say, due to a shadow cost of public funds) and not only on efficiency. Proposition 11 below applies also if the principal has lexicographic preferences, maximizing first social surplus and, if indifferent, maximizing its revenue. Summing up, we let $X^t\left[\Sigma_i \phi_i(\theta_i\,, s) + \lambda p_{i\,t}\right]$ denote the principal's flow payoff at date $t$; the polar cases are that of a for-profit platform ($\lambda \longrightarrow \infty$) and of a lexicographic social planner ($\lambda \longrightarrow 0$).

*Multi-sided platform*: Without loss of generality,[33] one can assume that announcements at date t are made to a machine, which reveals whether the partnership continues, and that the date-t transfers are made only at the end of date $T$. The question is whether information beyond the minimal information transmission on the continuation of the relationship should be forwarded to the principal or to the agents. As for the principal, we will shortly show that provided that the date-0 allocation is efficient, the principal can costlessly learn types at date 1 even if he has no information; this is a fortiori the case if he receives information at the end of date 0. On the agents' side, and even if the mechanism conveys no other information than the continuation decision, each agent $i$ knows from date-1 on information about the other agents that is not available to the principal; as (2) indicates, conditional on $X^{t-1} = 1$, a higher $\theta_i$ makes agent $i$ more pessimistic about the others' types: in the dyad case for instance, agent 1 has posterior distribution $f_2(\theta_2|\theta_1) \equiv f_2(\theta_2)/[1 - F_2(\theta_2^*(\theta_1))]$ with support $[\theta_2^*(\theta_1), \overline{\theta}_2]$ where $\theta_2^*(\cdot)$ is a strictly decreasing function of $\theta_1$. We call this information structure the *minimal* (or coarsest) *information structure*.

Proposition 11 (whose proof can be found in Appendix B), considers only minimal information transmission and shows that if the date-0 allocation corresponds to the Myerson-Satterthwaite allocation, the principal at date $t \geq 1$ can design a mechanism that allows him to extract all agents' information at no cost and appropriate the total surplus $\Sigma_i \phi_i(\theta_i\,, s)$ forever.[34]

**Proposition 11 (time inconsistency with multiple agents).** *Consider a multi-period n-agent partnership, which is dissolved whenever an agent quits. Suppose that Assumption 4 holds and that the principal's utility is either strictly increasing in money or lexicographic in efficiency and then money. There is no efficient and time-consistent allocation such that the agents learn at the end of each period only whether the relationship continues or not.*

Thus, even though agents consume zero or one unit of the partnership, they need to reveal more than just whether they want to stay in the partnership at the current price. And because each agent reveals fine information about himself, he necessarily learns from the partnership not being dissolved information about the other agents' types, even if he does not observe their

[33] See Myerson (1982).

[34] Like most of the static literature on the elicitation of correlated informations, we assume unlimited transfers and no collusion among the agents. Little is known outside this framework. Robert (1991) shows that unlimited transfers are needed if informations are nearly independent. Cremer (1996) provides results on coalitions in auctions with correlated values when both the auction and the coalition formation must be in dominant strategies.

reports. This impossibility result raises interesting research questions; in particular we leave for future research the characterization of the time-consistent solution.

## 7 Alleys for future research

This paper provides first insights on dynamic screening with positive selection. The main ones were summarized in the introduction. This conclusion therefore focuses on future research. At least four broad areas of research seem worth pursuing.

First, we saw that shifting types, common agency and partnerships all disconnect the resolution of the dynamic screening game from the simple optimization problem associated with commitment. While we showed that the simple structure of screening with positive selection allows for interesting characterizations, much work remains to be done in order to obtain general predictions for these environments.

Second, the model should be generalized to allow for competition among principals. Firms compete for employees and consumers, department for professors, religions for followers, municipalities and countries for plants and headquarters, languages for speakers, and so forth, and principals and their agents are engaged as in this paper in relationships of endogenous lengths. A richer model would formalize not only the retention policies studied here (human resource management, customer relationship management, evolution of financial and non-financial terms), but also how mobility affects the policies of competing principals and the agents' reservation utilities attached to splitting from their principal.

Third, the model could be enriched in several dimensions. It is hard to predict without further inquiry whether these extensions will deliver insights that go beyond a mere combination of existing insights. But it seems for example worthwhile to add private information held by the principal. The commitment case would involve mechanism design by an informed principal, and the non-commitment case repeated signaling. One could then study the role of commitment in this enlarged framework. Similarly, we assumed (finite or infinite) re-entry costs to be exogenous. While this assumption may be reasonable in a number of contexts, one could also allow the principal or the agent to impact this re-entry cost.

Fourth, the paradigm should be enlarged to accommodate political economy considerations. In a number of environments (such as religions or firms), the principal's preferences can be taken as exogenous in a first approximation. However, as Dewatripont and Roland (1992) stress, the principal's preferences may result from a vote or power relationships, and therefore change with the composition of the in- and out-groups; for instance, religious conversions may affect the balance of political power and quits may have a long-lasting effect on the orientation of an academic department. Political economy considerations add a new form of (positive or negative) network externalities, which are intrinsically dynamic rather than contemporaneous.

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# Appendix A. Sketch of the proof of Proposition 2

For notational simplicity only we will assume that the path of the state $s_t$ is deterministic.

*(i) Impatient agent ($\delta_A \le \delta$).* Consider the optimal allocation when the principal can commit and the agent cannot. Let $\{\theta_t^*\}_{t=0,\ldots,T}$ denote the sequence of optimal cutoffs and $\{p_t\}_{t=0,\ldots,T}$ the contributions. Because the agent cannot commit, these cutoffs must satisfy

$$\phi\left(\theta_T^*, \theta_T^*, s_T\right) - p_T \ge 0,$$
$$\phi\left(\theta_{T-1}^*, \theta_{T-1}^*, s_{T-1}\right) - p_{T-1} + \delta_A \max\left\{0, \phi\left(\theta_{T-1}^*, \theta_T^*, s_T\right) - p_T\right\} \ge 0$$
$$\ldots$$

Fix the cutoffs and optimize the principal's payoff (whose expression is the same as earlier, because we have taken $\delta$ to be the principal's discount factor) with respect to contributions. Suppose that $\phi\left(\theta_T^*, \theta_T^*, s_T\right) > p_T$. Then, consider new contributions $(\hat{p}_{T-1}, \hat{p}_T)$ such that

$$\hat{p}_{T-1} + \delta_A \hat{p}_T = p_{T-1} + \delta_A p_T$$

and

$$\hat{p}_T = \phi\left(\theta_T^*, \theta_T^*, s_T\right).$$

The new cutoffs satisfy $\hat{\theta}_T^* = \theta_T^*$ and $\hat{\theta}_t^* \le \theta_t^*$ for $t \le T-1$. From Assumption 2 and $\delta_A \le \delta$, the principal's payoff is increased. Repeat this reasoning; backward induction then shows that at the optimal allocation, the contributions satisfy:

$$p_t = \phi\left(\theta_t^*, \theta_t^*, s_t\right) \quad \text{for all } t.$$

Finally, the agent's discount factor is irrelevant under cutoff myopia as the cutoff's continuation valuation is always equal to 0. Hence $V^{nc}(\delta_A) = V^*$.

*(ii) Patient agent ($\delta_A > \delta$).* Suppose that the principal cannot commit. Then any price $p_T < \phi\left(\theta_T^*, \theta_T^*, s_T\right)$ is strictly suboptimal; hence $p_T = \phi\left(\theta_T^*, \theta_T^*, s_T\right)$. By backward induction, cutoff myopia prevails on the equilibrium path, and so again $\delta_A$ is irrelevant. But the commitment solution is not time-consistent: The principal would like to frontload contributions, which requires commitment, as already noted in the text. ■

# Appendix B. Proof of Proposition 11

Suppose that the agents tell the truth at date 0 (otherwise the optimal allocation cannot be implemented at date 0) and assume minimal information transmission. Let

$$\xi_i \equiv \sum_{j \ne i} \left[\phi(\theta_j\,, s) - \mu\left[\frac{1 - F_j(\theta_j)}{f_j(\theta_j)}\right] \frac{\partial \phi_j}{\partial \theta_j}\right] \tag{3}$$

denote a random variable, with distribution $H_i(\xi_i)$ on $\mathbb{R}$. Suppose that the Myerson-Satterthwaite allocation defined by (2) is time-consistent. For any announcement $\hat{\theta}_i^0$ at date 0 (not necessarily $\theta_i$, as we allow for a unilateral deviation) the partnership is not dissolved at that date if and only if

$$\xi_i \geq K_i(\hat{\theta}_i^0),$$

where

$$K_i(\theta_i) \equiv -\left[\phi(\theta_i\,, s) - \mu\left[\frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right]\frac{\partial \phi_i}{\partial \theta_i}\right],$$

is a decreasing function of $\theta_i$. Assume that at date 1 the principal and the agents know only that the relationship has continued at date 0. Choose an arbitrary function $w_i(\xi_i, \hat{\theta}_i^1)$ that is strictly decreasing in $\xi_i$ and strictly decreasing in $\hat{\theta}_i^1$ and such that agent $i$ breaks even provided that the announcement corresponds to the true conditional distribution of the $\xi_i$:

$$\int_{K_i(\hat{\theta}_i^1)}^{\infty} w_i\left(\xi_i, \hat{\theta}_i^1\right) dH_i(\xi_i) = 0.$$

Let the principal at date 1 offer the following side-bet mechanism, in which all agents reveal their types and agent $i$ receives a side-bet payment $\hat{p}_i(\hat{\theta}_i^1\,, \xi_i)$, where $\xi_i$ is computed as in (3) from the date-1 other reports and $\hat{\theta}_i^1$ is agent $i$'s report of his own type. Let $\hat{p}_i(\hat{\theta}_i^1\,, \xi_i) = -\infty$ if $\xi_i < K_i(\hat{\theta}_i^1)$ (this rules out under-reports $\hat{\theta}_i^1 < \hat{\theta}_i^0$) and for some $k > 0$,

$$\hat{p}_i\big(\hat{\theta}_i^1\,, \xi_i\big) = k\, w_i\big(\xi_i\,, \hat{\theta}_i^1\big) \quad \text{if} \quad \xi_i \geq K_i(\hat{\theta}_i^1).$$

The fact that $w_i$ is strictly decreasing in $\hat{\theta}_i^1$ rules out over-reports $\big(\hat{\theta}_i^1 > \hat{\theta}_i^0\big)$.

Note that with this mechanism, $\hat{\theta}_i^1 = \hat{\theta}_i^0$ is optimal on a stand-alone basis. But $\hat{\theta}_i^0 = \theta_i$ if the Myerson-Satterthwaite allocation is to be implemented already at date 0. So necessarily $\hat{\theta}_i^1 = \theta_i$. Furthermore the loss from lying for any $|\hat{\theta}_i - \theta_i| > \varepsilon$ (for $\varepsilon$ small) goes to infinity as $k$ goes to infinity. Last, link the mechanism with the demand of a payment for staying in the partnership

$$p_i(\hat{\theta}_i^1) = \phi(\hat{\theta}_i^1\,, s) - \varepsilon'$$

for some small $\varepsilon'$; $p_i$ is thus increasing in $\hat{\theta}_i^1$. The total date-1 payment is then $p_i\left(\hat{\theta}_i^1\right) - \hat{p}_i\left(\hat{\theta}_i^1\,, \xi_i\right)$. By taking $k$ to infinity, all agents report their date-0 report ($\hat{\theta}_j^1 = \hat{\theta}_j^0$ for all $j$) and the principal fully extracts the agents' surplus from date-1 on.

Thus if the agents tell the truth at date 0 and the Myerson-Satterthwaite allocation is implemented at that date, the principal perfectly extracts the agents' rent from date 1 on. Therefore, if $U_i^0(\theta_i)$ denotes the intertemporal payoff of type $\theta_i$ and $x_i^{\mathrm{MS}}(\theta_i)$ denotes her ex-ante probability of trade in the Myerson-Satterthwaite allocation (given by (2)), then

$$\frac{dU_i^0(\theta_i)}{d\theta_i} = x_i^{\mathrm{MS}}(\theta_i)\,\frac{\partial \phi_i}{\partial \theta_i}.$$

However , the consideration of repeated small under-announcements $\hat{\theta}_i^t = \hat{\theta}_i^0 = \theta_i - \varepsilon$ for $\varepsilon$ small, yields

$$\frac{dU_i^0(\theta_i)}{d\theta_i} = x_i^{\mathrm{MS}}(\theta_i)\big[1 + \delta + \cdots + \delta^T\big]\,\frac{\partial \phi_i}{\partial \theta_i},$$

a contradiction. ∎