Common Agency with Informed Principals[1]

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Abstract

The provision of public goods under asymmetric information has generally been viewed as a mechanism design problem under the aegis of an uninformed mediator. This paper focuses on institutional contexts where no such mediator is available. Contributors privately informed on their own willingness to pay non-cooperatively offer contributions to an agent who produces the public good on their behalf. This institutional setting is thus viewed as a common agency game with privately informed principals. Instead of reducing their marginal contributions to free-ride on others, principals do so to screen the agent's endogenous private information from observing other principals' offers. Under weak conditions, there always exists at least one differentiable equilibrium. Equilibria are always ex post inefficient. Interim efficiency most often fails except under rather stringent assumptions on the type distribution.

Keywords: Common agency, informed principals, public goods, ex post and interim efficiency.

1 Introduction

Since Green and Laffont (1979) the provision of public goods under asymmetric information has mostly been viewed as a mechanism design problem under the aegis of an uninformed mediator having a full commitment ability. This paper relaxes this assumption and focuses on cases where no such mediator is available. Contributors privately informed on their own willingness to pay non-cooperatively offer contributions to an agent who produces the public good on their behalf.

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Our first motivation comes from observing that, in much real-world settings, centralized mechanisms and uninformed mediators with a strong ability to commit to those mechanisms might not be available. Health, environment, global warming, terrorism, multilateral foreign aid and other transnational public goods are all examples of public goods which are voluntary provided by sovereign countries. There exists no mediator to design the mechanisms that those countries should play to reveal their preferences. Politics and games of influence among interest groups offer other important examples. Key decision-makers might not have much commitment power to organize and design ex ante the competition between interest groups. Instead, they only react ex post to the lobbying contributions they receive from those groups.[4] Similarly, politicians await for interest groups' contributions to their electoral campaigns. In those contexts, it is important to know whether a game of voluntary contributions fares well under asymmetric information.

A second motivation for our study is more theoretical. Even though the earlier works on asymmetric information (Clarke 1971, Groves 1973) studied particular mechanisms for the provision of public goods, the bulk of this literature has departed from the analysis of real-world institutions to characterize instead properties of the whole set of incentive-feasible allocations.[5] In the standard framework, often referred to as the centralized mechanism approach in what follows, an uninformed mediator moving first designs and offers a mechanism to informed players. This mechanism induces an equilibrium allocation which is Bayesian incentive compatible, feasible and might respect the agents' veto constraints. No other institutional constraint on the kind of mechanisms that can be used is considered. We add the extra requirement that such allocation comes from playing a game of voluntary contributions under asymmetric information. In such a game, a given contributor might want to offer a contribution scheme flexible enough to cope with different realizations of others' preferences. An a priori uninformed agent collects contributions, learns something about the contributors' preferences in the process and acts strategically in response. This institutional setting is thus viewed as a common agency game with privately informed principals non-cooperatively designing contributions.

At a best-response to what others offer, a given principal designs his own contribution not only to signal his preferences to the agent but also to extract information that this agent may have learned from observing others' offers. Applying the insights from the Theory of Informed Principals, signalling is costless in environments with private values and risk-neutrality.[6] Instead, screening is costly: Principals have to learn *"market information"* in equilibrium as pointed out by Epstein and Peters (1999) and Peters (2001).[7] However, this kind of information is endogenous: This is what other principals reveal to the common agent through their mere offers. Standard mechanism design techniques can nevertheless be used to compute best-responses. When choosing how much to contribute

[4] Grossman and Helpman (1994).

[5] This research strategy of the public good literature stands in sharp contrast with how the literature on auctions has evolved. There, an equal effort has been devoted to the study of particular auction formats on one side and to the characterization of the general properties of unrestricted auction mechanisms on the other. The present paper aims at filling this gap in the case of the public good literature.

[6] Maskin and Tirole (1990).

[7] Those authors derived Revelation Principles for multi-principal environments where principals' preferences are common knowledge. Market information might then capture the endogenous randomness that arises in mixed-strategy equilibria.

for q units of the public good, each principal behaves actually as a monopsonist in front of an agent endogenously privately informed on the preferences of others. By a standard argument of the screening literature,[8] this principal reduces his marginal contribution to decrease the agent's endogenous information rent.

There always exists a differentiable equilibrium of the game under some weak conditions on distributions. For a given equilibrium output, the marginal contribution in such equilibrium solves a complex functional equation with stringent boundary conditions. Those boundary conditions come from characterizing the bidding behavior of the two principals, who have either the highest or the lowest valuations, and who altogether induce the equilibrium output under scrutiny.

Importantly, the desire of each principal to screen the agent about his endogenously learned private information implies that any equilibrium is necessarily ex post inefficient. Principals contribute less at the margin than what the public good is worth to them. This is not because they want to hide their types to the common agent as the centralized mechanism design approach predicts in its interpretation of the "free-riding" problem.[9] Instead, principals induce less production of the public good to reduce the information rent that the agent gets from learning the preferences of others. Downward distortions below the first-best occur.

Being given the ex post inefficiency of equilibria, we may look for weaker efficiency properties and ask whether equilibria can be interim efficient, i.e., solutions to a centralized mechanism design problem under the aegis of an uninformed mediator.[10] This is an extremely stringent requirement. Beyond the case of type distributions with linear hazard rates, interim efficiency is unlikely. This suggests that public intervention under the aegis of an uniformed mediator is helpful in coordinating contributions.

Section 2 reviews the literature. Section 3 presents the model. Section 4 shows how to derive differentiable equilibria of the common agency game under asymmetric information. Section 5 presents the Lindahl-Samuelson conditions for games of voluntary provision under asymmetric information. We also give there tractable examples of equilibria and derive, more generally, the ex post inefficiency of any equilibrium. Section 6 derives existence and some properties of the equilibria. Section 7 investigates interim efficiency. Section 8 shows that differentiable Bayesian equilibria found are not dominant strategy equilibria. Section 9 briefly concludes. Proofs are relegated to an Appendix.

2 Review of the Literature

Following the seminal contributions of Wilson (1979) and Bernheim and Whinston (1986a), the common agency literature has developed an analytical framework to tackle a variety

[8] Laffont and Martimort (2002, Chapter 3).

[9] Laffont and Maskin (1982), Güth and Hellwig (1987), Rob (1989) and Mailath and Postlewaite (1990)

[10] Holmström and Myerson (1983) and Ledyard and Palfrey (1999). Because the agent might get a positive rent in equilibrium from his endogenous private information, he must also receive a positive weight in the social welfare function maximized by that mediator.

of important problems such as menu auctions,[11] public goods provision through voluntary contributions,[12] or policy formation with competing lobbying groups in complete information environments.[13] Imposing that contributions are "truthful", i.e., reflect the relative preferences of the principals among alternatives, Bernheim and Whinston (1986a) reduced the equilibrium indeterminacy of those games and selected efficient equilibria.[14] With such truthful schedules, what a principal pays at the margin for inducing a change in the agent's decision is exactly what it is worth to him and the "free-riding" problem in public good provision cannot arise. Modulo truthfulness, common agency aggregates preferences efficiently under complete information.[15] Modelling private information on the principals' side justifies the use of nonlinear contributions for screening purposes in the first place. The "truthfulness" requirement is then replaced by incentive compatibility conditions. The cost for the modeler of putting on firmer foundations the use of schedules is that ex post efficiency is lost and the conditions for interim efficiency quite severe.

Paralleling those complete information papers, Stole (1991), Martimort (1992, 1996), Mezzetti (1997), Biais, Martimort and Rochet (2000) and Martimort and Stole (2002, 2003) among others analyzed oligopolistic screening environments where different principals elicit information privately known by the common agent at the contracting stage. These papers stressed the impact of oligopolistic screening on the standard rent/efficiency trade-off. We focus instead on asymmetric information on the principals' side. The agent's private information vis à vis each principal is endogenous: It is what the agent may have learned from observing the other principals' offers.

Contrasting with the use of schedules stressed by Bernheim and Whinston (1986), the complete information literature on voluntary provision of public goods has highlighted inefficiency and "free-riding" in models where contributors are restricted to offer fixed contributions (Bergstom, Blume and Varian, 1986). Other solutions to this inefficiency problem include refunds (Bagnoli and Lipman, 1989) and multi-stage mechanisms in environments with partially verifiable information (Jackson and Moulin, 1992).

There exists a tiny literature on voluntary contributions by privately informed agents for a 0-1 public good. Menezes, Monteiro and Temini (2001) focused on the strong ex post inefficiency of those equilibria whereas Laussel and Palfrey (2003) found more positive results using interim efficiency. Our focus on a continuum level of public good invites the use of a differentiable approach. The requirement of interim efficiency is now much more stringent since it should apply not only on a line in the types space as in the 0-1 case (namely the set of types for which there is indifference between producing or not the

[11]Wilson (1979) and Bernheim and Whinston (1986a).

[12]Laussel and Lebreton (1998).

[13]Grossman and Helpman (1994) and Dixit, Grossman and Helpman (1997), among many others.

[14]Multiplicity might still come from the flexibility in sharing the aggregate surplus among the contributing principals and their common agent. The feasible redistributions of the aggregate surplus are described by means of simple inequalities. See Bernheim and Whinston (1986a).

[15]These results have been extended in many different directions. Dixit, Grossman and Helpman (1997) introduced redistributive concerns by relaxing the quasi-linearity assumption. Laussel and Lebreton (1998) studied incomplete information on the preferences of the common agent but focused on ex ante contracting when agency costs are null. Other extensions less directly relevant for the analysis of this paper include Prat and Rustichini (2003) who studied competition among principals trying to influence multiple agents and Bergemann and Välimäki (2003) who considered dynamic issues.

public good) but on the whole types space. This is in general too demanding.

Finally, it is also useful to place our contribution within the existing mechanism design literature. From Clarke (1971) and Groves (1973), it is well-known that ex post efficiency is possible under dominant strategy implementation. D'Aspremont and Gerard-Varet (1979) showed that one can maintain budget balance and efficiency under Bayesian implementation. Laffont and Maskin (1982), Güth and Hellwig (1987), Rob (1989) and Mailath and Postlewaite (1990) stressed the role of participation constraints to generate inefficiency. A game of voluntary contributions ensures participation by principals, relies on Bayesian strategies, and finally generates a positive surplus for the agent. Hence, ex post inefficiency necessarily arises. When a centralized mechanism is offered by an uninformed mediator, inefficiencies are due to the contributors' incentives to hide their own types to this mediator: the so-called "free-riding" problem. Under common agency, as we will see below, contributors reveal instead costlessly their types through their contract offers to the agent but want to screen this agent according to what he has learned from others. This is no longer contributors who underestimate their valuations but their common agent who wants to claim to each principal that the other has little willingness to pay: a somewhat different source of inefficiency in public goods provision.

3 The Model

Consider two risk-neutral principals P_i $(i = 1, 2)$ who derive utility from consuming an excludable public good which is produced in non-negative quantity q.[16,17] This public good may be an infrastructure of variable size, a charitable activity, or it may also have a more abstract interpretation as a policy variable in a lobbying game for instance. P_i gets a utility $V_i = \theta_i q - t_i$ from consuming q units of the good when he pays an amount t_i.

Principals are privately informed on their respective valuations θ_i. Those types are independently drawn from the same common knowledge and atomless distribution on $\Theta = [\underline{\theta}, \bar{\theta}]$ with c.d.f. $F(\cdot)$ and everywhere positive density $f = F'$.[18] Unless specified otherwise, we assume that $\underline{\theta} > 0$ and $\bar{\theta} < \infty$, i.e., the distribution has bounded support. The hazard rate $R(\theta) = \frac{1-F(\theta)}{f(\theta)}$ is non-increasing. $E_\theta[\cdot]$ denotes expectation w.r.t. θ.

Contributions are collected by a risk-neutral common agent A who produces at cost $C(q)$ the public good. The function $C(\cdot)$ is twice differentiable, increasing and convex with $C(0) = C'(0) = 0$ where the Inada condition avoids technicalities.

Benchmark: Let $q^{FB}(\theta_1, \theta_2)$ be the first-best level of public good. $q^{FB}(\cdot)$ is increasing in both arguments and satisfies the Lindahl-Samuelson conditions:

$$C'(q^{FB}(\theta_1, \theta_2)) = \sum_{i=1}^{2} \theta_i.$$

[16] Extension to the case of more than two principals increases significantly complexity and are left for further research.

[17] The public good can also be produced in quantity 0 and 1 and q is then viewed as its quality.

[18] The density may be zero at $\bar{\theta}$ only in Proposition 8 below.

Mechanisms: We do not want to put a priori any restriction on the kind of communication that may take place between each informed principal and the agent. From Myerson (1983) and Maskin and Tirole (1990), we know indeed that there might be some value in delaying information revelation in an informed principal setting. In our context, a contribution $\tilde{t}_i(q, m_i)$ specifies thus a monetary transfer from P_i to the agent depending both on the output level q and on a message m_i that the principal may choose to send to the agent, ex post, i.e., once the agent has already chosen the level of public good. The message m_i belongs to a set $\mathcal{M}'_i \cup \{\emptyset\}$ where $\mathcal{M}'_i$ is measurable but otherwise arbitrary.[19] The fact that the principal may choose not to send any message ex post is captured by including the null option $\{\emptyset\}$ in the message space. Let $\mathcal{M}_i = \mathcal{M}'_i \cup \{\emptyset\}$ be such extended communication space, $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ be the product set of these sets and m be a generic element of this product set. Contributions map thus $\mathbb{R}^+ \times \mathcal{M}_i$ into $\mathbb{R}$. A mechanism offered by P_i is an array of such contributions $\{\tilde{t}_i(\cdot, m_i)\}_{m_i \in \mathcal{M}_i}$.[20] Let $\mathcal{T}_i(\mathcal{M}_i)$ be the set of mechanisms available to P_i for a given message space, their product be $\mathcal{T}(\mathcal{M}) = \mathcal{T}_1(\mathcal{M}_1) \times \mathcal{T}_2(\mathcal{M}_2)$ and $\tilde{t} = (\tilde{t}_1, \tilde{t}_2)$ be any generic element of this product set.

Together with the principals' preferences and the information structure, these strategy spaces define the common agency game under incomplete information Γ.

Timing: The game Γ unfolds as follows:

• Stage 0: Principals learn their preferences.

• Stage 1: Principals non-cooperatively offer to the agent the mechanisms $\{\tilde{t}_i(q, m_i)\}_{m_i \in \mathcal{M}_i}$ for some communication spaces $\mathcal{M}_i$, $i = 1, 2$.

• Stage 2: The agent accepts or refuses those contracts. If he refuses all contracts, the game ends with zero payoffs for all players. If the agent refuses one of the principals' contribution, the latter gets zero payoff because the public good is excludable.[21]

• Stage 3: The agent produces an amount q.

• Stage 4: A principal P_i whose offer has been accepted sends message m_i to the agent.

• Stage 5: Payments are made according to the agent's acceptance decisions, his choice of output and the messages sent by principals.

We focus on pure-strategy Perfect Bayesian Equilibria (PBE) of this game.[22]

[19] To properly define strategy sets in the game, we assume that those message spaces are included into some larger set $\bar{\mathcal{M}}$.

[20] Our definitions can be straightforwardly extended to the case of random mechanisms and contributions. We focus in the sequel on pure strategy equilibria and choose accordingly to simplify presentation.

[21] Mechanisms could include also a communication stage where the principal sends messages to the agent after his acceptance but before he chooses an output. If the agent cannot be forced to play this mechanism after the first round of communication by the principal, we are back to our original timing.

[22] Because of the symmetry between the players, we focus on symmetric equilibrium contributions and we may omit subscripts.

4 Informed Principals

4.1 Preliminaries

Of particular importance are mechanisms such that all communication between principal P_i and the agent takes place at stage 1, i.e., through the mere offer he makes.

Definition 1 *: A mechanism* $\{\tilde{t}_i(q, m_i)\}_{m_i \in \mathcal{M}_i}$ *has direct communication if and only if* $\mathcal{M}_i = \{\emptyset\}$.[23]

This class of mechanisms is important because we have:

Lemma 1 *Assume that* P_{-i} *offers a direct communication mechanism in a given pure-strategy equilibrium of* Γ*. Then* P_i *has in his best-response correspondence a direct communication mechanism.*

Fix any direct communication mechanism offered by P_{-i} at an equilibrium, the design of P_i's best-response to such mechanism can be viewed as an informed principal problem under private values. Two facts follow from Maskin and Tirole (1990). First, instead of offering a mechanism to the agent with a communication stage after contract's acceptance and output choice, P_i is as well-off communicating his type only through the mere contract offer. This is the essence of Lemma 1 which justifies our focus on equilibria with direct communication mechanisms.[24]

Second, there is no loss of generality in having P_i to offer a contract to the agent exactly as if the latter was fully informed on the principal's type.[25] Because the agent's utility function does not depend directly on the principal's type, the agent's beliefs on that type do not matter and the agent's acceptance decisions and output choice depend only on the offered contributions. We have then:

Lemma 2 *Assume that* P_{-i} *offers a direct communication mechanism in any equilibrium of* Γ*. Then* P_i*'s best-response correspondence contains a direct communication mechanism which corresponds to* P_i*'s optimal contract had the agent known* P_i*'s type.*

[23] A direct communication mechanism is actually a nonlinear contribution, and the agent communicates to the corresponding principal through his output choice. See Section 4.2 for the discussion of direct revelation mechanisms where the agent communicates directly his endogenous information to principals.

[24] This is a reasoning already found in Bernheim and Whinston (1986a). To refine with the truthfulness criterion among all equilibria under complete information, they indeed first noticed that each principal has a best-response which is truthful and thus justified that focusing on equilibria in truthful schedules is meaningful. We apply a similar device to focus on equilibria with direct communication mechanisms.

[25] The mechanism consisting in piling up the various optimal contracts that would be signed by those different types if the agent was informed on P_i's preferences is incentive compatible from the principal's point of view and achieves a lower bound on the principal's payoffs. The key insight due to Maskin and Tirole (1990) is that higher payoffs can only be achieved if the principal is risk-averse. This is obtained by pooling those contracts at the offer stage and revealing the principal's type only at a later communication stage. Pooling relaxes the agent's incentive and participation constraints and improves risk-sharing among the different types. With risk-neutrality, this insurance motive disappears and the lowest bound on the principal's payoff is also an upper bound. See also Myerson (1983) and the "unscrutability principle".

We slightly abuse notations and rewrite the equilibrium choice of P_i when his type is θ_i as $\tilde{t}_i^*(q, \emptyset|\theta_i) = t_i(q, \theta_i)$. Let then $p_i(q, \theta_i) = \frac{\partial t_i}{\partial q}(q, \theta_i)$ be the equilibrium marginal contribution for such direct communication mechanism. Of particular importance are contributions such that an upward shift in P_i's valuation increases the equilibrium quantity. Such schedules satisfy the same Spence-Mirrlees property as the principals' preferences.[26]

Definition 2 *: A contribution $t_i(q, \theta_i)$ satisfies the Spence-Mirrlees Property (SMP) when $\frac{\partial p_i}{\partial \theta_i}(q, \theta_i) \geq 0$ for all (q, θ_i).*

4.2 Computing Best-Responses

From Lemmas 1 and 2 we know that, for any direct communication mechanism offered by P_i, P_{-i} has also in his best-response correspondence a direct communication mechanism whose offer reveals his type to the agent. Therefore, P_i should design his own contribution to screen P_{-i}'s type which is endogenously learned by the agent. This points at the role that contributions play in a common agency environment: learning over what Epstein and Peters (1999) and Peters (2001) call *market information*, i.e., other principals' preferences.

To compute P_i's best-response to any SMP contribution $t_{-i}(q, \theta_{-i})$ offered by P_{-i}, we now invoke the Revelation Principle[27] and restrict as well the analysis to direct truthful revelation mechanisms $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q^D(\hat{\theta}_{-i}|\theta_i)\}$ inducing the agent to reveal to P_i what he has learned by observing P_{-i}'s offer. Let $\hat{\theta}_{-i}$ denote the agent's report on θ_{-i} (that he has learned from observing P_{-i}'s offer). The agent's utility can then be written as:

$$\tilde{U}^D(\hat{\theta}_{-i}, \theta_{-i}|\theta_i) = t_i^D(\hat{\theta}_{-i}|\theta_i) + t_{-i}(q^D(\hat{\theta}_{-i}|\theta_i), \theta_{-i}) - C(q^D(\hat{\theta}_{-i}|\theta_i)).$$

From incentive compatibility, we get:

$$U^D(\theta_{-i}|\theta_i) = \tilde{U}^D(\theta_{-i}, \theta_{-i}|\theta_i) = \max_{\hat{\theta}_{-i}} \tilde{U}^D(\hat{\theta}_{-i}, \theta_{-i}|\theta_i). \quad (1)$$

Using standard techniques, we get:

Lemma 3 *Assume that P_{-i} offers a non-negative twice-differentiable contribution $t_{-i}(q, \theta_{-i})$ which satisfies SMP. The following properties are satisfied by the direct revelation mechanism $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q^D(\hat{\theta}_{-i}|\theta_i)\}$ offered by P_i at a best-response:*

- *$U^D(\theta_{-i}|\theta_i)$ is a.e. differentiable in θ_{-i} with*

$$\frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) = \frac{\partial t_{-i}}{\partial \theta_{-i}}(q^D(\theta_{-i}|\theta_i), \theta_{-i}). \quad (2)$$

- *$q^D(\theta_{-i}|\theta_i)$ is monotonically increasing and thus a.e. differentiable in θ_{-i} with*

$$\frac{\partial q^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0 \qquad a.e. \quad (3)$$

[26]For technical reasons, we focus on contributions which are three times piece-wise differentiable.

[27]See Martimort and Stole (2002, 2003) for this way of applying the Revelation Principle to compute best-responses in pure strategy equilibria of common agency games.

These conditions are also sufficient for global optimality of the agent's problem (1).

At a best-response to any non-negative contribution $t_{-i}(q,\theta_{-i})$, a principal P_i with type θ_i must solve the following mechanism design problem:

$$(\mathcal{P}_i): \quad \max_{\{U^D(\cdot|\theta_i);q^D(\cdot|\theta_i)\}} E_{\theta_{-i}}\left[\theta_i q^D(\theta_{-i}|\theta_i) + t_{-i}(q^D(\theta_{-i}|\theta_i),\theta_{-i}) - C(q^D(\theta_{-i}|\theta_i)) - U^D(\theta_{-i}|\theta_i)\right] \tag{4}$$

subject to (2)-(3) and

$$U^D(\theta_{-i}|\theta_i) \geq \hat{U}_{-i}(\theta_{-i}), \quad \text{for all } \theta_{-i} \in \Theta. \tag{5}$$

$\hat{U}_{-i}(\theta_{-i}) = \max_{q\in\{0\}\cup rg(q^D(\cdot|\theta_{-i}))} t_{-i}(q,\theta_{-i}) - C(q) \geq 0$[28] is the agent's rent when either accepting only P_{-i}'s contribution or none. The ex post participation constraint (5) ensures that the agent prefers to take both contributions at all profiles (θ_i,θ_{-i}).

A solution to $(\mathcal{P}_i)$ is an allocation $\{U^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}$ or equivalently a direct revelation mechanism $\{t_i^D(\theta_{-i}|\theta_i), q^D(\theta_{-i}|\theta_i)\}$[29] from which we can reconstruct a nonlinear contribution $t_i(q,\theta_i)$ when the output schedule $q(\theta_{-i}|\theta_i)$ is invertible.

The standard techniques for solving problems like $(\mathcal{P}_i)$ in monopolistic screening environments consist in first neglecting the second-order condition (3), second integrating by parts the expected rent left to the agent, and third maximizing pointwise with respect to output the virtual surplus function obtained thereby. A first difficulty is to ensure the concavity of this virtual function since it depends on the other principal's offer $t_{-i}(q,\theta_{-i})$ which is endogenous. A second difficulty comes from checking that (3) holds. It turns out that those difficulties can be handled altogether when principal P_{-i} offers a contribution $t_{-i}(q,\theta_{-i})$ such that the corresponding function $\psi_{-i}(q,\theta)$ defined as

$$\psi_{-i}(q,\theta_{-i}) = -p_{-i}(q,\theta_{-i}) + C'(q) + R(\theta_{-i})\frac{\partial p_{-i}}{\partial \theta_{-i}}(q,\theta_{-i})$$

satisfies $\frac{\partial \psi_{-i}}{\partial q}(q,\theta_{-i}) > 0$ and $\frac{\partial \psi_{-i}}{\partial \theta_{-i}}(q,\theta_{-i}) < 0$. Provided that both conditions hold, P_i's best-response is well-defined and entails an optimal output which is strictly increasing with respect to θ_{-i}. This output is defined by the following pointwise optimality condition:

$$\theta_i = \psi_{-i}(q^D(\theta_{-i}|\theta_i),\theta_{-i}). \tag{6}$$

4.3 Equilibria

At a symmetric equilibrium with marginal contribution $p(q,\theta)$, we denote respectively the output and the agent's rent as $q^D(\theta_{-i}|\theta_i) = q^D(\theta_i|\theta_{-i}) = q(\theta_1,\theta_2)$ and $U^D(\theta_{-i}|\theta_i) = U^D(\theta_i|\theta_{-i}) = U(\theta_1,\theta_2)$. Equation (6) becomes:

$$\theta_i + p(q(\theta_1,\theta_2),\theta_{-i}) - C'(q(\theta_1,\theta_2)) = R(\theta_{-i})\frac{\partial p}{\partial \theta_{-i}}(q(\theta_1,\theta_2),\theta_{-i}) \text{ for } i = 1,2. \tag{7}$$

[28] $rg(q(\cdot))$ denotes the range of $q(\cdot)$. We set $t_{-i}(0,\theta_{-i}) = 0$ because there should not be any contribution if the public good is not produced. This explains that we add the option $\{0\}$ to the possible outputs.

[29] We omit the implicit dependence on $t_{-i}(q,\theta_{-i})$.

At the last stage of the game, the agent chooses optimally the level of public good given that he has accepted both contributions. The interior level of public good must, on top of (7), also satisfy the following first-order condition:

$$\sum_{i=1}^{2} p(q(\theta_1,\theta_2),\theta_i) = C'(q(\theta_1,\theta_2)). \tag{8}$$

This condition is also sufficient provided that the local second-order condition for the agent's problem holds, namely:

$$\sum_{i=1}^{2} \frac{\partial p}{\partial q}(q(\theta_1,\theta_2),\theta_i) - C''(q(\theta_1,\theta_2)) \leq 0. \tag{9}$$

Proposition 1 *Take a non-negative marginal contribution $p(q,\theta)$ that satisfies (7), (8), (9) and such that*

$$\psi(q,\theta) = -p(q,\theta) + C'(q) + R(\theta)\frac{\partial p}{\partial \theta}(q,\theta) \tag{10}$$

satisfies

$$\frac{\partial \psi}{\partial q}(q,\theta) > 0 \text{ and } \frac{\partial \psi}{\partial \theta}(q,\theta) < 0. \tag{11}$$

Such a marginal contribution is part of a symmetric differentiable equilibrium which implements a non-negative output $q(\cdot)$ and a rent profile $U(\cdot)$ with the following properties:

- *A strictly monotonic output schedule:*

$$\frac{\partial q}{\partial \theta_i}(\theta_1,\theta_2) > 0 \text{ for } i = 1,2; \tag{12}$$

- *An efficient output when both principals have the highest valuation $q(\bar{\theta},\bar{\theta}) = q^{FB}(\bar{\theta},\bar{\theta})$;*
- *A rent profile for the agent such that*

$$U(\theta_1,\theta_2) \geq 0; \text{ with equality if } \theta_i = \underline{\theta} \text{ for at least one } i. \tag{13}$$

At a best-response, a principal induces a lower production from the agent than what is ex post efficient from the point of view of the bilateral coalition they form. This downward distortion reduces the information rent that the agent gets from his endogenous private knowledge of the other principal's type. This distortion is captured by the right-hand side of (7) which is positive thanks to SMP. Any source of inefficiency comes from the screening problem that each principal faces in contracting with an agent who is endogenously privately informed on the types of other principals by observing their mere offers.

This phenomenon should be contrasted with the usual "free-riding" problem for public good provision found in centralized Bayesian mechanisms à la Laffont and Maskin (1979), Rob (1989) and Mailath and Postlewaite (1990). Free-riding comes there from the contributors' incentives to underestimate their valuations. Under common agency instead, principals do not hide their own valuations to the agent but want to screen the agent about the preferences of others. This is no longer contributors themselves who hide information but the agent who is at the nexus of all information sets and might pretend having received less contributions from each principal.

Remark 1: Suppose that P_{-i}'s type is common knowledge. Then, the asymmetric information distortion on the right-hand side of (7) disappears. P_i contributes at the margin what it is worth to him. Using (8), (7) becomes $\theta_i = p(q, \theta_i)$ for all q and P_i may as well offer a "truthful" schedule where this equality holds at any q. When both principals' preferences are common knowledge, the unique equilibrium output with differentiable schedules is thus efficient as predicted by Bernheim and Whinston (1986).

5 Preliminary Properties

5.1 Modified Lindahl-Samuelson Condition

When the equilibrium output is increasing, we can uniquely define the *conjuguate* of type θ_i as $\psi(q, \theta_i)$ such that $q(\theta_i, \psi(q, \theta_i)) = q$. Condition (7) becomes:

$$\psi(q, \theta_i) + p(q, \theta_i) - C'(q) = R(\theta_i)\frac{\partial p}{\partial \theta_i}(q, \theta_i), \tag{14}$$

for all q in the range of $q(\theta_i, \cdot)$. This can be rewritten as:

$$\frac{\partial}{\partial \theta_i}[p(q, \theta_i)(1 - F(\theta_i))] = (\psi(q, \theta_i) - C'(q))f(\theta_i). \tag{15}$$

This differential equation in θ_i can be integrated to get $p(q, \theta_i)$. Since $p(q, \theta_i)$ must remain bounded around $\theta_i = \bar{\theta}$ for all q, we obtain:

$$p(q, \theta_i) = C'(q) - \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q, x)f(x)dx. \tag{16}$$

Taking into account (8) yields the following modified Lindahl-Samuelson conditions:

$$C'(q(\theta_1, \theta_2)) = \sum_{i=1}^{2} \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q(\theta_1, \theta_2), x)f(x)dx. \tag{17}$$

Conditions (16) and (17) might sometimes suffice to characterize the marginal contribution and output at an equilibrium:

Example 1: Consider a uniform distribution on an interval $[\underline{\theta}, \bar{\theta}]$, and let us look for a linear marginal contribution of the form

$$p^*(q, \theta) = \frac{\theta}{2} - \frac{\bar{\theta}}{6} + \frac{C'(q)}{3}. \tag{18}$$

One can check that condition (7) holds and that the equilibrium output satisfies:

$$C'(q^*(\theta_1, \theta_2)) = \frac{3}{2}(\theta_1 + \theta_2) - \bar{\theta}, \tag{19}$$

and thus $\psi^*(q, \theta) = \frac{2}{3}(\bar{\theta} + C'(q)) - \theta$ (with $\frac{\partial \psi^*}{\partial q}(q, \theta) > 0$ and $\frac{\partial \psi^*}{\partial \theta}(q, \theta) < 0$). Marginal contributions are always positive for any equilibrium output when $3\underline{\theta} > \bar{\theta}$. These marginal contributions increase with the amount of public good produced.[30]

Example 2: Consider an exponential distribution on $[\underline{\theta}, +\infty)$, with $1 - F(\theta) = exp\left(-r(\theta - \underline{\theta})\right)$ and $r > 0$. Let us still look for a linear marginal contribution of the form

$$p^*(q, \theta) = \theta - \frac{1}{r}. \tag{20}$$

Again (7) holds. It corresponds to a non-negative equilibrium output:

$$C'(q^*(\theta_1, \theta_2)) = \theta_1 + \theta_2 - \frac{2}{r}, \tag{21}$$

so that $\psi^*(q, \theta) = \frac{2}{r} + C'(q) - \theta$ (with again $\frac{\partial \psi^*}{\partial q}(q, \theta) > 0$ and $\frac{\partial \psi^*}{\partial \theta}(q, \theta) < 0$). Marginal contributions are positive for any equilibrium output when $\underline{\theta} > \frac{1}{r}$.

5.2 A Necessary Implementability Condition

Let us first provide a sufficient condition which makes it easy to check that a given output schedule cannot be implemented as a common agency equilibrium.

Proposition 2 *Any equilibrium output that is implemented in a common agency equilibrium under asymmetric information must satisfy the following implementability condition:*

$$E_{(\theta_1, \theta_2)}\left(\left(\sum_{i=1}^{2} \theta_i - R(\theta_i)\right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2))\right) \geq 2\underline{\theta} q(\underline{\theta}, \underline{\theta}) - C(q(\underline{\theta}, \underline{\theta})) > 0. \tag{22}$$

Condition (22) says that the expected virtual surplus (where marginal valuations θ_i have been replaced by their virtual values $\theta_i - R(\theta_i)$) is worth at least the whole surplus generated in the worst scenario where both principals have the lowest type. This is the standard feasibility condition that arises in asymmetric information models with independent types (bargaining or public goods) once Bayesian incentive compatibility, ex post budget balanced and individual rationality constraints are aggregated altogether.[31] In

[30] The corresponding marginal contribution satisfies $\frac{\partial p}{\partial \theta}(q, \theta) = \frac{1}{2}$. Bernheim and Whinston (1986) define a "truthful" contribution as such that the principal's marginal contribution reflects his own preferences, i.e., $\frac{\partial p_i}{\partial \theta_i}(q, \theta_i) = 1$. Locally, the symmetric equilibrium found above does not reflect everywhere the marginal preferences of the principals for the public good due to their incentives to induce downward distortions of the output for screening purposes.

[31] See Myerson and Satterthwaite (1983), Laffont and Maskin (1982), Güth and Hellwig (1987), Mailath and Postelwaite (1990), Ledyard and Palfrey (1999), and Hellwig (2003) among others.

those contexts, there is no restriction in the centralized mechanisms used to implement such allocation and this condition turns out to be also sufficient: Given any output schedule satisfying the implementability condition, one can find transfers which are ex post budget-balanced, Bayesian incentive compatible and individual rational for the informed players. Here, the added requirement is that the allocation should arise as the equilibrium of a common agency game, which may put significant structure on the allocations by imposing they solve also the functional equation (17). The necessary condition (22) is nevertheless already enough to get sharp results.

5.3 Ex Post Inefficiency

The numerical examples given in Section 5.1 already showed us inefficient equilibria. Equipped with condition (22), it is straightforward to check that ex post inefficiencies are actually pervasive in any equilibrium.

Proposition 3 *The first-best output $q^{FB}(\theta_1, \theta_2)$ never satisfies the implementability condition (22) and thus cannot be achieved at any equilibrium.*

This result echoes the discussion after Proposition 1. It contrasts sharply with the case of complete information where common agency games have efficient equilibria sustained with "truthful" schedules.

Remark 2: Although in the one principal case, the fact that the informed party moves first implies that there is no inefficiency in our private values setting; this is no longer enough with several informed parties moving first. Indeed, each of those parties want to screen the agent about what he has learned from the other.

6 Equilibrium Characterization

It is useful to describe an equilibrium in terms of its isoquant lines $\theta_2 = \psi(q, \theta_1)$. Coming back on (7) and (8) and writing those conditions along such isoquant yields:

$$p(q,\theta) + p(q,\psi(q,\theta)) = C'(q), \quad (23)$$

$$\psi(q,\theta) - p(q,\psi(q,\theta)) = R(\theta)\frac{\partial p}{\partial \theta}(q,\theta), \quad (24)$$

for all (q, θ), where q is in the range of the equilibrium schedule of outputs $q(\cdot)$.[32]

For a type distribution with finite support as assumed so far, those two equations are already quite informative on the shape of the marginal contribution at its boundaries.

[32]Note that, by definition of a conjugate type, it must also be that: $\psi(q, \psi(q, \theta)) = \theta$, for all θ in $[\underline{\theta}, \overline{\theta}]$ and q in the range of $q(\cdot)$.

Proposition 4 *Fix any q such that $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$. The lowest type $\underline{\theta}(q) \geq \underline{\theta}$ on the q-isoquant is increasing in q and such that:*

$$C'(q) = \bar{\theta} + \underline{\theta}(q) - \frac{1}{2}R(\underline{\theta}(q)). \tag{25}$$

Moreover, we have:

$$p(q, \underline{\theta}(q)) = \underline{\theta}(q); \quad \frac{\partial p}{\partial \theta}(q, \underline{\theta}(q)) = \frac{1}{2} \tag{26}$$

and

$$p(q, \bar{\theta}) = C'(q) - \underline{\theta}(q) < \bar{\theta} \text{ and } \tfrac{\partial p}{\partial \theta}(q, \bar{\theta}) > 0. \tag{27}$$

For $q \geq 0$ such that $C'(q) \leq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the lowest type on the $q-$isoquant is $\underline{\theta}$ and the highest one is $\bar{\theta}(q)$ with:

$$p(q, \underline{\theta}) < \underline{\theta}, \ \tfrac{\partial p}{\partial \theta}(q, \underline{\theta}) > 0; \tag{28}$$

$$p(q, \bar{\theta}(q)) = \bar{\theta}(q) - R(\underline{\theta})\frac{\partial p}{\partial \theta}(q, \underline{\theta}), \ \tfrac{\partial p}{\partial \theta}(q, \bar{\theta}(q)) = \tfrac{\theta - p(q,\theta)}{R(\bar{\theta}(q))} > 0. \tag{29}$$

These preliminary remarks on boundaries being made, solving for (23) and (24) amounts to looking for a function $p(q, \theta)$ increasing in θ (thus invertible) on a domain $[\underline{\theta}(q), \bar{\theta}]$ and solution to the following non-standard functional equation:

$$R(\theta)\dot{x}(\theta) - x(\theta) + C'(q) = x^{-1}(C'(q) - x(\theta)) \tag{30}$$

with the boundary conditions[33]

$$x(\underline{\theta}(q)) = \underline{\theta}(q) \text{ and } x(\bar{\theta}) = C'(q) - \underline{\theta}(q). \tag{31}$$

Standard theorems for differential equations do not apply to guarantee existence and uniqueness of such a solution since (30) depends not only on the function and its (non-decreasing) derivative but also on its inverse. Crucially, the boundary conditions are such that (30) has singularities at those boundary points. This makes any local approach for solving this equation away from these singularities of little help to prove existence.

An alternative and more tractable approach towards finding a global solution is to work with the equilibrium distributions of marginal prices.[34] For a given quantity q, let thus denote $G(p, q)$ the cumulative distribution of marginal price $p(q, \theta)$ on that isoquant, i.e., $G(p, q) = Proba(p(\theta, q) \leq p)$. Denote also $g(p, q) = \frac{\partial G}{\partial p}(p, q)$ the corresponding density. Formally, we have $G(p(q, \theta), q) = F(\theta)$ and $g(p(q, \theta))\frac{\partial p}{\partial \theta}(q, \theta) = f(\theta)$. The equilibrium condition (30) can be rewritten using the definition of $G(\cdot, q)$ as

$$F\left(C'(q) - p + \frac{1 - G(p, q)}{g(p, q)}\right) = G(C'(q) - p, q).$$

[33]We focus on the case $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, i.e., outputs close enough to the first-best when both principals have the highest valuation since it appears to be the most interesting case. Lower output levels correspond to less stringent boundary conditions which are thus less constraining for the equilibrium characterization.

[34]For a similar trick, see Leininger, Linhart and Radner (1989) for double-auctions and Wilson (1993) for nonlinear pricing.

From this, we obtain the following functional equation:

$$\frac{\frac{\partial G}{\partial p}(p,q)}{1-G(p,q)} = \frac{1}{F^{-1}(G(C'(q)-p,q)) - C'(q) + p}. \tag{32}$$

The boundary condition (31) gives us that $\sup G(\cdot, q) = [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$ with

$$G(\underline{\theta}(q), q) = F(\underline{\theta}(q)) \text{ and } G(C'(q) - \underline{\theta}(q), q) = 1. \tag{33}$$

Proposition 5 *Fix any q such that $2\bar{\theta} \geq C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$. Assume that $\frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) \leq -1$ for all $x \in \Theta$. Then a solution $G(p,q)$ to the system (32)-(33) (or alternatively a solution $p(q,\theta)$ to (30)-(31)) exists. This solution $G(p,q)$ (resp. $p(q,\theta)$) is increasing in p (resp. θ).*

It is interesting to understand how the proof of Proposition 5 works. The first step is to consider the sequence of distributions of marginal contributions that each principal plays in turn at a best-response to what the other offers starting from the simple case where one principal, say P_1, would myopically offer a marginal contribution always equal to his own valuation. Under the assumption $\frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) \leq -1$, we prove that the other principal P_2 reacts by offering distributions which, at each iteration, dominate in the sense of first-order stochastic dominance that offered at the round before. For P_1, this is the reverse; each iterate is dominated by the previous one. Intuitively, a principal finds it worth to offer higher marginal contributions if the other offers lower contributions and vice-versa; the best-response mapping is monotonically decreasing. The whole process converges towards a set of distributions which is stable in the following sense: If any principal offers a distribution of marginal prices from this set, the other principal's best-responses lies in this set also. The second step consists in applying Shauder Theorem[35] to guarantee existence of a distribution in that stable set which is its own best-response.[36]

Proposition 5 gives us existence of a solution $p(q,\theta)$ to (30) for a given isoquant q. We must also check that, as q increases, the corresponding $\psi(q,\theta)$ derived from the knowledge of $p(q,\theta)$ increases in q to ensure concavity of the principals' problem. Using that $\psi(q, \underline{\theta}(q)) = \bar{\theta}$ in any equilibrium and differentiating w.r.t. q yields $\frac{\partial \psi}{\partial q}(q,\theta) > 0$ in the neighborhood of $\underline{\theta}(q)$. Hence, this concavity property holds when $\Delta\theta$ is small enough. From Proposition 1 the monotonicity of output follows.

Let us now turn to the issue of uniqueness. The following proposition ensures that the distributions of marginal contributions $G_1(\cdot, q)$ and $G_2(\cdot, q)$ in two putative equilibria necessarily cross each other once on $(\underline{\theta}(q), C'(q) - \underline{\theta}(q))$.

[35]See Burton (2005, Chapter 3) for instance.

[36]Our existence result should be contrasted with the analysis of equilibria in the case of a discrete 0-1 public good. Menezes, Monteiro and Temini (2001) transform an equation close to (30) into a system of first-order differential equations and show existence with standard theorems away from the boundary conditions where those differential equations have singularities. A more global approach is needed to take care of those conditions and ours method does so.

Proposition 6 *Consider the distributions $G_1(\cdot, q)$ and $G_2(\cdot, q)$ in two putative distinct equilibria. One of these distributions cannot dominate the other in the sense of first-order stochastic dominance.*

Indeed, if any of those distributions dominates the other in the first-order sense, then the monotonicity of the best-responses to each of those distributions ensures that the dominance is reversed, a contradiction.

With an unbounded support, $\underline{\theta}(q)$ is not properly defined, and there is no boundary condition that must be satisfied by the price schedule at $\bar{\theta} = +\infty$. This indeterminacy opens the door to a multiplicity of equilibria as shown below.

Proposition 7 *Assume that types are distributed according to an exponential distribution $F(\theta) = 1 - exp(-r(\theta - \underline{\theta}))$ on $[\underline{\theta}, +\infty)$ with $\underline{\theta} > \frac{1}{r}$. There exists a whole continuum of equilibria $p(q, \theta)$ which solve (30). Those equilibria are such that $p(q, \theta) < \theta - \frac{1}{r}$. Inefficiencies in any of those equilibria are stronger than in the equilibrium of Example 2.*

7 Interim Efficiency?

Under complete information, it is well known that the "truthful" equilibria of common agency game are on the Pareto-frontier of what the principals could achieve by cooperating. Under asymmetric information, one can still be interested by the normative properties of common agency equilibria provided that *interim efficiency* is used as the welfare criterion since, from Proposition 3, ex post efficiency is hopeless. We now investigate under which circumstances an equilibrium might be *interim efficient*.

Interim efficient allocations are obtained as solutions of a centralized mechanism design problem. An uninformed mediator offers a single mechanism to both principals, who then report their types to this mediator. This mediator maximizes a weighted sum of both the principals' and the agent's utilities with the weights given to different types of the principals being possibly different. Because we want to replicate with such centralized mechanism a symmetric common agency equilibrium, we consider symmetric weights which do not depend on the principal's identity.

Lemma 4 *An interim efficient profile $q(\theta_1, \theta_2)$ is such that there exist non-negative social weights $\alpha(\theta) \geq 0$ such that $\int_{\underline{\theta}}^{\bar{\theta}} \alpha(\theta) f(\theta) d\theta \leq 1$*[37] *and*

$$C'(q(\theta_1, \theta_2)) = \sum_{i=1}^{2} b(\theta_i) \tag{34}$$

[37] The inequality $\int_{\underline{\theta}}^{\bar{\theta}} \alpha(x) f(x) dx \leq 1$ captures the possibility that the common agent receives a positive weight in the social welfare function maximized by the uninformed mediator. Remember that, in a common agency equilibrium, the agent gets a non-negative ex post rent $U(\theta_1, \theta_2)$ which should be accounted for when evaluating welfare. This distinguishes our notion of interim efficiency from the usual one where it is assumed that budget is always balanced ex post (Ledyard and Palfrey 1999).

where $b(\theta_i)=\theta_i - R(\theta_i)(1-\tilde{\alpha}(\theta_i))$ *is non-decreasing and* $\tilde{\alpha}(\theta_i)=\frac{1}{1-F(\theta_i)}\int_{\theta_i}^{\bar{\theta}}\alpha(x)f(x)dx$.

Coming back on Example 1, one can easily check that, for a uniform distribution having arbitrary support $[\underline{\theta},\bar{\theta}]$, the solution found in (19) remains interim efficient for the uniform weight $\alpha(\theta)=\frac{1}{2}$ for all θ. Positive results can also be found for Example 2 with the uniform weight $\alpha(\theta)=0$.[38]

Any equilibrium which would be also interim efficient links the equilibrium output $Q(\theta)=q(\theta,\theta)$ along the diagonal and the function $b(\theta)$ by the following identity

$$C'(Q(\theta))=2b(\theta) \tag{35}$$

with the condition that $Q(\bar{\theta})=q^{FB}(\bar{\theta},\bar{\theta})$ since $b(\bar{\theta})=\bar{\theta}$. Note that (35) puts no restriction on the increasing output profiles on the diagonal.

More generally, altogether (17) and (34) give us a global condition that must be satisfied by any increasing candidate function $b(\cdot)$:

$$\sum_{i=1}^{2} b(\theta_i)=\sum_{i=1}^{2}\frac{1}{1-F(\theta_i)}\int_{\theta_i}^{\bar{\theta}} b^{-1}(b(\theta_1)+b(\theta_2)-b(x))f(x)dx \quad \forall(\theta_1,\theta_2)\in\Theta^2. \tag{36}$$

This condition is very restrictive. It forces the output profile which is completely defined *off* the diagonal by the equilibrium conditions to satisfy *everywhere* a complex equation defined in terms of the output profile *on* the diagonal. As a result, it is not surprising that there are few candidates for the $b(\cdot)$ function:

Proposition 8 *The only candidate for an interim efficient equilibrium has an increasing output along the diagonal given by:*

$$C'(Q^c(\theta))=2b^c(\theta)=2\left(\theta-\frac{1}{1-F(\theta)}\int_{\theta}^{\bar{\theta}}(1-F(x))dx\right). \tag{37}$$

It solves (36) for the exponential case on an infinite support and for the density function $f(\theta)=\frac{(1+\eta)}{(\bar{\theta}-\underline{\theta})^{1+\eta}}(\bar{\theta}-\theta)^{\eta}$ *(for* $\eta\geq 0$*) for a finite support.*

Beyond the uniform and the exponential case, there exists a whole class of equilibria which are interim efficient provided distributions have linear hazard rates. For those equilibria, the function $b(\cdot)$ which satisfies (36) and (37) is linear (namely $b(\theta)=\frac{(\eta+3)\theta-\bar{\theta}}{\eta+2}$), isoquants are 45 degree line in the (θ_1,θ_2) space, and social weights are type-independent (namely $\alpha(\theta)=\frac{1}{\eta+2}$). Marginal contributions are given by:

$$p(q,\theta)=\frac{C'(q)}{\eta+3}+\bar{\theta}\left(\frac{\eta+1}{\eta+3}\right)-(\bar{\theta}-\theta)\left(\frac{\eta+1}{\eta+2}\right).$$

Perturbing even slightly the model and considering distributions which no longer have linear hazard rates makes checking (36) a complex numerical exercise. We conjecture that, beyond those cutting-edge cases, interim efficiency never holds.

[38] This last example is extreme. Together with Proposition 7, it shows that there exist equilibria which are not interim efficient and implement outputs with a greater downward distortion than in (21).

8 Dominant Strategy

Condition (7) is an ex post optimality condition, stating that each principal maximizes his virtual net utility pointwise. This is reminiscent of a dominant strategy requirement but, actually, it is quite distinct as we show below.

From the equilibrium nonlinear price $t(q, \theta_i)$ and the output schedule $q(\cdot)$, let indeed define a (symmetric) centralized mechanism $\{T((\theta_i, \theta_{-i}), q(\theta_i, \theta_{-i})\}$ with $T(\theta_i, \theta_{-i}) = t(q(\theta_i, \theta_{-i}), \theta_i)$.

Proposition 9 *There does not exist any symmetric differentiable equilibrium of the common agency game* $t(q, \theta_i)$ *such that the corresponding centralized mechanism* $\{T(\theta_i, \theta_{-i}), q(\theta_i, \theta_{-i})\}$ *can be implemented with dominant strategy.*

9 Conclusion

Let us summarize the main findings of our analysis.

First, introducing private information on the principals' preferences in a common agency game justifies the use of nonlinear contributions for screening purposes, whereas such strategy space is given a priori in previous complete information models. Doing so introduces incentive compatibility conditions which replace the "truthfulness" requirement used earlier on.

Second, under weak conditions on type distributions, there exists at least one differentiable equilibrium which solves a complex functional equation. However, ex post inefficiencies in any such equilibrium arise contrary to complete information models. The reason is not the standard "free-riding" phenomenon stressed by the standard centralized mechanism design approach but it comes now from the desire of each principal to screen the agent about the endogenous information he has learned from observing others' offers. The weaker criterion of interim efficiency may be satisfied by some of these equilibria but this is so only under very stringent assumptions on the hazard rate of the type distribution. This suggests that it might be quite often worth introducing uninformed mediators in those environments.

Finally, our approach in solving for the best-responses of each principal nicely uses the notion of "market information" pushed forward by Epstein and Peters (1999). It shows that this notion is tractable enough to model inefficiencies in multi-principals game.

Our model might be worth being extended along several directions. First, other information structures could be investigated, for instance, allowing correlation between the principals' types or common values aspects through a direct dependence of the agent's utility function on these types. Correlation may help to improve efficiency by using some sort of yardstick comparison between principals' contributions. Common values may restore a more significant role to the signalling problem that each principal faces in offering contracts. Second, the assumption of quasi-linear preferences could be relaxed.[39] Third, asymmetries both in terms of preferences and types distributions may have important impacts on output distortions. Fourth, extending the model to more than two principals raises difficulties related to the multi-dimensional adverse selection problem that each principal faces in trying to extract information on others contributors' preferences. Lastly, the techniques uncovered in this paper may also be used to understand equilibria of other bidding games where informed parties offer contribution schedules like multi-unit auctions on financial and electricity markets for instance.

We hope to address these issues in future research.

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Appendix

Proof of Lemma 1: Such equilibrium consists of the following strategies and beliefs:

• A strategy for the principal P_i with type θ_i is to choose optimally a communication space $\mathcal{M}_i^*(\theta_i)$ and a mechanism $\tilde{t}_i^*(\cdot|\theta_i)$. The strategy $\tilde{t}_i^*(\cdot|\theta_i)$ maps thus Θ into $\mathcal{T}_i(\mathcal{M}_i^*(\theta_i))$.

• The agent's beliefs on P_i's type at stage 2 following the offer $\tilde{t}_i$ are derived using Bayes' rule whenever possible, i.e., when $\tilde{t}_i = \tilde{t}_i^*(\cdot|\theta_i)$ for some θ_i, and are arbitrary otherwise.

• The agent's acceptance decision $a_i^*(\cdot)$ for P_i's offer maps $\mathcal{T}_i(\mathcal{M}_i)$ into $\{0, 1\}$. An array of acceptance decisions is denoted $a = (a_1, a_2)$.

• Principal P_i's updated beliefs on P_{-i}'s type following the observation of the acceptance decision a_i are obtained by Bayes' rule whenever possible, i.e., when $a_i = a_i^*(\tilde{t}_i^*(\cdot|\theta_i))$ and are arbitrary otherwise.

• The agent optimally chooses his output $q^*(a, \tilde{t})$ as a function of the array of mechanisms he receives and accepts.

• Principal P_i's updated beliefs on P_{-i}'s type following the observation of the acceptance decision and the output (a_i, q) are obtained by using Bayes' rule whenever possible, i.e., when $a_i = a_i^*(\tilde{t}_i^*)$, and are arbitrary otherwise.

• If $a_i = 1$, i.e., if his offer is accepted, principal P_i optimally chooses a probability distribution $d\mu_i^*(m_i|\theta_i)$ over the possible messages sent at stage 4. $d\mu_i^*(\cdot|\theta_i)$ maps Θ into $\Delta(\mathcal{M}_i^*(\theta_i))$ the set of measures on $\mathcal{M}_i^*(\theta_i)$.

Consider any such pure strategy equilibrium of Γ where P_{-i} offers a direct communication mechanism $\tilde{t}_{-i}^*(q, m_{-i}|\theta_{-i}) \equiv t_{-i}(q, \theta_{-i})$ for all $\theta_{-i} \in \Theta$ and this mechanism is accepted by the agent. We show that P_i has a direct communication mechanism which is also accepted in his best-response correspondence. Consider any P_i's equilibrium best-response when his type is θ_i. It consists in offering $\tilde{t}_i^*(q, m_i|\theta_i)$ at stage 1 and a mixture $d\mu^*(m_i|\theta_i)$ at stage 4 since P_i's offer is accepted. Define

$$t_i(q, \theta_i) = \int_{\mathcal{M}_i} \tilde{t}_i^*(q, m_i|\theta_i) d\mu^*(m_i|\theta_i).$$

Claim 1: The agent's acceptance and output decisions are unchanged when one replaces the equilibrium pair $(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i}))$ by $(t_i(\cdot, \theta_i), t_{-i}(\cdot, \theta_{-i}))$.

Indeed, if P_{-i} (resp. P_i) chooses $t_{-i}(q, \theta_{-i})$ (resp. $\tilde{t}_i^*(q, m_i|\theta_i)$), the agent's posterior beliefs on θ_{-i} are given by $dF(\theta_{-i}|t_{-i}(\cdot))$. For P_i's new offer $t_i(\cdot)$, let assume also that the agent's posterior beliefs on θ_i are unchanged with the new mechanism so that $dF(\theta_i|\tilde{t}_i^*(\cdot)) = dF(\theta_i|t_i(\cdot))$. Those new beliefs are consistent with equilibrium behavior given that P_i's payoff will be unchanged as we see below. Then, the agent's payoff when choosing an optimal output $q^*(\tilde{t}_i^*(\theta_i), t_{-i}(\theta_{-i}))$[40] satisfies:

$$\int_{\Theta\times\Theta} \Bigg(\int_{\mathcal{M}_i} \tilde{t}_i^*(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i})), m_i|\theta_i) d\mu_i^*(m_i|\theta_i) + t_{-i}(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i})), \theta_{-i})$$

$$-C(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i})))) dF(\theta_i|\tilde{t}_i^*(\cdot)) \Bigg) dF(\theta_{-i}|t_{-i}(\cdot))$$

$$\geq \int_{\Theta\times\Theta} \left(\int_{\mathcal{M}_i} \tilde{t}_i^*(q, m_i|\theta_i) d\mu_i^*(m_i|\theta_i) + t_{-i}(q, \theta_{-i}) - C(q) \right) dF(\theta_i|\tilde{t}_i^*(\cdot)) dF(\theta_{-i}|t_{-i}(\cdot)) \quad \text{for all } q.$$

Putting it differently,

$$\int_{\Theta\times\Theta} (t_i(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\theta_{-i})), \theta_i) + t_{-i}(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\theta_{-i})), \theta_{-i})$$

$$-C(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot|\theta_{-i})))) dF(\theta_i|t_i(\cdot)) F(\theta_{-i}|t_{-i}(\cdot))$$

$$\geq \int_{\Theta\times\Theta} (t_i(q, \theta_i) + t_{-i}(q, \theta_{-i}) - C(q)) dF(\theta_i|t_i(\cdot)) dF(\theta_{-i}|t_{-i}(\cdot)) \quad \text{for all } q$$

or[41]

$$q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i})) = q^*(t_i(\cdot, \theta_i), t_{-i}(\cdot, \theta_{-i})). \tag{A1}$$

The fact that the acceptance decisions are unchanged follows similar computations:

$$1 = a^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot, \theta_{-i})) = a^*(t_i(\cdot, \theta_i), t_{-i}(\cdot, \theta_{-i})).$$

[40] Slightly abusing notations and denoting $q^*(\tilde{t})$ the agent's output as a function of the accepted mechanisms.

$\blacksquare$

Claim 2: The principal's payoff is unchanged when one replaces the equilibrium pair $(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot,\theta_i))$ *by* $(t_i(\cdot,\theta_i), t_{-i}(\cdot,\theta_{-i}))$.

Let us write P_i's equilibrium payoff with the original mechanism as:

$$E_{\theta_{-i}}\left(\int_{\mathcal{M}_i}\left(\theta_i q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot,\theta_{-i})) - \tilde{t}_i^*(q^*(\tilde{t}_i^*(\cdot|\theta_i), t_{-i}(\cdot,\theta_{-i}))|\theta_i)\right) d\mu_i^*(m_i|\theta_i)\right).$$

Using (A1) and the definition of $t_i(\cdot|\theta_i)$, this payoff can also be written as:

$$E_{\theta_{-i}}(\theta_i q^*(t_i(\cdot,\theta_i), t_{-i}(\cdot,\theta_{-i})) - t_i(q^*(t_i(\cdot,\theta_i), t_{-i}(\cdot,\theta_{-i}))|\theta_i)).$$

Since payoffs are preserved, there is no problem in keeping posterior beliefs unchanged with the new direct communication mechanism. This ends the proof of Lemma 1. $\blacksquare$

Proof of Lemma 2: Observe that, in our private values environment, the agent's choice of output depends only on the contributions offered and not on his beliefs on the principals' types. Hence, everything happens as if the agent had complete information on the principal's type when he makes an offer and principals cannot do better than offering the optimal contracts they would offer had the agent known about their types. $\blacksquare$

Proof of Lemma 3: The proof is standard (see for instance Laffont and Martimort (2002, Chapter 3)) and is thus omitted. $\blacksquare$

Proof of Proposition 1: The proof goes through several steps. First, we show that there is no loss of generality in restricting the analysis to non-negative contributions extended over $rg(q(\cdot)) \cup \{0\}$. Second, we derive the optimality conditions (7) assuming quasi-concavity of the agent's problem and the fact that the participation constraint (5) binds only at $\theta_{-i} = \underline{\theta}$. Finally, we show that these conditions are indeed satisfied.

Lemma 5 *Assume that* P_{-i} *offers a non-negative contribution* $t_{-i}(q,\theta_{-i}) \geq 0 \quad \forall q \in rg(q(\cdot))$. *Then* P_i *offers also a non-negative contribution at a best-response:*

Proof: Note that acceptance of P_i's offer requires:

$$\sum_{i=1}^{2} t_i(q(\theta_1,\theta_2),\theta_{-i}) - C(q(\theta_1,\theta_2) \geq \hat{U}_i(\theta_{-i}) \geq t_{-i}(q(\theta_1,\theta_2),\theta_{-i}) - C(q(\theta_1,\theta_2)$$

for any θ_i. Hence, any equilibrium output $q(\theta_i,\theta_{-i})$ must be such that $t_i(q(\theta_i,\theta_{-i}),\theta_i) \geq 0$. We may choose as well to extend $t_i(q,\theta_i)$ with $t_i(0,\theta_i) = 0$ to capture the fact that there should not be any contribution for zero public good. $\blacksquare$

• **Pointwise optimization:** For equilibria where $\frac{\partial t_{-i}}{\partial \theta_{-i}}(q,\theta_{-i}) \geq 0$, $U^D(\theta_{-i}|\theta_i)$ is increasing in θ_{-i} and (5) is binding at $\theta_{-i} = \underline{\theta}$ only provided that the marginal contribution $p_{-i}(q,\theta_{-i})$ is positive (we show this last claim below). Integrating by parts, we obtain:

$$E_{\theta_{-i}}\left[U^D(\theta_{-i}|\theta_i)\right] = E_{\theta_{-i}}\left[R(\theta_{-i})\frac{\partial t_{-i}}{\partial \theta_{-i}}(q(\theta_{-i}|\theta_i),\theta_{-i})\right] + \hat{U}_{-i}(\underline{\theta}).$$

Inserting this latter expression into (4), we get to maximize pointwise

$$E_{\theta_{-i}}\left[S_i(q(\theta_{-i}|\theta_i),\theta_i,\theta_{-i})\right] \tag{A2}$$

where $S_i(q,\theta_i,\theta_{-i})$ denotes the virtual surplus of principal P_i:

$$S_i(q,\theta_i,\theta_{-i}) = \theta_i q + t_{-i}(q,\theta_{-i}) - C(q) - R(\theta_{-i})\frac{\partial t_{-i}}{\partial \theta_{-i}}(q,\theta_{-i}). \tag{A3}$$

This expression is strictly concave in q when

$$\frac{\partial^2 S_i}{\partial q^2}(q,\theta_i,\theta_{-i}) = -\frac{\partial \psi_{-i}}{\partial q}(q,\theta_{-i}) = \frac{\partial p_{-i}}{\partial q}(q,\theta_{-i}) - C''(q) - R(\theta_{-i})\frac{\partial^2 p_{-i}}{\partial \theta_{-i}\partial q}(q,\theta_{-i}) < 0$$

which is true by (11). Optimizing pointwise the virtual surplus in (A2) gives thus an output $q^D(\theta_{-i}|\theta_i)$ which satisfies

$$\frac{\partial S_i}{\partial q}(q^D(\theta_{-i}|\theta_i),\theta_i,\theta_{-i}) = 0.$$

This gives condition (7) at a symmetric equilibrium. Differentiating (7) with respect to θ_{-i}, we have:

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q^D}{\partial \theta_{-i}} = R(\theta_{-i})\frac{\partial^2 p_{-i}}{\partial \theta_{-i}^2} - \left(1-\dot{R}(\theta_{-i})\right)\frac{\partial p_{-i}}{\partial \theta_{-i}}. \tag{A4}$$

Differentiating (10) with respect to θ_{-i} allows us to simplify (A4) to get

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q^D}{\partial \theta_{-i}} = \frac{\partial \psi_{-i}}{\partial \theta_{-i}}.$$

Hence, $\frac{\partial q^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0$ when (11) holds at a symmetric equilibrium.

• **Quasi-concavity of the agent's problem:** It is also standard to show that the monotonicity condition (3) and the SMP property of the nonlinear contributions ensure global optimality of the agent's problem (1) when (8) holds. Fix any q such that $q \leq q(\theta_1,\theta_2)$ (the case $q \geq q(\theta_1,\theta_2)$ can be treated similarly). We have:

$$\sum_{i=1}^{2} t(q(\theta_1,\theta_2),\theta_i) - C(q(\theta_1,\theta_2)) - \left(\sum_{i=1}^{2} t(q,\theta_i) - C(q)\right) = \int_q^{q(\theta_1,\theta_2)} \left(\sum_{i=1}^{2} p(x,\theta_i) - C'(x)\right) dx.$$

For any $x \in [q, q(\theta_1,\theta_2)]$, note that from (8) $\psi(x,\theta_1)$ defined from (10) satisfies $p(x,\theta_1) + p(x,\psi(x,\theta_1)) = C'(x)$. Then (10) implies that:

$$\int_q^{q(\theta_1,\theta_2)} \left(\sum_{i=1}^{2} p(x,\theta_i) - C'(x)\right) dx = \int_q^{q(\theta_1,\theta_2)} (p(x,\theta_2) - p(x,\psi(x,\theta_1)))dx \geq 0$$

since $\psi(x,\theta_1) \leq \psi(q(\theta_1,\theta_2),\theta_1) = \theta_2$ from (11) and the SMP holds. This shows that the local second-order condition (12) holds.

• **First-best at the top**: Using (7) for $\theta_1 = \theta_2 = \bar{\theta}$ and (8) yields the result.

• **The agent's participation constraints**: We want to show that (5) binds only at $\underline{\theta}$. We proceed with several lemmata.

Define first the output level when the agent takes only P_{-i}'s contribution and chooses a positive production with this schedule as:[42]

$$\hat{q}_{-i}(\theta_{-i}) = arg \max_{q \in rg(q(\cdot,\theta_{-i}))} \{t(q,\theta_{-i}) - C(q)\}.$$

Lemma 6 *Assume that $p(q,\theta) \geq 0$ for all (q,θ) such that $q \in dom(t(\cdot,\theta))$. Then, for any (θ_i,θ_{-i}), we have:*

$$q(\theta_1,\theta_2) \geq \hat{q}_{-i}(\theta_{-i}) = q(\underline{\theta},\theta_{-i}). \tag{A5}$$

Proof: When $p(q,\theta) \geq 0$, (8) implies $p(q,\theta_{-i}) - C'(q) \leq 0$ for any $q \in rg(q(\cdot,\theta))$. Hence, $t(q,\theta_{-i}) - C(q)$ decreases over this range and $q(\underline{\theta},\theta_{-i}) = \hat{q}_{-i}(\theta_{-i})$. Moreover, using the SMP we get $p(q(\theta_i,\theta_{-i}),\theta_i) \geq p(q(\theta_i,\theta_{-i}),\underline{\theta})$ and, from (8), we obtain:

$$p(q(\theta_i,\theta_{-i}),\underline{\theta}) + p(q(\theta_i,\theta_{-i}),\theta_{-i}) - C'(q(\theta_i,\theta_{-i})) \leq 0.$$

Quasi-concavity of the agent's problem at $(\underline{\theta},\theta_{-i})$ yields the first inequality in (A5). ■

Lemma 7 *$U^D(\theta_{-i}|\theta_i) \geq \hat{U}_{-i}(\theta_{-i})$ for any (θ_i,θ_{-i}) if $U(\underline{\theta}|\theta_i) \geq \hat{U}_{-i}(\underline{\theta})$ holds.*

Proof: Using the Envelope Theorem, we get

$$\frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) = \frac{\partial t}{\partial \theta_{-i}}(q(\theta_1,\theta_2),\theta_{-i}).$$

Moreover,

$$\frac{\partial \hat{U}_{-i}}{\partial \theta_{-i}}(\theta_{-i}) = \frac{\partial t}{\partial \theta_{-i}}(\hat{q}_{-i}(\theta_{-i}),\theta_{-i}) + (p(\hat{q}_{-i}(\theta_{-i}),\theta_{-i}) - C'(\hat{q}_{-i}(\theta_{-i}))\frac{\partial \hat{q}_{-i}}{\partial \theta_{-i}}(\theta_{-i}).$$

Note first that $\frac{\partial \hat{q}_{-i}}{\partial \theta_{-i}}(\theta_{-i}) \geq 0$ from (A5) and the fact that $q(\cdot)$ is monotonically increasing. Second, we have necessarily $p(\hat{q}_{-i}(\theta_{-i}),\theta_{-i}) - C'(\hat{q}_{-i}(\theta_{-i})) \leq 0$. Hence, we get:

$$\frac{\partial \hat{U}_{-i}}{\partial \theta_{-i}}(\theta_{-i}) \leq \frac{\partial t}{\partial \theta_{-i}}(\hat{q}_{-i}(\theta_{-i}),\theta_{-i}) \leq \frac{\partial t}{\partial \theta_{-i}}(q(\theta_1,\theta_2),\theta_{-i})$$

where the last inequality follows from (A5) and the SMP. Then, we have:

$$\frac{\partial U^D}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq \frac{\partial \hat{U}}{\partial \theta_{-i}}(\theta_{-i}).$$

■

The next lemma fully characterizes the contributions in any symmetric equilibrium.

[42] If there are several such maximizers, we take the largest one.

Lemma 8 $U^D(\underline{\theta}|\theta_i) = 0 > \hat{U}_{-i}(\underline{\theta})$ *for all* θ_i *and all* i.

Proof: From the binding participation constraints for both principals, we get the system

$$U^D(\underline{\theta}|\theta_i) = t(q(\underline{\theta}, \theta_i), \underline{\theta}) + t(q(\underline{\theta}, \theta_i), \theta_i) - C(q(\underline{\theta}, \theta_i)) = \hat{U}_{-i}(\underline{\theta}) \text{ for all } \theta_i$$

where

$$\hat{U}_{-i}(\underline{\theta}) = t(q(\underline{\theta}, \underline{\theta}), \underline{\theta}) - C(q(\underline{\theta}, \underline{\theta})).$$

For $\theta_i = \underline{\theta}$, we get:

$$U^D(\underline{\theta}|\underline{\theta}) = 2t(q(\underline{\theta}, \underline{\theta}), \underline{\theta}) - C(q(\underline{\theta}, \underline{\theta})) = \max\{0, t(q(\underline{\theta}, \underline{\theta}), \underline{\theta}) - C(q(\underline{\theta}, \underline{\theta}))\}$$

which admits the solution

$$t(q(\underline{\theta}, \underline{\theta})) = \frac{C(q(\underline{\theta}, \underline{\theta}))}{2} > 0.$$

For $\theta_i \geq \underline{\theta}$, consider:

$$t(q(\underline{\theta}, \theta_i), \theta_i) = C(q(\underline{\theta}, \theta_i)) - \frac{C(q(\underline{\theta}, \underline{\theta}))}{2} - \int_{q(\underline{\theta}, \underline{\theta})}^{q(\underline{\theta}, \theta_i)} p(x, \underline{\theta}) dx.$$

Given the marginal contribution $p(\cdot, \underline{\theta})$, one recovers $t(q(\underline{\theta}, \theta_i), \theta_i)$ with the property that $\frac{\partial t}{\partial \theta_i}(q(\underline{\theta}, \theta_i), \theta_i) = 0$. ■

Together Lemmata 7 and 8 ensure that the relevant binding participation constraint in any SMP equilibrium is at $\underline{\theta}$.

• **Implementation of the optimal direct mechanism through a nonlinear contribution:** At a symmetric equilibrium, given that P_{-i} makes the offer $t(q, \theta_{-i})$ defined through its non-negative margin $p(q, \theta_{-i})$ (up to a constant) satisfying (7), (8), (9) and (10), P_i cannot do better than offering himself a direct revelation mechanism which implements the non-decreasing output $q^D(\cdot|\theta_i)$ satisfying (7). From this and the transfer $t^D(\cdot|\theta_i)$, we can reconstruct the nonlinear schedule $t(q, \theta_i)$ that P_i could as well offer.[43] ■

Proof of Proposition 2: Denote P_i's ex post payoff for a given pair (θ_i, θ_{-i}) as:

$$V_i(\theta_i, \theta_{-i}) = \theta_i q(\theta_i, \theta_{-i}) - t(q(\theta_i, \theta_{-i}), \theta_i).$$

A simple condition gives:

$$U(\theta_1, \theta_2) + \sum_{i=1}^{2} V_i(\theta_i, \theta_{-i}) = (\sum_{i=1}^{2} \theta_i) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)). \tag{A6}$$

From the fact that $U(\underline{\theta}, \underline{\theta}) = 0$ in any symmetric equilibrium, we must have:

$$2V(\underline{\theta}, \underline{\theta}) = 2\underline{\theta} q(\underline{\theta}, \underline{\theta}) - C(q(\underline{\theta}, \underline{\theta})) > 0. \tag{A7}$$

[43] See Laffont and Martimort (2002, Chapter 9) for instance.

Indeed, we have $\underline{\theta} - p(q(\underline{\theta}, \underline{\theta}), \underline{\theta}) > 0$ from SMP and (7). Hence, we get using (8):

$$2\underline{\theta} > C'(q(\underline{\theta}, \underline{\theta})) > \frac{C(q(\underline{\theta}, \underline{\theta}))}{q(\underline{\theta}, \underline{\theta})}$$

where the last inequality follows from the strict convexity of $C(\cdot)$, $C(0) = 0$ and the fact that $q(\underline{\theta}, \underline{\theta}) > 0$ when $p(q, \underline{\theta}) > 0$ and $C'(0) = 0$.

We also obtain the following expressions of the partial derivatives of $V(\cdot)$:

$$\frac{\partial V_i}{\partial \theta_{-i}}(\theta_i, \theta_{-i}) = (\theta_i - p(q(\theta_i, \theta_{-i}), \theta_i)) \frac{\partial q}{\partial \theta_{-i}}(\theta_i, \theta_{-i}) = R(\theta_{-i}) \frac{\partial p}{\partial \theta}(q(\theta_i, \theta_{-i}), \theta_{-i}) \frac{\partial q}{\partial \theta_{-i}}(\theta_i, \theta_{-i}). \tag{A8}$$

and

$$\frac{\partial V_i}{\partial \theta_i}(\theta_i, \theta_{-i}) = q(\theta_i, \theta_{-i}) + (\theta_i - p(q(\theta_i, \theta_{-i}), \theta_i)) \frac{\partial q}{\partial \theta_i}(\theta_i, \theta_{-i}) - \frac{\partial t}{\partial \theta}(q(\theta_i, \theta_{-i}), \theta_i)$$
$$= q(\theta_i, \theta_{-i}) + R(\theta_{-i}) \frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}(\theta_1, \theta_2) - \frac{\partial U}{\partial \theta_i}(\theta_1, \theta_2). \tag{A9}$$

Integrating (A9) yields

$$V_i(\theta_i, \theta_{-i}) = \phi(\theta_{-i}) + \int_{\underline{\theta}}^{\theta_i} q(x, \theta_{-i}) dx + R(\theta_{-i}) \frac{\partial U}{\partial \theta_{-i}}(\theta_i, \theta_{-i}) - U(\theta_i, \theta_{-i}) \tag{A10}$$

for some function $\phi(\cdot)$. Because $U(\underline{\theta}, \theta_{-i}) = 0$ for all θ_{-i}, one gets

$$V_i(\underline{\theta}, \theta_{-i}) = \phi(\theta_{-i}). \tag{A11}$$

Inserting the expressions obtained from (A10) and (A11) into (A6) yields:

$$-U(\theta_i, \theta_{-i}) + \sum_{i=1}^{2} R(\theta_i) \frac{\partial U}{\partial \theta_i}(\theta_i, \theta_{-i})$$
$$= \left(\sum_{i=1}^{2} \theta_i \right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) - \left(\sum_{i=1}^{2} \phi(\theta_i) + \int_{\underline{\theta}}^{\theta_i} q(x, \theta_{-i}) dx \right). \tag{A12}$$

Simple integrations by parts show that:

$$E_{(\theta_1, \theta_2)} \left(-U(\theta_1, \theta_2) + \sum_{i=1}^{2} R(\theta_i) \frac{\partial U}{\partial \theta_i}(\theta_1, \theta_2) \right) = E_{(\theta_1, \theta_2)}(U(\theta_1, \theta_2)).$$

Because in any equilibrium $U(\theta_1, \theta_2) \geq 0$, we must have from (A12):

$$E_{(\theta_1, \theta_2)} \left(\left(\sum_{i=1}^{2} \theta_i \right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) - \left(\sum_{i=1}^{2} \int_{\underline{\theta}}^{\theta_i} q(x, \theta_{-i}) dx \right) \right) \geq \sum_{i=1}^{2} E_{\theta_i}(\phi(\theta_i)).$$

Integrating by parts the left-hand side above yields the following inequality:

$$E_{(\theta_1, \theta_2)} \left(\left(\sum_{i=1}^{2} \theta_i - R(\theta_i) \right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) \right) \geq \sum_{i=1}^{2} E_{\theta_i}(\phi(\theta_i)).$$

To get (22), note that $\phi'(\theta_{-i}) \geq 0$ from (A8) and that $\phi(\underline{\theta}) > 0$ is given by (A7).

In passing, using (A10), integrating by parts and taking into account that $U(\theta_i, \underline{\theta}) = 0$ shows also that

$$E_{\theta_{-i}}(V(\theta_i, \theta_{-i})) = E_{\theta_{-i}}(\phi(\theta_{-i})) + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}(q(x, \theta_{-i}))dx \geq \phi(\underline{\theta}) > 0.$$

Hence, the principals' interim participation constraints are satisfied. ■

Proof of Proposition 3: Define first

$$J(\theta_2) = E_{\theta_1}\left(\left(\sum_{i=1}^{2} \theta_i - R(\theta_i)\right) q^{FB}(\theta_1, \theta_2) - C(q^{FB}(\theta_1, \theta_2))\right)$$

so that $I = E_{\theta_2}(J(\theta_2))$. Integrating by parts and using $\frac{d}{dx}(x(F(x)-1)) = xf(x)-1-F(x)$, we have:

$$J(\theta_2) = (\underline{\theta} + \theta_2 - R(\theta_2))\, q^{FB}(\underline{\theta}, \theta_2) - C(q^{FB}(\underline{\theta}, \theta_2))$$
$$+ \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1, \theta_2) \left(\theta_1 + \theta_2 - R(\theta_2) - C'(q^{FB}(\theta_1, \theta_2))\right)(1 - F(\theta_1))d\theta_1.$$

Using the definition of $q^{FB}(\cdot)$ to simplify the last integral, this yields:

$$J(\theta_2) = (\underline{\theta} + \theta_2 - R(\theta_2))\, q^{FB}(\underline{\theta}, \theta_2) - C(q^{FB}(\underline{\theta}, \theta_2)) - R(\theta_2) \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1, \theta_2)(1 - F(\theta_1))d\theta_1.$$

Therefore, taking expectations w.r.t. θ_2 yields:

$$I = E_{(\theta_1,\theta_2)}\left(\sum_{i=1}^{2} (\theta_i - R(\theta_i))\, q^{FB}(\theta_1, \theta_2) - C(q^{FB}(\theta_1, \theta_2))\right) = E_{\theta_2}(J(\theta_2))$$
$$= E_{\theta_2}\left((\underline{\theta} + \theta_2 - R(\theta_2))\, q^{FB}(\underline{\theta}, \theta_2) - C(q^{FB}(\underline{\theta}, \theta_2))\right)$$
$$- \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1, \theta_2)(1 - F(\theta_2))(1 - F(\theta_1))d\theta_1 d\theta_2.$$

The first term can again be integrated by parts to get:

$$E_{\theta_2}\left((\underline{\theta} + \theta_2 - R(\theta_2))\, q^{FB}(\underline{\theta}, \theta_2) - C(q^{FB}(\underline{\theta}, \theta_2))\right) = 2\underline{\theta} q^{FB}(\underline{\theta}, \underline{\theta}) - C(q^{FB}(\underline{\theta}, \underline{\theta}))$$
$$- \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_2}(\underline{\theta}, \theta_2)(\underline{\theta} + \theta_2 - C'(q^{FB}(\underline{\theta}, \theta_2)))(1 - F(\theta_2))d\theta_2$$

where the last integral is zero by definition of $q^{FB}(\cdot)$. Gathering everything, we get:

$$I = 2\underline{\theta} q^{FB}(\underline{\theta}, \underline{\theta}) - C(q^{FB}(\underline{\theta}, \underline{\theta})) - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial q^{FB}}{\partial \theta_1}(\theta_1, \theta_2)(1 - F(\theta_2))(1 - F(\theta_1))d\theta_1 d\theta_2$$
$$< 2\underline{\theta} q^{FB}(\underline{\theta}, \underline{\theta}) - C(q^{FB}(\underline{\theta}, \underline{\theta})).$$

Hence, (22) does not hold for the first-best. ■

Proof of Proposition 4: First consider a q-isoquant which crosses the vertical axis at $\bar{\theta}$. Define $\underline{\theta}(q)$ such that $\underline{\theta}(q) = \psi(q, \bar{\theta})$ (and thus $\bar{\theta} = \psi(q, \underline{\theta}(q))$). From the equilibrium conditions (24) taken respectively at $\underline{\theta}(q)$ and $\bar{\theta}$, we get:

$$\underline{\theta}(q) = p(q, \underline{\theta}(q)) \text{ and } \bar{\theta} - C'(q) + p(q, \underline{\theta}(q)) = R(\underline{\theta}(q))\frac{\partial p}{\partial q}(q, \underline{\theta}(q)).$$

From SMP and $\underline{\theta}(q) < \bar{\theta}$, we get $\bar{\theta} > p(q, \bar{\theta})$.

Taking (A19) to evaluate $\frac{\partial \psi}{\partial \theta}(q, \theta)$ at $\underline{\theta}(q)$ and using Lhospital's rule yields

$$\frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) = \frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) \frac{\frac{\partial p}{\partial \theta}(q, \underline{\theta}(q))}{1 - \frac{\partial p}{\partial \theta}(q, \underline{\theta}(q))}.$$

The only possibility for having $\frac{\partial \psi}{\partial \theta}(q, \underline{\theta}(q)) < 0$ is $\frac{\partial p}{\partial \theta}(q, \underline{\theta}(q)) = \frac{1}{2}$. Therefore, $\underline{\theta}(q)$ is defined by (25) and, given that $R(\cdot)$ is decreasing, this can only be possible when $C'(q) \geq \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$. For $C'(q) < \bar{\theta} + \underline{\theta} - \frac{1}{2f(\underline{\theta})}$, the conditions coming from the equilibrium behavior of types $\underline{\theta}$ and $\bar{\theta}$ are given by (28). ■

Proof of Proposition 5: Note that the boundary condition (A14) can be used to integrate (32) and get $G(\cdot, q)$ as a solution to:

$$1 - G(p, q) = (1 - F(\underline{\theta}(q)))exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q) - x, q)) - C'(q) + x}\right). \quad \text{(A13)}$$

Consider now the mapping $\Phi(\cdot)$ such that:

$$1 - \Phi(G)(p) = (1 - F(\underline{\theta}(q)))exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q) - x)) - C'(q) + x}\right). \quad \text{(A14)}$$

An equilibrium distribution $G(\cdot, q)$ is thus a fixed-point of the mapping $\Phi(\cdot)$.

Several facts immediately follow from the definition (A14):

- Boundary conditions: $\Phi(G)(\underline{\theta}(q)) = F(\underline{\theta}(q))$, and $\Phi(G)(C'(q) - \underline{\theta}(q)) = 1$ when $G(\underline{\theta}(q), q) = F(\underline{\theta}(q))$;
- $\Phi(\cdot)$ is monotonically decreasing and thus $\Phi^2(\cdot)$ is monotonically increasing: $G_1 \leq G_2$ implies $\Phi(G_1) \geq \Phi(G_2)$.

Consider $\Phi(1)(\cdot)$. For $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q))$, we have:

$$1 - \Phi(1)(p) = (1 - F(\underline{\theta}(q)))exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{\bar{\theta} - C'(q) + x}\right) = (1 - F(\underline{\theta}(q)))\left(\frac{\bar{\theta} + \underline{\theta} - C'(q)}{\bar{\theta} - C'(q) + p}\right)$$

with

$$\Phi(1)(C'(q) - \underline{\theta}(q)) = 1 \text{ and } lim_{p \to C'(q) - \underline{\theta}(q)} \Phi(1)(p) < 1.$$

One can check that

$$\Phi(F)(p) = \begin{cases} 1 & \text{if } p > \underline{\theta}(q) \\ \underline{\theta}(q) & \text{if } p = \underline{\theta}(q) \end{cases} \text{ and } \Phi^2(F) = \Phi(1). \tag{A15}$$

Moreover, we want to find a condition ensuring that the mapping $\Phi(\cdot)$ will be onto and that the distribution of price at any iteration starting from $\Phi(1)(\cdot)$ never crosses $F(\cdot)$ to avoid infinite terms in the denominator on the right-hand side of (A14). A sufficient condition is that $\Phi(1)(p) \geq F(p)$ for all $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$. This amounts to:

$$\chi(p) = (1 - F(p))(\bar{\theta} + p - C'(q)) - (1 - F(\underline{\theta}(q)))(\bar{\theta} + \underline{\theta}(q) - C'(q)) \geq 0. \tag{A16}$$

Note that $\chi(\underline{\theta}(q)) = 0$ and that $\chi(\cdot)$, which is quasi-concave under the assumption $\frac{d}{dp}\left(\frac{1-F(p)}{f(p)}\right) \leq 0$, achieves its maximum at $p^* < C'(q) - \underline{\theta}(q)$ such that $\bar{\theta} + p^* - C'(q) = \frac{1-F(p^*)}{f(p^*)}$. Hence, (A16) holds when $\chi(C'(q) - \underline{\theta}(q)) > 0$. This last condition holds when $\frac{1-F(x)}{\bar{\theta}-x}$ increases with x, a sufficient condition is $\frac{d}{dp}\left(\frac{1-F(p)}{f(p)}\right) \leq -1$.

Consider now the following sequence $\phi_n = \Phi^n(\phi_0)$ with $\phi_0 = F$. One can easily show that ϕ_{2k} is increasing whereas ϕ_{2k+1} is decreasing. Moreover, $\phi_2 < 1 = \phi_1$ and thus, by iterating, we get $\phi_{2k} \leq \phi_{2k+1}$. Moreover, as soon as $n \geq 2$, $\phi_n(\underline{\theta}(q)) = F(\underline{\theta}(q))$ and $\phi_n(C'(q) - \underline{\phi}(\underline{\theta}(q)) = 1$. Now, denote by $\underline{\phi}$ and $\bar{\phi}$ the respective limits of ϕ_{2k} and ϕ_{2k+1}. We have: $\underline{\phi} \leq \bar{\phi}$, $\quad \underline{\phi} = \Phi(\bar{\phi})$ and $\bar{\phi} = \Phi(\underline{\phi})$. Note that $\underline{\phi}(\underline{\theta}(q)) = \bar{\phi}(\underline{\theta}(q)) = F(\underline{\theta}(q))$ and $\underline{\phi}(C'(q) - \underline{\theta}(q)) = \bar{\phi}(C'(q) - \underline{\theta}(q)) = 1$. Moreover, $\underline{\phi}(\cdot)$ and $\bar{\phi}(\cdot)$ are by definition both differentiable at $C'(q) - \underline{\theta}(q)$ with $\dot{\underline{\phi}}_{2k}(C'(q) - \underline{\theta}(q))$ decreasing and $\dot{\underline{\phi}}_{2k+1}(C'(q) - \underline{\theta}(q))$ increasing whereas $+\infty > \dot{\underline{\phi}}(C'(q) - \underline{\theta}(q)) \geq \dot{\bar{\phi}}(C'(q) - \underline{\theta}(q)) > 0 = \dot{\phi}_1(C'(q) - \underline{\theta}(q))$.

Define first $N = \{G(\cdot)|G(\cdot)$ is increasing and $\underline{\phi}(p) \leq G(p) \leq \bar{\phi}(p)$ for all $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]\}$. Clearly, N is convex and non-empty. Let us also define:

$$N^* = \{G(\cdot)|G(\cdot) \text{ is increasing and } \underline{\phi}(p) \leq G(p) \leq \bar{\phi}(p) \text{ for all } p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$$

$$\text{and } |G(p) - G(p')| \leq K|p - p'|\}$$

where $K < +\infty$ is chosen below. $\Phi(\cdot)$ maps N into N^*. Indeed, from the Theorem of Intermediate Values, we have:

$$|\Phi(G)(p) - \Phi(G)(p')| = |\Phi(\dot{G})(\zeta)||p - p'|$$

for some $\zeta \in [p, p']$ where

$$|\Phi(\dot{G})(\zeta)| = \frac{(1 - F(\underline{\theta}(q)))}{F^{-1}(G(C'(q) - \zeta)) - C'(q) + \zeta} exp\left(-\int_{\underline{\theta}(q)}^{\zeta} \frac{dx}{F^{-1}(G(C'(q) - x)) - C'(q) + x}\right).$$

Using that $\underline{\phi} \leq G \leq \bar{\phi}$, we get

$$|\Phi(\dot{G})(\zeta)| \leq \frac{(1 - F(\underline{\theta}(q)))}{F^{-1}(\underline{\phi}(C'(q) - \zeta)) - C'(q) + \zeta} exp\left(-\int_{\underline{\theta}(q)}^{\zeta} \frac{dx}{F^{-1}(\bar{\phi}(C'(q) - x)) - C'(q) + x}\right)$$

$$= \frac{1-\underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q)-\zeta)) - C'(q) + \zeta}.$$

The right-hand side above is in fact a bounded function of ζ over $[\underline{\theta}(q), C'(q) - \underline{\theta}(q)]$. Indeed, using Lhospital rule, we have:

$$lim_{\zeta \to C'(q)-\underline{\theta}(q)} \frac{1-\underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q)-\zeta)) - C'(q) + \zeta} = \frac{\dot{\underline{\phi}}(C'(q)-\underline{\theta}(q))}{1 - \frac{\dot{\underline{\phi}}(\underline{\theta}(q))}{f(\underline{\theta}(q))}}.$$

Now, notice that

$$\dot{\underline{\phi}}(\underline{\theta}(q)) = \frac{(1-F(\underline{\theta}(q)))}{\bar{\theta} + \underline{\theta}(q) - C'(q)} = \frac{f(\underline{\theta}(q))}{2}.$$

Hence, we get

$$lim_{\zeta \to C'(q)-\underline{\theta}(q)} \frac{1-\underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q)-\zeta)) - C'(q) + \zeta} = 2\dot{\underline{\phi}}(C'(q)-\underline{\theta}(q)).$$

Finally, denote $K' = \sup_{\zeta \in [\underline{\theta}(q), C'(q)-\underline{\theta}(q)]} \frac{1-\underline{\phi}(\zeta)}{F^{-1}(\underline{\phi}(C'(q)-\zeta)) - C'(q)+\zeta} < +\infty$. Take now $K = \sup\{K', \sup_\zeta \dot{\underline{\phi}}(\zeta), \sup_\zeta \dot{\bar{\phi}}(\zeta)\}$. Such value of K ensures that N^* is non-empty because at least $\underline{\phi}$ and $\bar{\phi}$ are in it. Moreover, by Ascoli Theorem, N^* is compact.

Finally, $\Phi(\cdot)$ is continuous on N. To show that consider two distributions G and H in N. We have:

$$\Phi(G)(p) - \Phi(H)(p) = (1 - F(\underline{\theta}(q))$$
$$\times \left(exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(H(C'(q)-x)) - C'(q) + x}\right) - exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q)-x)) - C'(q) + x}\right)\right). \quad \text{(A17)}$$

First, note that $H \leq \bar{\phi}$ implies

$$exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(H(C'(q)-x)) - C'(q) + x}\right) \leq exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(\bar{\phi}(C'(q)-x)) - C'(q) + x}\right)$$

and similarly, $G \leq \bar{\phi}$ implies

$$exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(G(C'(q)-x)) - C'(q) + x}\right) \leq exp\left(-\int_{\underline{\theta}(q)}^{p} \frac{dx}{F^{-1}(\bar{\phi}(C'(q)-x)) - C'(q) + x}\right).$$

Now fix ϵ arbitrarily small. There exists η such that for $p \geq C'(q) - \underline{\theta}(q) - \eta$, both right-hand sides above are less than ϵ and thus $|\Phi(G)(p) - \Phi(H)(p)| \leq 2\epsilon$. For $p \in [\underline{\theta}(q), C'(q) - \underline{\theta}(q) - \eta]$, the right-hand side of (A17) can be made arbitrarily small, say less than 2ϵ by taking H close enough to G with respect to $||\cdot||_\infty$. Gathering everything $||\Phi(G) - \Phi(H)||_\infty = sup_p|\Phi(G)(p) - \Phi(H)(p)| \leq 2\epsilon$ which ensures continuity.

Therefore $\Phi(\cdot)$ is a compact map from N onto N^*. We get existence of $G(\cdot, q)$ using then Schauder's Second Theorem[44] which states that a compact mapping on a convex non-empty subset of a Banach space N has a fixed point. ■

[44] See Burton (2005, p.184).

Proof of Proposition 6: Let us suppose that there exists two fixed-points G_1 and G_2 for $\Phi(\cdot)$ with $G_1(\underline{\theta}(q)) = G_2(\underline{\theta}(q)) = F(\underline{\theta}(q))$, $G_1(C'(q) - \underline{\theta}(q)) = G_2(C'(q) - \underline{\theta}(q)) = 1$ and $G_1(p) \leq G_2(p)$ for $p \in (\underline{\theta}(q), C'(q) - \underline{\theta}(q))$. But then, using that $\Phi(\cdot)$ is monotonic, we get $G_1 = \Phi(G_1) \geq G_2 = \Phi(G_2)$ and, finally, $G_1 = G_2$. Hence, the equilibrium is unique.■

Proof of Proposition 7: Equations (23) and (24) can be first transformed into a system of first-order differential equations to get both the marginal contribution of a given type and the identity of his conjugate. Using (23), we get:

$$\frac{\partial p}{\partial \theta}(q, \theta) = r(\psi(q, \theta) + p(q, \theta) - C'(q)). \tag{A18}$$

Now differentiating (23) with respect to θ yields:

$$\frac{\partial p}{\partial \theta}(q, \theta) = -\frac{\partial p}{\partial \theta}(q, \psi(q, \theta))\frac{\partial \psi}{\partial \theta}(q, \theta).$$

Using (24), we get:

$$\frac{\partial \psi}{\partial \theta}(q, \theta) = -\frac{\psi(q, \theta) + p(q, \theta) - C'(q)}{\theta - p(q, \theta)}. \tag{A19}$$

Taking $\psi(q, \theta)$ from (A19), differentiating w.r.t. θ and inserting into (A18) yields:

$$\frac{\partial p}{\partial \theta}(q, \theta)\,(1 - r(\theta - p(q, \theta))) + (\theta - p(q, \theta))\frac{\partial^2 p}{\partial \theta^2}(q, \theta) = 0. \tag{A20}$$

The solutions to this differential equation do not depend on q and we denote $u(\theta) = \theta - \frac{1}{r} - p(q, \theta)$. We look for such non-negative solutions $u(\cdot)$ with $0 < \dot{u}(\theta) \leq 1$ where the last inequality is needed to satisfy the SMP. (A20) can also be written as

$$\ddot{u}(\theta)(ru(\theta) + 1) + ru(\theta)(1 - \dot{u}(\theta)) = 0.$$

Defining $\phi(\cdot)$ as $\dot{u}(\theta) = \phi(u(\theta))$, we get:

$$\phi'(u)\frac{\phi(u)}{1 - \phi(u)} = -\frac{ru}{1 + ru}.$$

A first quadrature yields:

$$\phi(u) + log(1 - \phi(u)) = -\lambda + u - \frac{1}{r}log(1 + ru)$$

where λ is some constant. Since the function $\phi + log(1 - \phi)$ is monotonically decreasing on $[0, 1)$, it is invertible. Denote $G(\cdot)$ its inverse defined over $\mathbb{R}^-$. We obtain:

$$\dot{u}(\theta) = G\left(-\lambda + u(\theta) - \frac{1}{r}log(1 + ru(\theta))\right). \tag{A21}$$

Take now any initial value $u(\underline{\theta}) \in (0, \underline{\theta} - \frac{1}{r})$ and consider the solution to (A21) with this initial condition when $\lambda > u(\underline{\theta}) - \frac{1}{r}log(1 + ru(\underline{\theta}))$. It is easy to show that the solution $u(\cdot)$ is, non-negative, strictly increasing and has a slope less than 1, so that it never reaches the boundary $v(\theta) = \theta - \frac{1}{r}$. Using the Theorem of Uniqueness for the solution to such differential equation,[45] it can also be shown that such solution converges without reaching it towards a limit u_∞ defined as $\lambda = u_\infty - \frac{1}{r}log(1 + ru_\infty)$. ■

Proof of Lemma 4: The uninformed mediator offers a centralized mechanism $\{T_1(\theta_i, \theta_{-i}), T_2(\theta_i, \theta_{-i}), q(\theta_i, \theta_{-i})\}$. Denote P_i's expected payoff when his type is θ_i as:

$$V_i(\theta_i) = \theta_i E_{\theta_{-i}}(q(\theta_i, \theta_{-i}) - T_i(\theta_i, \theta_{-i})).$$

Denote also the agent's payoff as

$$U(\theta_1, \theta_2) = \sum_{i=1}^{2} T_i(\theta_i, \theta_{-i}) - C(q(\theta_i, \theta_{-i})).$$

Incentive compatibility implies

$$\dot{V}_i(\theta_i) = E_{\theta_{-i}}(q(\theta_i, \theta_{-i})) \tag{A22}$$

and

$$E_{\theta_{-i}}[q(\theta_i, \theta_{-i})] \text{ non-decreasing in } \theta_i. \tag{A23}$$

Voluntary participation by the principals and the agent requires respectively:

$$V_i(\theta_i) \geq 0 \quad \forall \theta_i. \tag{A24}$$

$$U(\theta_1, \theta_2) \geq 0 \quad \forall (\theta_1, \theta_2). \tag{A25}$$

The uninformed mediator maximizes now the following objective function:[46]

$$E_{(\theta_1, \theta_2)} \left(\sum_{i=1}^{2} \alpha'(\theta_i) f(\theta_i) V(\theta_i) + \beta U(\theta_1, \theta_2) \right)$$

subject to (A22) to (A25)

for some weights $\alpha'(\cdot)$ to be made precise below. The characterization of those interim efficient allocations follows then closely Ledyard and Palfrey (1999). First, note that (A22) implies

$$V_i(\theta_i) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}(q(x, \theta_{-i}))dx$$

where we use symmetry to set $V_1(\underline{\theta}) = V_2(\underline{\theta}) = V(\underline{\theta}) \geq 0$. Then, observe that

$$E_{(\theta_1, \theta_2)}((\sum_{i=1}^{2} \theta_i) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) - \sum_{i=1}^{2} V_i(\theta_i)) = E_{(\theta_1, \theta_2)}(U(\theta_1, \theta_2)) \geq 0$$

where the last inequality follows from (A25). Integrating by parts the left-hand side above, one gets

$$E_{(\theta_1, \theta_2)} \left(\left(\sum_{i=1}^{2} \theta_i - R(\theta_i) \right) q(\theta_1, \theta_2) - C(q(\theta_1, \theta_2)) \right) \geq 2V(\underline{\theta}). \tag{A26}$$

[45] Hirsh and Smale (1974, p.164).

[46] We neglect (A23) which is checked ex post.

Integrating by parts the mediator's objective function, we get:

$$\beta\left(E_{(\theta_1,\theta_2)}\left(\left(\sum_{i=1}^{2}\theta_i - R(\theta_i)\right)q(\theta_1,\theta_2) - C(q(\theta_1,\theta_2))\right)\right)$$

$$+\sum_{i=1}^{2}\int_{\underline{\theta}}^{\bar{\theta}}(1-F(\theta_i))\tilde{\alpha}'(\theta_i)E_{\theta_{-i}}(q(\theta_i,\theta_{-i}))d\theta_i + 2V(\underline{\theta})\left(\int_{\underline{\theta}}^{\bar{\theta}}\alpha'(\theta)f(\theta)d\theta - \beta\right) \quad \text{(A27)}$$

where $\tilde{\alpha}'(\theta_i) = \frac{1}{1-F(\theta_i)}\int_{\theta_i}^{\bar{\theta}}\alpha'(\theta)f(\theta)d\theta$. Hence, any interim efficient allocation must maximize (A27) subject to (A26). Denote λ the multiplier of this last constraint. Optimizing the corresponding Lagrangean pointwise yields:

$$C'(q(\theta_1,\theta_2)) = \sum_{i=1}^{2}\theta_i - R(\theta_i)\left(1-\frac{\tilde{\alpha}'(\theta_i)}{\lambda+\beta}\right)$$

which is the solution when the monotonicity condition (A23) holds; and $V(\underline{\theta})$ is not infinite when $\frac{\tilde{\alpha}'(\bar{\theta})}{\beta+\lambda} \leq 1$. Denoting $\alpha(\theta) = \frac{\tilde{\alpha}'(\theta)}{\beta+\lambda}$ yields (34).

Reciprocally, the fact that a common agency equilibrium satisfies (34) implies that one can find transfers which implement the corresponding output. Take $T_i(\theta_i,\theta_{-i}) = t(q(\theta_i,\theta_{-i}),\theta_i)$ where $t(\cdot)$ is the symmetric contribution schedule. ■

Proof of Proposition 8: We first prove a lemma which significantly restricts the kind of equilibrium schedules which may be looked for.

Lemma 9 *Any equilibrium of a common agency game which is interim efficient must be such that:*

$$\frac{\partial^2 p}{\partial\theta\partial q}(Q(\theta),\theta) = 0 \quad \forall\theta\in\Theta. \quad \text{(A28)}$$

Proof: First notice that along the diagonal where both principals have the same type θ, we must have:

$$b(\theta) = p(Q(\theta),\theta) \text{ and } \theta - b(\theta) = R(\theta)\frac{\partial p}{\partial\theta}(Q(\theta),\theta). \quad \text{(A29)}$$

Let us now fix an isoquant having equation $\theta_2 = \psi(Q(\tilde{\theta}),\theta_1)$. From (A29), we have:

$$\sum_{i=1}^{2}\theta_i - C'(Q(\tilde{\theta})) = \sum_{i=1}^{2}R(\theta_i)\frac{\partial p}{\partial\theta}(Q(\theta_i),\theta_i). \quad \text{(A30)}$$

Along such isoquant, we have also

$$\theta_i - p(Q(\tilde{\theta}),\theta_i) = R(\theta_{-i})\frac{\partial p}{\partial\theta_{-i}}(Q(\tilde{\theta}),\theta_{-i}).$$

Therefore, we get:

$$\sum_{i=1}^{2} \theta_i - C'(Q(\tilde{\theta})) = \sum_{i=1}^{2} R(\theta_i) \frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \theta_i). \tag{A31}$$

Gathering (A30) and (A31) yields along the isoquant:

$$R(\theta_1)\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \theta_1) + R(\psi(Q(\tilde{\theta}), \theta_1))\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \psi(Q(\tilde{\theta}), \theta_1))$$

$$= R(\theta_1)\frac{\partial p}{\partial \theta}(Q(\theta_1), \theta_1) + R(\psi(Q(\tilde{\theta}), \theta_1))\frac{\partial p}{\partial \theta}(Q(\psi(Q(\tilde{\theta}), \theta_1)), \psi(Q(\tilde{\theta}), \theta_1)). \tag{A32}$$

This identity should hold for all θ_1. We now look at the Taylor expansions of both the right- and left-hand sides of (A32) around $\tilde{\theta}$. Note that

$$\frac{\partial \psi}{\partial \theta_1}(Q(\tilde{\theta}), \tilde{\theta}) = -1 \text{ and } \frac{\partial^2 \psi}{\partial \theta_1^2}(Q(\tilde{\theta}), \tilde{\theta}) = -2\left(\frac{\dot{R}(\tilde{\theta})}{R(\tilde{\theta})} - \frac{1 - \frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta})}{\tilde{\theta} - \frac{C'(Q(\tilde{\theta}))}{2}}\right).$$

For an interim efficient equilibrium (if any), it must be that

$$2\tilde{\theta} - C'(Q(\tilde{\theta})) = 2R(\tilde{\theta})(1 - \tilde{\alpha}(\tilde{\theta})) \leq 2R(\tilde{\theta}).$$

Since $\dot{R}(\theta) < 0$, we have $\frac{\partial^2 \psi}{\partial \theta_1^2}(Q(\tilde{\theta}), \tilde{\theta}) > 0$. The right- and left-hand sides of (A32) are equal at $\tilde{\theta}$ and have both zero first-order derivative at this point. The second-order derivative for the left-hand side is

$$\frac{\partial^2 \psi}{\partial \theta_1^2}(Q(\tilde{\theta}), \tilde{\theta})\left(\dot{R}(\tilde{\theta})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) + R(\tilde{\theta})\frac{\partial^2 p}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta})\right).$$

The second-order derivative for the right-hand side is

$$\frac{\partial^2 \psi}{\partial \theta_1^2}(Q(\tilde{\theta}), \tilde{\theta})\left(\dot{R}(\tilde{\theta})\frac{\partial p}{\partial \theta}(Q(\tilde{\theta}), \tilde{\theta}) + R(\tilde{\theta})\left(\frac{\partial^2 p}{\partial \theta^2}(Q(\tilde{\theta}), \tilde{\theta}) + \frac{\partial^2 p}{\partial \theta \partial q}(Q(\tilde{\theta}), \tilde{\theta})\dot{Q}(\tilde{\theta})\right)\right).$$

Since $\dot{Q}(\tilde{\theta}) > 0$ holds, these second-order derivatives can only be equal when (A28) holds. ■

Condition (A28) is of course very demanding since, taken with the equilibrium conditions, it fully characterizes the equilibrium $Q(\cdot)$ along the diagonal.

From (14) that we differentiate w.r.t. q, we have:

$$\frac{\partial \psi}{\partial q}(Q(\theta), \theta) - C''(Q(\theta)) + \frac{\partial p}{\partial q}(Q(\theta), \theta) = R(\theta)\frac{\partial^2 p}{\partial \theta \partial q}(Q(\theta), \theta) = 0.$$

Using also the identity $\psi(Q(\theta), \theta) = \theta$ and differentiating w.r.t. θ yield:

$$\frac{\partial \psi}{\partial q}(Q(\theta), \theta)\dot{Q}(\theta) + \frac{\partial \psi}{\partial \theta}(Q(\theta), \theta) = 1.$$

Using $\frac{\partial \psi}{\partial \theta}(Q(\theta),\theta) = -1$, we finally find

$$\frac{\partial p}{\partial q}(Q(\theta),\theta) = C''(Q(\theta)) + \frac{2}{\dot{Q}(\theta)}.$$

Moreover, using $2p(Q(\theta),\theta) = C'(Q(\theta))$ and differentiating w.r.t. θ yields

$$\left(2\frac{\partial p}{\partial q}(Q(\theta),\theta) - C''(Q(\theta))\right)\dot{Q}(\theta) + \frac{\partial p}{\partial \theta}(Q(\theta),\theta) = 0.$$

Finally, we have:

$$2\theta - C'(Q(\theta)) = R(\theta)\frac{\partial p}{\partial \theta}(Q(\theta),\theta) = (4 - C''(Q(\theta))\dot{Q}(\theta))R(\theta). \tag{A33}$$

Integrating (A33) with the boundary condition requested by interim efficiency (i.e., $C'(Q(\bar{\theta})) = 2\bar{\theta}$) yields (37). ■

Proof of Proposition 9: Imposing that the centralized (symmetric) mechanism $\{T(\theta_i,\theta_{-i}), q(\theta_i,\theta_{-i})\}$ is implemented with dominant strategy amounts to

$$\theta_i \frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}) = \frac{\partial T}{\partial \theta_i}(\theta_i,\theta_{-i}).$$

Using the definition of $T(\cdot)$ yields:

$$\left(\theta_i - \frac{\partial t}{\partial q}(q(\theta_i,\theta_{-i}),\theta_i)\right)\frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}) = \frac{\partial t}{\partial \theta}(q(\theta_i,\theta_{-i}),\theta_i). \tag{A34}$$

Remember that the agent's rent can be written as:

$$U(\theta_i,\theta_{-i}) = arg\max_q \sum_{i=1}^{2} t(q,\theta_i) - C(q).$$

Using the Envelope Theorem, we immediately get

$$\frac{\partial U}{\partial \theta_i}(\theta_i,\theta_{-i}) = \frac{\partial t}{\partial \theta}(q(\theta_i,\theta_{-i}),\theta_i), \tag{A35}$$

and

$$\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}(\theta_1,\theta_2) = \frac{\partial^2 t}{\partial \theta \partial q}(q(\theta_i,\theta_{-i}),\theta_{-i})\frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}). \tag{A36}$$

Using the first-order condition for the agent's problem (8), inserting into (A34) and using the ex post optimality condition (7) yields

$$(\theta_i - C'(q(\theta_i,\theta_{-i})) + p(q(\theta_i,\theta_{-i}),\theta_{-i}))\frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}) = R(\theta_{-i})\frac{\partial p}{\partial \theta}(q(\theta_i,\theta_{-i}),\theta_{-i})\frac{\partial q}{\partial \theta_i}(\theta_i,\theta_{-i}).$$

Using now (A35) and (A36), we get the following partial derivative equation satisfied by the non-negative $U(\cdot)$ function:

$$R(\theta_{-i})\frac{\partial^2 U}{\partial \theta_1 \partial \theta_2}(\theta_1,\theta_2) = \frac{\partial U}{\partial \theta_i}(\theta_1,\theta_2) \text{ for } i = 1,2. \tag{A37}$$

Taking into account the boundary conditions $U(\underline{\theta},\theta_i) = 0$, the only possible solution is $U(\theta_1,\theta_2) = 0$ for all (θ_1,θ_2). But then, the first-best would be implemented, a contradiction with Proposition 3. ■