On Competitive Nonlinear Pricing*

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April 9, 2014

Abstract

Many financial markets rely on a discriminatory limit-order book to balance supply and demand. We study these markets in a static model in which uninformed market makers compete in nonlinear tariffs to trade with an informed insider, as in Glosten (1994), Biais, Martimort, and Rochet (2000), and Back and Baruch (2013). We analyze the case where tariffs are unconstrained and the case where tariffs are restricted to be convex. In both cases, we show that pure-strategy equilibrium tariffs must be linear and, moreover, that such equilibria only exist under exceptional circumstances. These results cast doubt on the stability of even well-organized financial markets.

Keywords: Adverse Selection, Competing Mechanisms, Limit-Order Book.
JEL Classification: D43, D82, D86.

*We thank Bruno Biais, John Hardman Moore, Alessandro Pavan, Jean-Charles Rochet, Dimitri Vayanos, and Michael Whinston for very valuable feedback. We also thank seminar audiences at Aalto School of Economics, ETH Zürich, LSE, Northwestern University, Toulouse School of Economics, Università degli Studi di Milano-Bicocca, Université de Montpellier, as well as conference participants at the 7th ENSAI Economic Day, the 2013 Conference of the Society for the Advancement of Economic Theory, the 2013 Cowles Foundation Annual Conference on General Equilibrium and its Applications, the 2013 Glasgow SIRE Workshop in Microeconomic Theory, the 2013 LUISS Workshop on Macroeconomics and Financial Frictions, and the 2013 Université Paris-Dauphine Workshop in Honor of Rose-Anne Dana for many useful discussions. Elena Panova provided excellent research assistance. Financial support from the Chaire Marchés des Risques et Création de Valeur and the European Research Council (Starting Grant 203929–ACAP) is gratefully acknowledged.

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1 Introduction

Important financial markets, such as EURONEXT or NASDAQ, rely on a discriminatory limit-order book to balance supply and demand. This book gathers the limit orders posted by market makers.[1] Any upcoming order is then matched with the best offers in the book. Pricing is discriminatory, in that each market maker gets paid at the price he has quoted for a given volume of shares.[2] The aim of this paper is to analyze the formation of prices on such a discriminatory market.

Models of the discriminatory limit-order book typically feature an insider with superior information about the fundamental value of the traded asset.[3] This makes less informed market makers reluctant to sell, as they suspect that the fundamental value is likely to be high when the asset is in high demand. Market makers compete by posting sequences of limit orders or, equivalently, convex tariffs.[4] The insider then hits the resulting limit-order book with a market order that reflects her private information.[5] The problem of price formation thus amounts to characterizing the tariffs posted by market makers in equilibrium, in anticipation of the insider's trading strategy.

In a well-known article, Glosten (1994) proposed a candidate nonlinear tariff, meant to describe the limit-order book as a whole, and that can be interpreted as a marginal version of Akerlof (1970) pricing. Namely, this tariff specifies that an additional share beyond any quantity Q can be bought at a price equal to the expected value of the asset, conditional on the event that the insider buys at least Q shares. By construction, this tariff is convex and yields zero expected profit to the market makers. Glosten (1994) additionally shows that this tariff is the only one to resist entry by an uninformed market maker. As acknowledged in Glosten (1998), however, a natural question is whether this tariff can be sustained in an equilibrium of a competitive game with strategic market makers.

This issue was first addressed in Biais, Martimort, and Rochet (2000) in a model where a risk-averse insider with private but imperfect information about the fundamental value of an asset may trade for informational or hedging purposes. When the insider's marginal valuation

[1]A limit order allows to trade at the specified price any quantity up to a specified limit.

[2]By contrast, uniform limit-order books compute a single price that balances supply and demand and that applies to all trades that are matched. The appropriate modeling tool is supply-function equilibria (see, for instance, Wilson (1979), Grossman (1981), Klemperer and Meyer (1989), Kyle (1989), and Vives (2011)).

[3]See, for instance, Glosten (1994), Biais, Martimort, and Rochet (2000, 2013), and Back and Baruch (2013)).

[4]This is unlike the discriminatory models of Treasury-bill auctions (Wilson (1979), Back and Zender (1993)), in which the bidders are the holders of private information.

[5]This timing differs from Kyle's (1989) uniform-price auction, in which both informed and uninformed traders simultaneously post supply functions.

for the asset—an aggregate of the insider's informational and hedging motivations to trade—is continuously distributed, Biais, Martimort, and Rochet (2000) exhibit a unique pure-strategy equilibrium with strictly convex tariffs. The equilibrium outcome is reminiscent of Cournot competition: with a finite number of market makers, the equilibrium is symmetric and each market maker earns a strictly positive expected profit. In the limit when the number of market makers grows large, the equilibrium aggregate tariff converges to the Glosten (1994) tariff. Back and Baruch (2013) complement this study by focusing on a slightly different game in which market makers are restricted to post convex tariffs from the outset. Focusing on symmetric equilibria with strictly convex tariffs, they identify the same equilibrium tariff as in Biais, Martimort, and Rochet (2000). These results have been interpreted as a strategic foundation to Glosten's (1994) original approach, in the spirit of Cournotian foundations of competitive equilibrium (Parlour and Seppi (2008), Vayanos and Wang (2011)).

However, such equilibria exist only under quite restrictive assumptions. In a clarifying note, Biais, Martimort, and Rochet (2013) acknowledge that their existence result requires several conditions on the distribution of the insider's marginal valuation for the asset and on the expected value of the asset conditional on this marginal valuation. Back and Baruch (2013) provide an alternative set of sufficient conditions and emphasize that existence obtains only if the adverse selection problem is severe enough. This contrasts with the generality of Glosten's (1994) construction and thus raises the question of how robust the findings of this literature are to specifications of the model.

In this paper, we address this question by setting up a general model of trade under asymmetric information, in which a privately informed insider trades with several market makers. Unlike previous contributions, our model does not rely on particular specifications of the traders' preferences. We also depart from the existing literature by assuming that the insider's private information, or type, can take an arbitrary but finite number of values. This apparently minor change leads to strikingly different results. First, any pure-strategy equilibrium with weakly convex tariffs actually requires linear tariffs, at odds with the above-mentioned strictly convex tariffs. Second, such linear equilibria essentially exist only in the special Bertrand case, that is, when there is no adverse selection and market makers have a constant unit cost of serving demand. In all other cases, pure-strategy linear equilibria do not exist, apart from exceptional cases where a single insider type trades in equilibrium. These results hold true independently of the distribution of types and under very general assumptions on payoff functions, both in the game in which market makers are allowed to deviate by posting arbitrary tariffs (as in Biais, Martimort, and Rochet (2000, 2013)) and, in the absence of wealth effects, in the game in which market makers are required to post convex tariffs from the outset (as in Back and Baruch (2013)). In the former case, our analysis suggests that organized exchanges such as limit-order books can be destabilized by decentralized exchanges, such as over-the-counter markets, that allow for arbitrary offers. In the latter case, we point at an inherent instability of organized exchanges relying on discriminatory limit-order books.

The paper is organized as follows. Section 2 describes the model. Section 3 states our main results. Section 4 establishes that equilibria with convex tariffs and nondecreasing individual quantities feature linear pricing. Section 5 shows that such equilibria only exist in exceptional cases when there is adverse selection or when market makers have strictly convex costs. Section 6 extends these results to all equilibria with convex tariffs. Section 7 concludes by discussing the interpretation of our results and their relationship to the literature.

2 The Model

Our model features a privately informed insider who can purchase nonnegative amounts of an asset from several market makers. Shares are homogeneous, so the insider cares only about her aggregate trade. Unless otherwise stated, we allow for general nonparametric payoff functions and arbitrary discrete distributions for the insider's type.

2.1 The Insider

The insider is privately informed of her preferences. Her type i can take a finite number $I \geq 1$ of values with positive probabilities m_i such that $\sum_i m_i = 1$. Each insider type cares only about the aggregate quantity $Q \geq 0$ she purchases from the market makers and the aggregate transfer T she makes in return.[6] Type i's preferences over aggregate quantity-transfer bundles $(Q, T) \in \mathbb{R}_+ \times \mathbb{R}$ are represented by a utility function $U_i(Q, T)$, which is assumed to be continuous and strictly quasiconcave in (Q, T) and strictly decreasing in T. The following strict single-crossing property is the main determinant of the insider's behavior in our model.

Assumption SC-*U* *For all $i < i'$, $Q < Q'$, T, and T',*

$$U_i(Q, T) \leq U_i(Q', T') \text{ implies } U_{i'}(Q, T) < U_{i'}(Q', T').$$

[6]Thus, in the limit-order-book interpretation of our model, we focus on the ask side of the book, in line with Back and Baruch (2013) and Biais, Martimort, and Rochet (2013).

In words, a higher type is more eager to increase her purchases than lower types are. As an illustration, for any price p and type i, consider the demand function $D_i(p)$, that is, the unique solution to

$$\max\{U_i(Q, pQ) : Q \in \mathbb{R}_+ \cup \{\infty\}\}.$$

The continuity and strict quasiconcavity of U_i ensure that $D_i(p)$ is uniquely defined and continuous in p. Moreover, Assumption SC-U implies that, for each p, $D_i(p)$ is nondecreasing in i. We strengthen this property by requiring that demand be strictly increasing in the insider's type, in the following sense.

Assumption ID-*U* *For all* $i < i'$ *and* $p \in \mathbb{R}$,

$$0 < D_i(p) < \infty \text{ implies } D_i(p) < D_{i'}(p).$$

A sufficient condition for both Assumptions SC-U and ID-U to hold is that the marginal rate of substitution $MRS_i(Q, T)$ of shares for transfers be well defined and strictly increasing in i for all (Q, T). Assumptions SC-U and ID-U are maintained throughout the paper.

Some of our results are valid for such general utility functions for the insider, allowing for risk aversion and wealth effects (Theorem 1). Others require quasilinearity (Theorem 2), though none relies on a particular parametrization of the insider's utility function. The corresponding assumption is as follows.

Assumption QL-*U* *The insider has quasilinear utility* $U_i(Q, T) = u_i(Q) - T$, *where* $u_i(Q)$ *is differentiable and strictly concave in* Q.

Under this additional assumption, Assumption SC-U requires that the derivatives $u_i'(Q)$ be nondecreasing in i for all Q. For instance, in Biais, Martimort, and Rochet (2000, 2013), $U_i(Q, T) \equiv \theta_i Q - (\alpha\sigma^2/2)Q^2 - T$, reflecting that the insider has CARA utility with absolute risk-aversion parameter α and faces Gaussian noise with variance σ^2. Assumption SC-U holds if θ_i is nondecreasing in i. Similarly, Back and Baruch (2013) assume that the insider's demand function is independent of her wealth.

2.2 The Market Makers

There are $K \geq 2$ market makers. Each market maker cares only about the quantity $q \geq 0$ he provides the insider with and the transfer t he receives in return. Such pair (q, t) we call a *trade*. Market maker k's preferences over trades with type i are represented by a profit function $v_i^k(q,t)$, which is assumed to be continuous and weakly quasiconcave in (q,t) and strictly increasing in t. Note that this profit can depend on the insider's type, a common-value case that has received a lot of attention in the market-microstructure literature (Glosten and Milgrom (1985), Kyle (1985), Glosten (1994)). This contrasts with the private-value case, in which the market maker's profit does not depend on i. We allow for both cases by imposing that each market maker weakly prefers to sell lower quantities to higher types.

Assumption SC-*v* *For all k, $i < i'$, $q < q'$, t, and t',*

$$v_i^k(q,t) \geq v_i^k(q',t') \quad \textit{implies} \quad v_{i'}^k(q,t) \geq v_{i'}^k(q',t').$$

Assumptions SC-U and SC-v together introduce an element of adverse selection in the model: an insider with a higher type is willing to buy more shares, but faces market makers who are more reluctant to sell.

The assumptions we impose at this stage on the profit functions for the market makers are very general, allowing for risk aversion and inventory costs (Stoll (1978), Ho and Stoll (1981, 1983)). Again, whereas some of our results are valid for such general profit functions, others require more structure, typically in the form of symmetry and quasilinearity (Theorems 1–2). One may first require the market makers' profit function to be linear, as in Glosten (1994), Biais, Martimort, and Rochet (2000, 2013), and Back and Baruch (2013).

Assumption L-*v* *For each i, each market maker k earns a profit $v_i^k(q,t) = t - c_i q$ when he trades (q,t) with type i, where c_i is the unit cost of serving type i.*

Here, the market makers are assumed to be risk neutral and c_i may be thought of as the liquidation value of the asset when the insider is of type i. Under this assumption, Assumption SC-v amounts to imposing that $c_{i'} \geq c_i$ when $i' > i$. Alternatively, one may, in line with Roll (1984), assume that each market maker incurs a strictly increasing and strictly convex order-handling cost when selling shares.

Assumption C-v *For each i, each market maker k earns a profit $v_i^k(q,t) = t - c_i(q)$ when he trades (q,t) with type i, where the cost $c_i(q)$ is strictly convex in q, with $c_i(0) = 0$.*

Under this assumption, Assumption SC-v amounts to imposing that $\partial^- c_{i'}(q') \geq \partial^+ c_i(q)$ whenever $i' > i$ and $q' > q$.[7] Note that Assumption C-v generalizes Roll (1984) by allowing for both order-handling and adverse-selection costs.

[7] For any convex function f defined over a convex subset of $\mathbb{R}$, we use the notation $\partial f(x)$, $\partial^- f(x)$, and $\partial^+ f(x)$ to denote respectively the subdifferential of f at x, the minimum element of $\partial f(x)$, and the maximum element of $\partial f(x)$. Hence $\partial f(x) = [\partial^- f(x), \partial^+ f(x)]$.

We shall state our main results for the symmetric case where the market makers' profit functions satisfy Assumption L-v or Assumption C-v. We will, however, indicate in the course of the formal analysis to which extent our results can be extended to more general, possibly asymmetric profit functions.

2.3 Timing and Strategies

The game unfolds as follows:

1. The market makers $k = 1, \ldots, K$ simultaneously post tariffs t^k. Each tariff t^k is defined over a domain $A^k \subset \mathbb{R}_+$ that contains 0 and is such that $t^k(0) = 0$.

2. After privately learning her type, the insider purchases a quantity $q^k \in A^k$ from each market maker k, for which she pays in total $\sum_k t^k(q^k)$.

A pure strategy s for the insider maps any tariff profile $(t^1, \ldots, t^K)$ and any type i into a quantity profile $(q^1, \ldots, q^K)$. To ensure that type i's problem

$$\max \left\{ U_i \left(\sum_k q^k, \sum_k t^k(q^k) \right) : (q^1, \ldots, q^K) \in A^1 \times \cdots \times A^K \right\} \tag{1}$$

always has a solution, we require the domains A^k to be compact and each tariff t^k to be lower semicontinuous over A^k. These requirements are light enough to allow market makers to offer menus $\mu \equiv \{(0,0), \ldots, (q_i, t_i), \ldots\}$ containing a finite number of trades, including the null trade $(0,0)$.

We call the above game the *game with arbitrary tariffs*. In this game, market makers can post basically arbitrary tariffs, as in Biais, Martimort, and Rochet (2000, 2013) and Attar, Mariotti, and Salanié (2011, 2014). It is also interesting to study the *game with convex tariffs*, in which market makers can post only convex tariffs, as in Back and Baruch (2013). That is, it is required of any admissible strategy for market maker k that the domain A^k be a compact interval and that the tariff t^k be convex over A^k. Any such tariff can then be interpreted as a sequence of limit orders posted by market maker k.

2.4 Equilibria in Convex Tariffs

We shall hereafter focus on pure-strategy perfect-Bayesian equilibria $(t^1, \ldots, t^K, s)$ in which market makers post convex tariffs. This restriction is hardwired in the market makers' strategy spaces in the game with convex tariffs, whereas it is an additional constraint on equilibrium strategies in the game with arbitrary tariffs. The focus on convex tariffs intends to describe an idealized discriminatory limit-order book in which market makers post limit orders, or sequences of limit orders. Such instruments are known to have nice efficiency properties under complete information.[8] It is thus natural to ask whether they perform as well under adverse selection. In line with this interpretation, characterizing the equilibria in convex tariffs of the game with arbitrary tariffs amounts to studying the robustness of the book to the side trades that may take place in the dark, outside the book (Theorem 1), whereas characterizing the equilibria of the game with convex tariffs amounts to studying the inherent stability of the book (Theorem 2). We perform the latter exercise under stronger assumptions than the former, so that the two sets of results are not nested.

The focus on equilibria with convex tariffs also ensures that on the equilibrium path, the insider's preferences over collections of individual trades are well behaved, as we now show. Recall first that convexity of tariffs is preserved under aggregation. In particular, the minimum aggregate transfer the insider has to make in return for an aggregate quantity Q,

$$T(Q) \equiv \min \left\{ \sum_k t^k(q^k) : q^k \in A^k \text{ for all } k \text{ and } \sum_k q^k = Q \right\}, \tag{2}$$

is convex in Q in equilibrium.[9] As a consequence, and because the utility functions U_i are strictly quasiconcave, any type i has a uniquely determined aggregate equilibrium demand Q_i, which is nondecreasing in i under Assumption SC-U. Similarly, if the insider wishes to trade an aggregate quantity $Q^{-k} \in \sum_{k' \neq k} A^{k'}$ with the market makers other than k, the minimum transfer she has to make in return is

$$T^{-k}(Q^{-k}) \equiv \min \left\{ \sum_{k' \neq k} t^{k'}(q^{k'}) : q^{k'} \in A^{k'} \text{ for all } k' \neq k \text{ and } \sum_{k' \neq k} q^{k'} = Q^{-k} \right\}$$

and once more the aggregate tariff T^{-k} is convex in equilibrium. In turn, each type i evaluates any bundle (q, t) she may trade with market maker k through the indirect utility function

$$z_i^{-k}(q,t) \equiv \max \left\{ U_i(q + Q^{-k}, t + T^{-k}(Q^{-k})) : Q^{-k} \in \sum_{k' \neq k} A^{k'} \right\}. \tag{3}$$

Observe that the maximum in (3) is always attained and that $z_i^{-k}(q,t)$ is strictly decreasing in t and continuous in (q,t).[10] The convexity of the tariff T^{-k} and the quasiconcavity of

[8] Biais, Foucault, and Salanié (1998) show in the single-type case that equilibria of the game with convex tariffs exist and are efficient (see also Dubey (1982)). A difference with our setting, though, is that they assume that the insider's demand for shares is perfectly inelastic.

[9] Formally, T is the infimal convolution of the individual tariffs t^k posted by the market makers (see Rockafellar (1970, Theorem 5.4)). In this case $\sum_k A^k = [0, \sum_k \max A^k]$ as each domain A^k is a compact interval that contains 0.

[10] The last statement follows from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)).

the utility function U_i imply that $z_i^{-k}(q,t)$ is weakly quasiconcave in (q,t). Moreover, the convexity of the tariffs T^{-k} and Assumption SC-U together imply that the family of functions z_i^{-k} satisfy the following weak single-crossing property.[11]

Property SC-$\boldsymbol{z}$ *For all k, $i < i'$, $q \leq q'$, t, and t',*

$$z_i^{-k}(q,t) \leq z_i^{-k}(q',t') \quad \textit{implies} \quad z_{i'}^{-k}(q,t) \leq z_{i'}^{-k}(q',t'), \tag{4}$$

$$z_i^{-k}(q,t) < z_i^{-k}(q',t') \quad \textit{implies} \quad z_{i'}^{-k}(q,t) < z_{i'}^{-k}(q',t'). \tag{5}$$

Overall, our focus on equilibria in convex tariffs implies that the indirect utility functions z_i^{-k}, which are endogenous objects, satisfy regularity properties that they inherit from the primitive utility functions U_i.[12] It should be noted, however, that the functions z_i^{-k} satisfy quasiconcavity and single-crossing only in a weak sense, unlike the functions U_i. For instance, if the market makers other than k offer to sell any quantity at the same unit price p, then some insider type may be indifferent between two trades with market maker k taking place at price p.

3 The Main Results

Our central results are the following theorems.

Theorem 1 *Consider the game with arbitrary tariffs. The following statements are satisfied in any equilibrium in convex tariffs*:

(i) When market makers have linear costs (Assumption L-v), all trades take place at some constant unit price p. Each type i trades $D_i(p)$ and all types who trade have the same unit cost c_i, equal to c_I and p.

(ii) When market makers have strictly convex costs (Assumption C-v), all trades take place at some constant unit price p. Each type i trades $D_i(p)$, but only type I may trade and in that case $p \in \partial c_I(D_I(p)/K)$.

Theorem 2 *Consider the game with convex tariffs. When the insider has quasilinear utility (Assumption QL-U), statement (i) in Theorem 1 is satisfied in any equilibrium.*

[11]The proofs of these two results, and more generally all the proofs not given in the text, can be found in the Appendix.

[12]This contrasts with the analysis in Attar, Mariotti, and Salanié (2011), where the presence of a capacity constraint and the absence of restrictions on equilibrium menus could result in indirect utility functions that were discontinuous and did not satisfy any single-crossing property.

The remainder of the paper consists in proving and discussing these two theorems. From now on, we presuppose the existence of an equilibrium (in convex tariffs) $(t^1, \dots, t^K, s)$ in either game and we investigate its properties. In the game with arbitrary tariffs, this equilibrium should be robust to deviations by market makers to arbitrary tariffs, whereas in the game with convex tariffs, this equilibrium should only be robust to deviations by market makers to convex tariffs.

4 The Linear-Pricing Result

In this section, we prove the linear-pricing result for equilibria in which the quantity q_i^k traded by the insider with any market maker k is nondecreasing in her type i. Such equilibria we call *equilibria with nondecreasing individual quantities.* This first step is motivated by the fact that, under Assumption SC-U, aggregate quantities traded in equilibrium cannot decrease with the insider's type. Section 6 extends the result to all equilibria.

4.1 The Game with Arbitrary Tariffs

We first consider the game with arbitrary tariffs, in line with Biais, Martimort, and Rochet (2000, 2013). We first establish a tie-breaking lemma, which gives a lower bound for each market maker's equilibrium expected profit given the tariffs posted by his opponents. We then use this lemma to establish our linear-pricing result.

4.1.1 How the Market Makers Can Break Ties

Consider an equilibrium $(t^1, \dots, t^K, s)$ in convex tariffs of the game with arbitrary tariffs. Suppose that market maker k deviates to a menu $\{(0,0), \dots, (q_i, t_i), \dots\}$, designed so that type i selects the alternative (q_i, t_i). For this to be the case, it must be that the following incentive-compatibility and individual-rationality constraints hold for any types i and i':

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(q_{i'}, t_{i'}), \tag{6}$$

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(0, 0). \tag{7}$$

These constraints are formulated in terms of the insider's indirect utility functions, which are endogenous objects. Fortunately, under Property SC-z, we need to consider only a subset of these constraints. Specifically, we will focus on the *downward local constraints*

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(q_{i-1}, t_{i-1}) \tag{8}$$

for all i, where by convention $(q_0, t_0) \equiv (0, 0)$ to handle the individual-rationality constraint of type 1. Clearly these constraints are not sufficient to ensure that each type i will choose to trade (q_i, t_i) after the deviation. Indeed, local upward incentive constraints need not hold. More importantly, a given type may be indifferent between two trades, thus creating some ties. Nevertheless, as we shall now see, as long as he sticks to menus with nondecreasing quantities, market maker k can secure the expected profit he would obtain if he could break such ties in his favor. Define

$$V^k(t^{-k}) \equiv \sup \left\{ \sum_i m_i v_i^k(q_i, t_i) \right\} \tag{9}$$

over all menus $\{(0,0), \ldots, (q_i, t_i), \ldots\}$ that satisfy (8) and that have nondecreasing quantities, that is, $q_{i+1} \geq q_i$ for all $i < I$.

Lemma 1 *In any equilibrium $(t^1, \ldots, t^K, s)$ in convex tariffs of the game with arbitrary tariffs, market maker k's expected profit is at least $V^k(t^{-k})$.*

The idea here is that, from any menu verifying the constraints in (9), one can play both with transfers (which can be increased if (8) does not bind) and with quantities (so as to avoid cycles of binding incentive-compatibility constraints) to build another menu with no lower payoffs verifying (6)–(7). In the absence of cycles, transfers can then be perturbed slightly to make these constraints strict inequalities, which ensures that the insider has a unique best response. It should be noted that this result relies only on Assumptions SC-U and SC-v. In particular, each market maker need not have a quasilinear or even quasiconcave profit function and profit functions may differ across market makers.

4.1.2 Equilibria with Nondecreasing Individual Quantities

The above tie-breaking lemma suggests that we first focus on equilibria with nondecreasing individual quantities, that is, $q_{i+1}^k \geq q_i^k$ for all k and $i < I$. Suppose, therefore, that such an equilibrium exists. The equilibrium trades of market maker k then verify all the constraints in (9). An immediate consequence of Lemma 1 is thus that these trades must be solution to (9). Because the functions z_i^{-k} are strictly decreasing in transfers and weakly quasiconcave, it follows in turn that all downward local constraints (8) must bind. With convex tariffs, this seems very demanding. Indeed, consider an insider type who exhausts aggregate supply at some price p. When facing a given market maker, this type never wants to mimic another type who does not exhaust this market maker's supply at price p, because she would end up paying too much to get her aggregate quantity. In these circumstances, one may wonder how to build a chain of binding downward local constraints (8) that goes all the way down to the null trade $(0,0)$.

Let us make this point more formally. Because equilibrium tariffs are convex, one can define $\overline{s}^k(p)$ as the quantity supplied at price p by market maker k and $\overline{S}(p)$ as the aggregate quantity offered by the market makers at this price.[13] Now, suppose that there exists i and p such that the quantity Q_i traded by type i is no less than $\overline{S}(p)$ and that the latter aggregate supply is positive:

$$Q_i \geq \overline{S}(p) > 0.$$

For this value of p, consider the smallest such i. Because type i has convex preferences and exhausts aggregate supply at price p, any of her best responses must be such that she trades at least $\overline{s}^k(p)$ with each market maker k. Because the downward local constraints for type i must bind for all k, two cases may then arise:

(i) Either $i > 1$ and $q^k_{i-1} \geq \overline{s}^k(p)$ for all k, so that $Q_{i-1} \geq \overline{S}(p)$, in contradiction with the assumption that i is the smallest type such that $Q_i \geq \overline{S}(p)$.

(ii) Or $i = 1$ and, because at least one market maker k has $\overline{s}^k(p) > 0$, the participation constraint of type 1 cannot bind for this market maker, once again a contradiction.

This shows that, for any price p at which aggregate supply is positive, all types must trade an aggregate quantity below this level: $Q_i < \overline{S}(p)$ for all i if $\overline{S}(p) > 0$. Because $\overline{S}$ is right-continuous, we can safely consider the infimum of the set of such prices; call it again p. At price p, either aggregate supply is zero and there is no trade; or aggregate supply is positive and the insider faces a linear tariff with slope p. Because Q_i is then strictly less than $\overline{S}(p)$ for all i, each type i must trade $D_i(p)$ in the aggregate. We, therefore, have established the following result.

Proposition 1 *In the game with arbitrary tariffs, for any equilibrium in convex tariffs and with nondecreasing individual quantities, there exists a price p such that all trades take place at unit price p and each type i purchases $D_i(p)$ in the aggregate.*

[13] Note that p is a marginal, or limit price. When market maker k selects a convex tariff t^k, his supply correspondence is the inverse of the subdifferential of t^k (Biais, Martimort, and Rochet (2000, Definition 2)): for each $p \in \mathbb{R}$, the supply of market maker k at the marginal price p is the set $\{q : p \in \partial t^k(q)\}$. This set is a nonempty compact interval with lower and upper bounds $\underline{s}^k(p)$ and $\overline{s}^k(p)$ that are nondecreasing in p. When this interval is nontrivial, t^k is affine over it, with slope p. We let $\underline{S}(p) \equiv \sum_k \underline{s}^k(p)$ and $\overline{S}(p) \equiv \sum_k \overline{s}^k(p)$. Observe finally that $\overline{s}^k$ is right-continuous for all k and that $\overline{S}$ inherits this property.

The upshot of Proposition 1 is that the possibility of side trades leads to linear pricing, at least when attention is restricted to equilibria in convex tariffs and with nondecreasing individual quantities. This shows the disciplining role of competition in our model: although market makers can propose arbitrary tariffs, they end up trading at the same price. The role of binding downward local constraints is graphically clear, as illustrated in Figure 1: when such a constraint binds for type i and market maker k, market maker k's equilibrium tariff must be linear over $[q^k_{i-1}, q^k_i]$, because z^{-k}_i represents convex preferences. But considering a market maker in isolation is not enough: as seen in the above argument, it is because any type i's downward local constraints must bind for *all* market makers that the linear-pricing result holds.

This result is quite general: as pointed out in our discussion of Lemma 1, we need not postulate that the market makers have symmetric, quasilinear, or even quasiconcave profit functions. This result also markedly differs from those obtained in the continuous-type case by Biais, Martimort, and Rochet (2000, 2013), who show that an equilibrium with strictly convex tariffs and nondecreasing individual quantities exists under certain conditions on players' valuations and distribution functions.

4.2 The Game with Convex Tariffs

The above analysis relied on the market makers' ability to post arbitrary tariffs, including finite menus of trades. One may thus wonder whether this does not give market makers too much freedom to deviate and ultimately drives the linear-pricing result. To examine this question, we now consider the game with convex tariffs, in line with Back and Baruch (2013). We conduct the analysis under two additional assumptions. First, we assume that each insider type has a quasilinear utility function, that is, Assumption QL-U is satisfied. Second, we assume that the market makers have identical linear profit functions, that is, Assumption L-v is satisfied. These assumptions are not without loss of generality as they exclude wealth effects and insurance considerations. Yet they are general enough to encompass prominent examples studied in the literature such as the CARA-Gaussian example studied in Back and Baruch (2013, Example 1).

Focusing on convex tariffs has two main advantages. First, it allows us to rely on simple tools such as supply functions and first-order conditions, the properties of which are well-known under convexity assumptions. This contrasts with using arbitrary menus, with their cohort of incentive-compatibility constraints, and makes for more intuitive proofs. (Some of our arguments are in fact quite direct when considering figures.) Second, compared to the game with arbitrary tariffs, we reduce the set of deviations available to the market makers. This change can a priori only enlarge the set of equilibria. Yet we shall derive a linear-pricing result similar to Proposition 1 for the game with convex tariffs. The structure of the argument is similar to that in Section 4.1: we first establish a tie-breaking lemma, which we then use to establish our linear-pricing result.

4.2.1 How the Market Makers Can Break Ties

We first reformulate Lemma 1. Consider an equilibrium $(t^1, \ldots, t^K, s)$. Suppose that market maker k deviates to a convex tariff t with domain A. For type i to select the quantity q_i in this tariff, it must be that

$$q_i \in \arg\max \{z_i^{-k}(q, t(q)) : q \in A\}. \tag{10}$$

This constraint is not sufficient to ensure that type i will choose to purchase q_i from market maker k after the deviation. Indeed, type i may be indifferent between two quantities in the tariff t, thus creating some ties. Nevertheless, as we shall now see, as long as he sticks to nondecreasing quantities, market maker k can secure the expected profit he would obtain if he could break such ties in his favor. Define

$$V_{\text{co}}^k(t^{-k}) \equiv \sup \left\{ \sum_i m_i v_i^k(q_i, t(q_i)) \right\} \tag{11}$$

over all convex tariffs t and all families of quantities q_i that satisfy (10) for all i and that are nondecreasing, that is, $q_{i+1} \geq q_i$ for all $i < I$.

Lemma 2 *In any equilibrium $(t^1, \ldots, t^K, s)$ of the game with convex tariffs, market maker k's expected profit is at least $V_{\text{co}}^k(t^{-k})$.*

When the insider's preferences are quasilinear, only the slope of the tariff t matters for q_i to be a best response of type i. As illustrated in Figure 2, one can therefore replace the tariff t by a piecewise linear tariff inducing the same best response for the insider and yielding market maker k an expected profit at least equal to that he obtained by posting t. Moreover, consider a segment of this piecewise linear tariff with slope p and the set of types who trade on this segment. If there exists a quantity $\overline{q}$ on this segment such that market maker k would prefer all types who trade above $\overline{q}$ to trade $\overline{q}$, then market maker k could raise his profits by truncating this segment at $\overline{q}$, as illustrated in Figure 3. Indeed, this would reduce the quantities traded by those types, with transfers that are as least as high. Finally, market maker k can reduce the slope p slightly. This ensures that all the relevant insider types buy the maximum quantity $\overline{q}$ at price p. Proceeding in this way for each segment of his tariff, market maker k can secure the announced expected profit.

4.2.2 Equilibria with Nondecreasing Individual Quantities

Lemma 2 implies that, in any equilibrium with nondecreasing individual quantities, market makers post piecewise linear tariffs that can be interpreted as finite sequences of limit orders. Another feature of such an equilibrium, which follows from Lemma 2, is that, if there is a kink in the aggregate tariff, there exists at least one insider type who trades exactly at this kink. In other words, this type exhausts the aggregate supply $\overline{S}(p)$ at some price p for which $\overline{S}(p) > 0$, which means that she exhausts the supply $\overline{s}^k(p)$ of each market maker k at price p. This implies that each market maker offering some trades at price p is indispensable for this type to reach her equilibrium payoff. However, it can be shown using Bertrand-type arguments that the tariff resulting from the aggregation of all market makers' tariffs shares with Glosten's (1994) tariff the property that any increase in quantity must be priced at the corresponding increase in costs, which implies zero expected profit by construction. As one can hardly be indispensable and yet earn zero expected profit, we get that, if some insider type were to exhaust the aggregate supply $\overline{S}(p)$ at some price p for which $\overline{S}(p) > 0$, at least one of the market makers could slightly increase his tariff in a profitable way. That is, the following result holds.

Proposition 2 *In the game with convex tariffs, when the insider has quasilinear utility (Assumption QL-U) and the market makers have linear costs (Assumption L-v), for any equilibrium with nondecreasing individual quantities, there exists a price p such that all trades take place at unit price p and each type i purchases $D_i(p)$ in the aggregate.*

5 Market Breakdown

We now determine equilibrium prices and quantities in the candidate equilibria with linear tariffs and nondecreasing individual quantities characterized in Propositions 1–2. We show that, both in the game with arbitrary tariffs and in the game with convex tariffs, such equilibria when they exist typically exhibit an extreme form of market breakdown and that, moreover, equilibria exist only under exceptional circumstances.

5.1 Linear Costs

The case where market makers have linear costs (Assumption L-v) is easily handled, thanks to two arguments. First, the standard Bertrand undercutting argument implies that market makers must make zero expected profit: otherwise, because the functions D_i are continuous, any market maker k could claim almost all profits for himself by charging a uniform unit price slightly below the equilibrium price p. This implies that, if trade takes place in equilibrium, the price p cannot lie above the highest possible cost c_I. Second, in equilibrium p cannot lie below c_I either. Indeed, if it did, then market makers would want to limit the quantities they sell to type I, which they can do by posting a limit order at the equilibrium price with a well-chosen maximum quantity. Formally, in the game with arbitrary tariffs, any market maker k could deviate to a menu that would allow types $i < I$ to purchase the equilibrium quantity q_i^k at unit price p, whereas type I would be asked to purchase only q_{I-1}^k at unit price p. Such an offer is incentive compatible and individually rational, with nondecreasing quantities. Similarly, in the game with convex tariffs, any market maker k could deviate to a limit order $t(q) = p \min\{q, q_{I-1}^k\}$. A best response for any type $i < I$ is then to purchase q_i^k as before, whereas a best response for type I is to purchase q_{I-1}^k, preserving nondecreasing quantities. In either case, it follows from Lemmas 1–2 that the variation in market maker k's expected profit is at most zero,

$$m_I(p - c_I)(q_{I-1}^k - q_I^k) \leq 0.$$

Summing on k yields $m_I(p - c_I)[D_{I-1}(p) - D_I(p)] \leq 0$ and under Assumption ID-U this implies that $p \geq c_I$ if $D_I(p) > 0$. Because aggregate expected profits are zero, we get that $p = c_i = c_I$ for any type i who trades. Hence the following result.

Proposition 3 *Suppose that the market makers have linear costs (Assumption L-v), and consider an equilibrium in linear tariffs and with nondecreasing individual quantities of either the game with arbitrary tariffs or the game with convex tariffs. If trade takes place, then the equilibrium price is equal to the highest cost c_I and all types who trade have the same unit cost $c_i = c_I$.*

This result highlights a tension between zero expected profits in the aggregate and the high equilibrium price c_I. In the pure private-value case where the cost c_i is independent of the insider's type i, this tension is easily relaxed and we obtain the usual Bertrand result, leading to an efficient outcome. By contrast, in the pure common-value case where the cost c_i is strictly increasing in the insider's type i, only the highest type I may trade in equilibrium, whereas all types $i < I$ must be excluded from trade. This market breakdown is much more dramatic than in Akerlof (1970) or Rothschild and Stiglitz (1976), as at most one type trades in equilibrium. Moreover, conditions for the existence of an equilibrium become very restrictive: one must have $D_i(c_I) = 0$ for all $i < I$ if an equilibrium is to exist at all.

5.2 General Profit Functions

Thanks to the simplicity of our setting under linear tariffs, we can extend the above analysis to the case where market makers are endowed with general, convex preferences. This allows us to encompass the case where market makers are risk neutral with respect to transfers but have strictly convex order-handling costs, as in Roll (1984), or even more general cases allowing for risk aversion, as in Stoll (1978) and Ho and Stoll (1981, 1983). As above, two arguments are used.

5.2.1 Limit Orders as Best Responses

The first argument is a characterization result that does not depend on the game under study and may, therefore, be of some independent interest. Consider a situation in which all trades take place at some price p and suppose that the demands $D_i(p)$ are bounded. A natural deviation for any market maker k consists in offering a limit order at a price $p' < p$, with a maximum quantity $\overline{q}$. As he posts the best price, he trades a quantity $\min\{D_i(p'), \overline{q}\}$ with each type i. Because demand functions and profit functions are continuous, by making p' go to p, market maker k can claim the profits associated to the quantities

$$\min\{D_i(p), \overline{q}\}$$

for all i, where $\overline{q}$ remains to be chosen. For further reference, we call such quantities *limit-order quantities at price p*. On the other hand, and as suggested by Lemmas 1–2, one may also want to characterize market maker k's most preferred trades at price p, assuming that he sticks to nondecreasing quantities. These trades solve

$$\sup\left\{\sum_i m_i v_i^k(q_i, pq_i)\right\} \tag{12}$$

under the feasibility constraints

$$0 \leq q_i \leq D_i(p) \tag{13}$$

for all i, and the constraint that quantities be nondecreasing, that is, $q_{i+1} \geq q_i$ for all $i < I$. Under our assumptions, the mappings $q \mapsto v_i^k(q, pq)$ are continuous and weakly quasiconcave for all i and, from SC-v, they satisfy a single-crossing property. The following result characterizes the solutions to problem (12)–(13).

Lemma 3 *Let p be such that the demands $D_i(p)$ are bounded. Then problem (12)–(13) has a solution with limit-order quantities at price p. Moreover, if the mappings $q \mapsto v_i^k(q, pq)$ are strictly quasiconcave, then all solutions to (12)–(13) are limit-order quantities at price p.*

The proof relies on a very simple reasoning: if the price is high enough to convince a market maker to supply a positive quantity to high types, then from Assumption SC-v market maker k will want to provide the highest possible quantities to lower types.[14] The result itself is a neat characterization of limit orders: they are the optimal tool to use under linear pricing when a market maker faces adverse selection.

5.2.2 Equilibria

Our second argument relies on equilibrium considerations. Note first that, in an equilibrium with linear tariffs at price p and nondecreasing individual quantities, each market maker k's expected profit cannot lie above the expected profit from his most preferred trades at price p. Because, by Lemma 3, this expected profit can be approximated by a well-chosen limit order at a price arbitrarily close to p, it follows that, in such an equilibrium, the expected profit of market maker k is equal to the value of (12)–(13). Therefore, the quantities sold by market maker k in equilibrium must be solution to (12)–(13). Moreover, when his profit functions are strictly quasiconcave, such solutions must be limit-order quantities.

For simplicity, assume, moreover, that market makers have identical profit functions and strictly convex costs (Assumption C-v). Then all problems (12)–(13) are identical. By strict convexity, they admit a single, common solution, which must be a family of limit-order quantities. Each market maker k thus trades in equilibrium the quantities $\min\{D_i(p), \overline{q}\}$, for some well-chosen $\overline{q}$. But as any type i cannot trade more than $D_i(p)$, it must be that each market maker k sells the same quantity $\overline{q}$ to all types of the insider who trades and, therefore, that the aggregate demand of all types who trade is the same. Because, by Assumption ID-U, $D_I(p) > D_{I-1}(p)$ if trade takes place, the following result holds.

Proposition 4 *Suppose that market makers have identical, strictly convex cost functions (Assumption C-v). In any equilibrium in linear tariffs and with nondecreasing individual quantities of either the game with arbitrary tariffs or the game with convex tariffs, only type I can trade. If $D_I(p) > 0$, then $p \in \partial c_I(D_I(p)/K)$ and all market makers trade the same quantity $D_I(p)/K$ with type I only.*

[14] The proof given in the Appendix also allows for a continuous set of types. Concerning the generality of the result, notice that the ordering of the demands $D_i(p)$ does not play any role. One could as well allow for arbitrary bounded values, provided that the nondecreasing quantities constraint is replaced by the constraint that individual quantities be comonotonic with total demand, that is, $D_i(p) \leq D_{i'}(p)$ implies $q_i \leq q_{i'}$.

Proposition 4 is stated for the case where Assumption C-v holds (Roll (1984)). The result, however, readily extends to the case to the case where the market makers have identical profit functions v_i such that the mappings $q \mapsto v_i^k(q, pq)$ are strictly concave. This is for instance the case if market makers are risk averse, as when $v_i(q,t) \equiv v(t - c_i q)$ for some strictly concave von Neumann-Morgenstern utility function v (Stoll (1978), Ho and Stoll (1981, 1983)).

When there is a single insider type, that is, when $I = 1$, Proposition 4 states that any equilibrium is competitive in the usual sense: (i) the insider purchases her optimal demand $D_1(p)$ at price p; (ii) the market makers maximize their profit $v_1(q, pq)$ at price p; (iii) the equilibrium price p equalizes the insider's demand and the sum of the market makers' supplies. Equilibrium outcomes are hence first-best efficient, as in the case of linear costs with pure private values.

With multiple insider types, the unique candidate equilibrium outcome remains that which would prevail in an economy populated by type I only. A necessary condition for equilibrium is thus that all types $i < I$ demand a zero quantity at the equilibrium price p. The market breakdown effect is thus similar to the one characterized by Proposition 3 in the linear-cost case. A novel insight of Proposition 4 is that the result that no trade may take place except perhaps at the top of the insider's type distribution now holds whether or not the environment features common values. To illustrate this point, consider for instance the case of convex costs (Assumption C-v) and suppose that the cost function is the same for each type, $c_i \equiv c$ for all i, whereas demands $D_i(p)$ are strictly increasing in i. As a market maker's profit $t - c(q)$ on a given trade (q,t) does not depend on the identity of the insider, we are in a private-value setting, so that only risk sharing matters. Still, oligopolistic competition threatens the existence of equilibria: each market maker would like to reduce his maximum supply if the equilibrium price were too low; but a high equilibrium price strengthens competition to attract lower types. Thus competition is strong enough to imply that, in equilibrium, at most one type can trade.

6 Other Equilibrium Outcomes

In this section, we show that the focus on equilibria with nondecreasing individual quantities is without loss of generality: one can turn any equilibrium in convex tariffs into an equilibrium with the same tariffs and the same payoffs, but now with nondecreasing individual quantities. This result holds both in the game with arbitrary tariffs and in the game with convex tariffs. The proof is in fact very general and only relies on a property specifying that, in some sense, allocations with nondecreasing quantities are efficient.

To understand why, notice that market makers have to choose tariffs before demand realizes. How risk is collectively shared then becomes a central question. Given a profile $(t^1, \ldots, t^K)$ of convex tariffs, recall that each type has a uniquely determined aggregate trade (Q_i, T_i). Define a feasible allocation $(q_1^1, \ldots, q_1^K, \ldots, q_I^1, \ldots, q_I^K)$ as an allocation that satisfies

$$\sum_k q_i^k = Q_i$$

and

$$\sum_k t^k(q_i^k) = T_i$$

for all i; in other words, this allocation describes a best response of the insider to the tariffs $(t^1, \ldots, t^K)$. Define an efficient risk-sharing allocation as a feasible allocation that is not Pareto-dominated by any other feasible allocation from the market makers' viewpoint: there is no other feasible allocation that yields as much expected profit to each market maker, and strictly more expected profit to at least one market maker. Our result relies on the following property.

Property P *For any profile of convex tariffs $(t^1, \ldots, t^K)$, there exists an efficient risk-sharing allocation with nondecreasing individual quantities.*

This property is reminiscent of the usual risk-sharing result (Borch (1962)): efficiency requires that any increase in the aggregate quantity to be shared should translate into an increase in the individual shares of market makers. Nevertheless, notice that, in our setting, the market makers' payoff functions are state dependent because they directly depend on the insider's type. Moreover, the convexity of the tariffs $(t^1, \ldots, t^K)$ may make the risk-sharing problem nonconvex. To bypass these difficulties, we have to impose more restrictions on the market maker's profit functions than in the previous sections. Notable special cases are Assumptions L-v and C-v used in Theorems 1–2.[15]

Lemma 4 *Assume that all market makers have the same profit function, given by*

$$v_i^k(q, t) = t - c_i(q),$$

where for each i the cost function c_i is convex. Then Property P holds.

[15] One can more generally show that Lemma 4 holds for market makers with heterogenous cost functions c_i^k, the derivatives of which satisfy $c_i^{k\prime} = f_i \circ a^k$, where f_i is strictly increasing and a^k is nondecreasing. This would in particular allow handling the case of market makers with heterogeneous inventories, in which one has $c_i^k(q) = c_i(q - I^k)$ for some given inventories I^k.

We can now turn to the study of an arbitrary equilibrium $(t^1, \ldots, t^K, s)$ of either the game with arbitrary tariff or the game with convex tariffs. Let v^k be the equilibrium expected profit of market maker k. Depending on the game studied, Lemma 1 (respectively Lemma 2) offers a lower bound $V^k(t^{-k})$ (respectively $V^k_{\text{co}}(t^{-k})$) for this expected profit. We can build another lower bound by imposing in problem (9) (respectively problem (11)) the additional constraint that the transfers to market maker k be computed using the equilibrium tariff t^k; this defines $\underline{V}^k(t^1, \ldots, t^K)$. We therefore have

$$v^k \geq \underline{V}^k(t^1, \ldots, t^K) \tag{14}$$

for all k. On the other hand, under Property P, we know that there exists an efficient risk-sharing allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$ with nondecreasing individual quantities. In particular, for each k, $(q^k_1, \ldots, q^k_I)$ satisfies the constraints in the problem that defines $\underline{V}^k(t^1, \ldots, t^K)$. This implies that, for each k, we have

$$\underline{V}^k(t^1, \ldots, t^K) \geq \sum_i m_i v^k_i(q^k_i, t^k(q^k_i)). \tag{15}$$

Chaining inequalities (14)–(15), we get that each market maker k's equilibrium expected profit lies above his expected profit from the allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$. As the latter is a Pareto optimum, this is impossible unless all inequalities are equalities. Hence, for each k, we have

$$v^k = \sum_i m_i v^k_i(q^k_i, t^k(q^k_i)). \tag{16}$$

We now build an equilibrium that implements the allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$. Let us define s^* as the insider's strategy that selects this allocation if the tariff profile $(t^1, \ldots, t^K)$ is posted; otherwise, s^* selects the same quantities as s. We claim that $(t^1, \ldots, t^K, s^*)$ forms an equilibrium. Indeed, the insider plays a best response to any tariff profile. Moreover, in the initial equilibrium $(t^1, \ldots, t^K, s)$, no market maker has a profitable deviation, so that, for each k and for any tariff $\hat{t}^k \neq t^k$, we have[16]

$$v^k \geq \sum_i m_i v^k_i(s^k_i(\hat{t}^k, t^{-k}), \hat{t}^k(s^k_i(\hat{t}^k, t^{-k}))).$$

But from (16) and the definition of s^*, this can be rewritten as

$$\sum_i m_i v^k_i(s^{*k}_i(t^k, t^{-k}), t^k(s^{*k}_i(t^k, t^{-k}))) \geq \sum_i m_i v^k_i(s^{*k}_i(\hat{t}^k, t^{-k}), \hat{t}^k(s^{*k}_i(\hat{t}^k, t^{-k}))),$$

[16]In the game with convex tariffs, $\hat{t}^k$ must additionally be convex.

which expresses that market maker k has no profitable deviation when the other market maker post the tariffs t^{-k} and the insider plays her best response s^*. Hence the following result, which holds both in the game with arbitrary tariffs and in the game with convex tariffs.

Proposition 5 *Let $(t^1, \ldots, t^K, s)$ be an equilibrium (in convex tariffs) such that there exists an efficient risk-sharing allocation with nondecreasing individual quantities (Property P). Then there exists a strategy s^* for the insider such that $(t^1, \ldots, t^K, s^*)$ is an equilibrium with nondecreasing individual quantities that yields the same expected profit to each market maker.*

Given Propositions 1–2, a direct implication of Proposition 5 is that all equilibria must involve linear pricing. Theorems 1–2 are then immediate consequences of Propositions 3–4. Note also from (16) that equilibria when they exist support efficient risk-sharing allocations among market makers.

7 Discussion

In this section, we put our main results in perspective, relate them to the literature, and discuss alternative avenues of research.

1. The model we use is standard, one may even say canonical. It can be seen as the adverse-selection extension of the Bertrand competition model. We consider very general environments, allowing for arbitrary finite distributions of types for the insider and a rich set of convex preferences for the insider and the market makers. The restriction to equilibria in convex tariffs is motivated by our focus on discriminatory pricing in a limit-order book. The strict convexity of the insider's preferences implies that the aggregate quantity of the asset she is ready to purchase responds continuously to variations in prices. This may reflect that she trades the asset partly for hedging purposes, as in Glosten (1989), Biais, Martimort, and Rochet (2000, 2013), and Back and Baruch (2013), and partly for informational purposes. News traders, that is, insiders who are perfectly informed of the liquidation value of the asset and trade only on this information, as in Dennert (1993) or Baruch and Glosten (2013), are a limiting case of our analysis. Finally, the model is fully strategic, in that it does not rely on noise traders who are insensitive to prices, unlike much of the market-microstructure literature.

2. The first insight of our analysis is that equilibria exhibit a strong Bertrand property: in both the game with arbitrary tariffs and in the game with convex tariffs, no market maker is indispensable for providing any type with her equilibrium trades. The reason is that, otherwise, a market maker would have an incentive to raise his price on the marginal trade he makes with some type. We use standard mechanism-design techniques (Lemma 1) or standard price-theory arguments (Lemma 2) to show that he can do so without reducing the expected profit he makes when dealing with the other types. Discreteness of the type set is crucial for this logic. Indeed, in models with a continuum of types, Biais, Martimort, and Rochet (2000, 2013) and Back and Baruch (2013) show how to construct an equilibrium in which all market makers offers the same, strictly convex tariff and thus are indispensable as each type has a unique best response. Although strictly convex tariffs are not consistent with equilibrium in the discrete-type case—as the consideration of the one-type case readily shows—, they can be sustained in the continuous-type case because a local change in the tariff affects the behavior of all neighboring types, unlike in the discrete-type case. Suppose for instance that, over some interval of quantities, a market maker deviates by proposing, instead of the strictly convex equilibrium tariff, the corresponding chord. This would increase the deviator's profit if the insider's behavior remained the same. But such a change increases (decreases) the marginal price for relatively low-cost (high-cost) types who used to trade in this interval and thus, under common values or strictly convex costs, trades change in an unfavorable way. This last effect is reinforced whenever the buyer simultaneously trades with several sellers: any increase in the quantity purchased from a single seller is compensated by a reduction in the quantity she purchases from the others. The equilibrium in Biais, Martimort, and Rochet (2000, 2013) and Back and Baruch (2013) strikes a delicate balance between these two effects and, as in any Cournot-like equilibrium, the elasticity of demand for each type comes to play a crucial role. This is why their construction requires complex and quite restrictive joint conditions on the distribution of the insider's type and on the expected value of the asset conditional on her type.[17] By contrast, our results hold for general discrete-type environments and do not rely on such conditions.

3. The main determinant of equilibrium in our model is that market makers want to hedge against the adverse-selection risk or, when they have strictly convex costs, against the high-demand risk. A strictly convex tariff performs this role by making high-cost and, therefore,

[17] It is interesting to contrast these insights with those arising from models of exclusive competition under adverse selection. Indeed, independently of the assumptions made on the buyer's type space, sellers are typically not indispensable in pure-strategy equilibria of such models. A case in point is the insurance model of Rothschild and Stiglitz (1976), in which the insured agent has multiple best responses, so that each insurance company can be dispensed with.

high-demand types trade at a higher marginal price than low-cost and, therefore, low-demand types. However, whereas such tariffs arise naturally in the continuous-type environments of Glosten (1994), Biais, Martimort, and Rochet (2000, 2013), or Back and Baruch (2013), they are ruled out in our discrete-type environment as any equilibrium must feature linear pricing. Simpler tariffs such as limit orders then play a key role. In a situation in which all market makers but one offer linear tariffs, we have shown that using a well-chosen limit order is the best way for the remaining market maker to limit his exposure to the adverse-selection and the high-demand risks. (This logic is general and also applies in a candidate linear-price equilibrium of a model with a continuum of types. Alternatively, it implies that marginal profits in an equilibrium with strictly convex tariffs must be nonnegative at the upper end of the distribution of types.) However, limit orders are consistent with equilibrium only under exceptional circumstances. This is because the equilibrium price must be high enough to convince market makers to serve high-cost types. But such a high price means that each market maker would like to serve all the demand emanating from low-cost types, which is inconsistent with equilibrium unless these types do not wish to trade at that price. This confirms and extends in a radical way earlier results obtained by Attar, Mariotti, and Salanié (2014), who show in the two-type case that at most one type trades in any equilibrium. Overall, our results suggest that equilibrium existence for the discriminatory limit-order book is problematic in common-value environments. A novel insight of our analysis is that the market may break down or an equilibrium may fail to exist altogether even in private-value environments, as long as a market maker's marginal cost is not constant in the quantity of the asset that he trades with the insider.

4. In Theorem 1, we showed that, under a wide range of circumstances, the limit-order book can be destabilized by side trades that take place outside the book. To do so, we considered a game with arbitrary tariffs, but we restricted attention to equilibria in which market makers post convex tariffs. One may wonder whether this game admits other equilibria involving nonconvex tariffs. This question might be relevant to analyze competition on less regulated markets, such as over-the-counter-markets, in which trading is bilateral and nonexclusive. However, the above-mentioned work by Attar, Mariotti, and Salanié (2014) shows that, even in the two-type case, there is no hope in that direction. A more promising avenue of research might be to consider mixed-strategy equilibria. A first issue is existence. One can adapt the arguments of Carmona and Fajardo (2009) to show that the convex game admits a mixed-strategy equilibrium. In the game with arbitrary tariffs, however, it is unclear that there exists a mixed-strategy equilibrium in which market makers only randomize over convex tariffs, as required by the rules of the limit-order book. In any case, characterizing such equilibria appears to be a difficult task. Dennert (1993) and Baruch and Glosten (2013) construct mixed-strategy equilibria for related games, but they make the extreme assumption that the insider is a perfectly informed news trader. As a result, in equilibrium, the insider totally empties the book when the price is different from the liquidation value of the asset. This is different from our analysis, which, as mentioned above, incorporates other motives to trade, such as hedging, that make the insider's demand for the asset continuous in prices. A related point is that, because their models do not feature any gains from trade, both Dennert (1993) and Baruch and Glosten (2013) must resort to noise traders for trade to take place in the mixed-strategy equilibria they characterize.

5. Our negative existence results tell us that looking for an exact strategic foundation for the limit-order book may be too demanding. A natural alternative candidate nevertheless stands out, namely, Glosten's (1994) aggregate tariff, foreshadowed by the early contributions of Jaynes (1978) and Hellwig (1988). This tariff, a marginal version of Akerlof (1970), is by construction robust to entry. Yet, according to our analysis, it is not inherently stable, because some market maker providing part of it would have an incentive to deviate and take advantage of his competitors' tariffs. The reason for this is that, in this aggregate tariff, market makers trading with low-cost insiders are indispensable for providing these types with their equilibrium trades. Yet these market makers make zero expected profits, which does not square with their being indispensable. (We exploited this logic in the proof of Proposition 2.) A natural question is how much profits they forego by not playing a best response. The answer turns out to depend on the market structure, that is, on the number of market makers. Specifically, we show in the appendix that, in the two-type version of our model, the maximum deviation expected profit is of the order of $1/K^2$ as the number K of market makers goes to infinity. This result suggests that one can rationalize the Glosten (1994) aggregate tariff as an approximate equilibrium outcome when there are many market makers. This reconciles in the limit our findings with those of Biais, Martimort, and Rochet (2000), who show in the continuous-type case that their equilibrium aggregate tariff converges to the Glosten tariff as the number of market makers grows large. Yet the puzzle remains that discrete- and continuous-type models yield strikingly different predictions in the oligopolistic case with a fixed number of market makers.

Appendix

Proof that the Functions z_i^{-k} Are Weakly Quasiconcave. Fix a type i and a market maker k. For the sake of clarity, we hereafter omit the indexes i and k in this proof. Let (q,t) and (q',t') be two trades, and let Q^- and $Q^{-\prime}$ be the associated solutions to (3). For each $\lambda \in [0,1]$, $z^-(\lambda q + (1-\lambda)q', \lambda t + (1-\lambda)t')$ is at least

$$U(\lambda q + (1-\lambda)q' + \lambda Q^- + (1-\lambda)Q^{-\prime}, \lambda t + (1-\lambda)t' + T^-(\lambda Q^- + (1-\lambda)Q^{-\prime}))$$

because $\lambda Q^- + (1-\lambda)Q^{-\prime}$ is an admissible candidate in (3). Because T^- is convex and U is decreasing in transfers, this lower bound is itself at least

$$U(\lambda(q+Q^-) + (1-\lambda)(q'+Q^{-\prime}), \lambda[t + T^-(q+Q^-)] + (1-\lambda)[t' + T^-(q'+Q^{-\prime})]),$$

and because U is quasiconcave this expression is at least

$$\min\{U(q+Q^-, t+T^-(Q^-)), U(q'+Q^{-\prime}, t'+T^-(Q^{-\prime}))\},$$

which is $\min\{z^-(q,t), z^-(q',t')\}$ by construction. The result follows. ■

Proof of Property SC-z. Fix some k, $q < q'$, t, and t'. Let $\mathcal{T}(Q) \equiv t + T^{-k}(Q-q)$, defined for $Q \geq q$. Similarly, let $\mathcal{T}'(Q) \equiv t' + T^{-k}(Q-q')$, defined for $Q \geq q'$. According to (3), for each i, computing $z_i^{-k}(q,t)$ amounts to maximizing $U_i(Q, \mathcal{T}(Q))$ with respect to $Q \geq q$. Let $\mathcal{Q}_i \geq q$ be the solution to this problem; it is unique as U_i is strictly quasiconcave and strictly decreasing in aggregate transfers, and $\mathcal{T}(Q)$ is convex in Q. Similarly, computing $z_i^{-k}(q',t')$ amounts to maximizing $U_i(Q, \mathcal{T}'(Q))$ with respect to $Q \geq q'$. Let $\mathcal{Q}'_i \geq q'$ be the unique solution to this problem. The proof consists of two steps.

Step 1 We first prove (5). Suppose that

$$z_i^{-k}(q,t) < z_i^{-k}(q',t')$$

for some $i < I$ and let $i' > i$. Because $\mathcal{Q}_{i'} \geq q$ is an admissible candidate in the problem that defines $z_i^{-k}(q,t)$, we must have

$$U_i(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) \leq z_i^{-k}(q,t) < z_i^{-k}(q',t') = U_i(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i)). \tag{17}$$

Two cases may now arise:

(i) Suppose first that $\mathcal{Q}_{i'} < \mathcal{Q}'_i$. Then

$$z_{i'}^{-k}(q,t) = U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) < U_{i'}(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i)) \leq z_{i'}^{-k}(q',t'),$$

where the first inequality follows from (17), Assumption SC-U, and the assumptions that $i < i'$ and $\mathcal{Q}_{i'} < \mathcal{Q}'_i$, and the second inequality follows from the fact that $\mathcal{Q}'_i \geq q'$ is an admissible candidate in the problem that defines $z_{i'}^{-k}(q', t')$. This shows (5).

(ii) Suppose next that $\mathcal{Q}_{i'} \geq \mathcal{Q}'_i$. Because $\mathcal{Q}'_i \geq q' > q$ is an admissible candidate in the problem that defines $z_i^{-k}(q, t)$, we have

$$U_i(\mathcal{Q}'_i, \mathcal{T}(\mathcal{Q}'_i)) \leq z_i^{-k}(q, t) < z_i^{-k}(q', t') = U_i(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i))$$

which shows $\mathcal{T}'(\mathcal{Q}'_i) < \mathcal{T}(\mathcal{Q}'_i)$. Moreover, because $q < q'$ and T^{-k} is convex, $\mathcal{T}'(Q) - \mathcal{T}(Q)$ is nonincreasing in Q for $Q \geq q'$. Because $\mathcal{Q}_{i'} \geq \mathcal{Q}'_i$, this shows $\mathcal{T}'(\mathcal{Q}_{i'}) < \mathcal{T}(\mathcal{Q}_{i'})$. Now, as $\mathcal{Q}_{i'} \geq \mathcal{Q}'_i \geq q'$, $\mathcal{Q}_{i'}$ is an admissible candidate in the problem that defines $z_{i'}^{-k}(q', t')$ and thus

$$U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}'(\mathcal{Q}_{i'})) \leq z_{i'}^{-k}(q', t').$$

Hence, as $\mathcal{T}'(\mathcal{Q}_{i'}) < \mathcal{T}(\mathcal{Q}_{i'})$, we directly obtain

$$z_{i'}^{-k}(q, t) = U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) < U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}'(\mathcal{Q}_{i'})) \leq z_{i'}^{-k}(q', t').$$

This shows (5).

Step 2 The proof of (4) follows from (5) by continuity. Indeed, suppose that $z_i^{-k}(q, t) = z_i^{-k}(q', t')$ for some $i < I$ and let $i' > i$. Then, for each $\varepsilon > 0$, $z_i^{-k}(q, t + \varepsilon) < z_i^{-k}(q', t')$ and thus $z_{i'}^{-k}(q, t + \varepsilon) < z_{i'}^{-k}(q', t')$ from (5) as $i < i'$ and $q < q'$. Because $z_{i'}^{-k}$ is continuous, one can take limits as ε goes to zero to obtain (4). The result follows. ■

Proof of Lemma 1. Fix a market maker k. For the sake of clarity, we hereafter omit the index k in this proof. Fix a menu $\mu^* = \{(0, 0), \ldots, (q_i^*, t_i^*), \ldots\}$ with nondecreasing quantities that satisfies (8) for all i. The proof consists of two steps.

Step 1 First, we establish that there exists a menu $\mu = \{(0, 0), \ldots, (q_i, t_i), \ldots\}$ with nondecreasing quantities that satisfies the following properties:

(a) $\sum_i m_i v_i(q_i, t_i) \geq \sum_i m_i v_i(q_i^*, t_i^*)$.

(b) For each $i \geq 1$, $z_i^-(q_i, t_i) \geq z_i^-(q_{i-1}.t_{i-1})$.

(c) For each $i \geq 2$, if $q_i > q_{i-1}$, then $z_{i-1}^-(q_{i-1}, t_{i-1}) > z_{i-1}^-(q_i, t_i)$.

Notice that (b) is identical to (8), whereas (c) is a strict version of the upward local incentive-compatibility constraints. We proceed by contradiction and assume that there is no menu that satisfies (a), (b), and (c). Nevertheless, the set of menus with nondecreasing quantities that satisfy (a) and (b) is nonempty, as it contains μ^*. Therefore one can select in this set a menu μ that maximizes the index $i' \geq 2$ of the first violation of (c). For this index i', we have $q_{i'} > q_{i'-1}$.

One can even impose that the menu μ satisfy (b) as an equality at $i = i'$. Indeed, if (b) is a strict inequality at i', one can increase $t_{i'}$ until reaching an equality: this is feasible because $z^-_{i'}$ is weakly quasiconcave and strictly decreasing in t. This change in $t_{i'}$ defines a new menu that still satisfies (a), (b) for all i (with an equality at $i = i'$), and (c) for all $i < i'$; but, according to our definition of μ, (c) is violated at $i = i'$. With a slight abuse of notation, we call this menu $\mu = \{(0,0), \dots, (q_i, t_i), \dots\}$ again.

Now, because (b) holds as an equality at i' and because $q_{i'} > q_{i'-1}$, from the contraposition of (5) in property SC-z we get $z^-_{i'-1}(q_{i'-1}, t_{i'-1}) \geq z^-_{i'-1}(q_{i'}, t_{i'})$. Recall, however, that (c) is violated at i'. The only remaining possibility is thus that this inequality is in fact an equality. So (b) and (c) are equalities at i' and we face a cycle of binding incentive constraints that we now eliminate by pooling both types on the same quantity. Two cases may arise:

(i) Suppose first that $v_{i'}(q_{i'}, t_{i'}) \leq v_{i'}(q_{i'-1}, t_{i'-1})$. Then one can build a new menu μ' from μ by allocating $(q_{i'-1}, t_{i'-1})$ to types $i'-1$ and i'. (a) is relaxed by construction. (b) and (c) are unaffected for $i < i'$ and trivially hold at $i = i'$ as types $i'-1$ and i' are pooled on the same trade. Finally, (b) also holds for $i > i'$, because, by Property SC-z, the downward incentive-compatibility constraints are satisfied as soon as the downward local incentive-compatibility are satisfied. But then any violation of (c) for the new menu μ' would have to take place for types strictly above i', in contradiction with our definition of μ.

(ii) So it must be that $v_{i'}(q_{i'}, t_{i'}) > v_{i'}(q_{i'-1}, t_{i'-1})$. Then one can build a new menu μ' from μ by allocating $(q_{i'}, t_{i'})$ to types $i'-1$ and i'. (a) is relaxed because, as $q_{i'} > q_{i'-1}$, we can apply the contraposition of SC-v to obtain $v_{i'-1}(q_{i'}, t_{i'}) > v_{i'-1}(q_{i'-1}, t_{i'-1})$. (b) and (c) are unaffected for $i < i'-1$ and trivially hold at $i = i'$ as types $i'-1$ and i' are pooled on the same trade. (b) is unaffected for $i > i'$. At $i = i'-1$, because (c) was an equality at $i = i'$ for the menu μ, the change from μ to μ' does not affect type $i'-1$'s payoff and so (b) holds at $i'-1$. There remains to check that (c) holds at $i = i'-1$ (in the case $i' \geq 3$). As (c) at i' is an equality in the menu μ, the contraposition of (5) in SC-z implies that

$$z_{i'-2}^{-}(q_{i'-1}, t_{i'-1}) \geq z_{i'-2}^{-}(q_{i'}, t_{i'}).$$

We also know that (c) holds for the menu μ at $i = i' - 1$ and hence

$$z_{i'-2}^{-}(q_{i'-2}, t_{i'-2}) > z_{i'-2}^{-}(q_{i'-1}, t_{i'-1}).$$

These inequalities together imply that (c) holds at $i = i' - 1$. But once more we get a contradiction, as μ' verifies (a), (b), and (c) for $i \leq i'$.

Step 2 In Step 1, we established that, for any menu μ^* with nondecreasing quantities that satisfies (8), there exists a menu μ with nondecreasing quantities that yields market maker k at least as much expected profit as μ^* and that satisfies properties (b) and (c). By continuity of the functions z_i^-, one can then slightly reduce each transfer in the menu μ to get a menu μ' so that both (b) and (c) now hold as strict inequalities. Hence the local incentive-compatibility and type 1's individual-rationality constraint in μ' are slack. Property SC-z together with the fact that quantities in the menu μ' are nondecreasing then ensure that, when faced with μ', the insider has a unique best response. As the reduction in transfers in μ' relative to μ is arbitrarily small, we get that market maker k can approximate his expected profit in μ and, a fortiori, his expected profit in μ^*. The result follows. ■

Proof of Lemma 2. We begin with some preliminary remarks on the insider's best response when facing an arbitrary profile of convex tariffs $(t^1, \ldots, t^K)$.

Step 0 Recall that, given an arbitrary profile $(t^1, \ldots, t^K)$ of convex tariffs, the aggregate demand Q_i of type i is uniquely defined and nondecreasing in i. Given Q_i, type i's utility-maximization problem (1) reduces to minimizing her total payment for Q_i, $T(Q_i)$, as defined by problem (2). This is a convex problem, so that, by the Kuhn–Tucker theorem one can associate to any of its solutions $(q_i^1, \ldots, q_i^K)$ a Lagrange multiplier p_i such that $p_i \in \partial t^k(q_i^k)$ for all k. If there were two different solutions $(q_i^1, \ldots, q_i^K)$ and $(q_i'^1, \ldots, q_i'^K)$ to (1) with different multipliers $p_i < p_i'$, then, because each tariff is convex, one would obtain $q_i^k \leq q_i'^k$ for all k and because both solutions must sum to the same Q_i they would be identical, a contradiction. This shows that two different solutions must share the same p_i. Thus one can associate to each type i a price p_i such that, whatever the solution $(q_i^1, \ldots, q_i^K)$ to (2), one has $p_i \in \partial t^k(q_i^k)$ for all k. Finally, we can without loss of generality adopt the convention that p_i is nondecreasing in i. Indeed, if $p_i > p_{i+1}$ for some $i < I$, then, because $p_i \in \partial t^k(q_i^k)$ and $p_{i+1} \in \partial t^k(q_{i+1}^k)$ for all k, one has $q_i^k \geq q_{i+1}^k$ for all k. Because these quantities sum respectively to Q_i and Q_{i+1} and because $Q_i \leq Q_{i+1}$, it actually follows that $q_i^k = q_{i+1}^k$ for all k. Hence $p_i \in \partial t^k(q^k_{i+1})$ for all k and one may replace p_{i+1} by p_i. Given this convention, $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ are nondecreasing in i for all k.

Now, suppose that $(t^1, \ldots, t^K)$ are equilibrium tariffs and that market maker k deviates to some convex tariff t. Let $(q_1, \ldots, q_I)$ be a nondecreasing family of quantities such that (10) holds for all i. We know from Property SC-z that such a family exists. Letting $p_i \in \partial t(q_i)$ be a Lagrange multiplier for type i's problem of minimizing her total payment, one may according to Step 0 impose that p_i be nondecreasing in i. In the present quasilinear setting with differentiable strictly concave utility functions u_i, we actually have that each type i purchases $D_i(p_i) = (u'_i)^{-1}(p_i)$ in the aggregate, which uniquely pins down the value of p_i. The proof consists of four steps.

Step 1 Letting $\boldsymbol{p} \equiv (p_1, \ldots, p_I)$ and $\boldsymbol{q} \equiv (q_1, \ldots, q_I)$, construct the piecewise linear tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ such that $t_{\boldsymbol{p},\boldsymbol{q}}(0) = 0$ and

$$t_{\boldsymbol{p},\boldsymbol{q}}(q) = t_{\boldsymbol{p},\boldsymbol{q}}(q_{i-1}) + p_i(q - q_{i-1})$$

for all i and $q \in (q_{i-1}, q_i]$, with $q_0 \equiv 0$ by convention. Because the price and quantity families $(p_1, \ldots, p_I)$ and $(q_1, \ldots, q_I)$ are nondecreasing, the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ is convex. It is straightforward to check that $t_{\boldsymbol{p},\boldsymbol{q}}(q_i) \geq t(q_i)$ for all i.[18] Moreover, because $p_i = \partial^- t_{\boldsymbol{p},\boldsymbol{q}}(q_i)$, it remains a best response for any type i to purchase q_i from market maker k if the tariffs $(t_{\boldsymbol{p},\boldsymbol{q}}, t^{-k})$ are posted. In fact, under quasilinearity, $t_{\boldsymbol{p},\boldsymbol{q}}$ is the highest convex tariff with the property that the family $(q_1, \ldots, q_I)$ is a best response of the insider to this tariff, given the equilibrium tariffs t^{-k} of the market makers other than k (see Figure 2).

Step 2 According to Step 1, we can henceforth consider that market maker k deviates to the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$. As in Footnote 13, define the interval $[\underline{s}^k(p_i), \overline{s}^k(p_i)] \equiv \{q : p_i \in \partial t_{\boldsymbol{p},\boldsymbol{q}}(q)\}$ for any type i. Define also a family $(\overline{q}_1, \ldots, \overline{q}_I)$ as follows:

(i) If $\underline{s}^k(p_i) < \overline{s}^k(p_i)$ and if $I^+_i \equiv \{i' : p_{i'} = p_i > c_{i'}\} \neq \emptyset$, set $\overline{q}_i \equiv \max\{q_{i'} : i' \in I^+_i\}$.

(ii) Otherwise, set $\overline{q}_i \equiv \underline{s}^k(p_i)$.

Observe that the family $(\overline{q}_1, \ldots, \overline{q}_I)$ is nondecreasing. Intuitively, there is a single value of $\overline{q}$ for each value of p in $\{p_1, \ldots, p_I\}$: below $\overline{q}$, one finds all the types with $p > c_i$ who trade at price p and for which market maker k would like to increase trade. Above $\overline{q}$, the opposite

[18] An important observation is that one will have $t_{\boldsymbol{p},\boldsymbol{q}}(q_i) > t(q_i)$ for some i if and only if t is not itself of the form $t_{\boldsymbol{p},\boldsymbol{q}}$ given a nondecreasing family of quantities $(q_1, \ldots, q_I)$ such that the constraints (10) hold for all i. Thus t must be of the form $t_{\boldsymbol{p},\boldsymbol{q}}$ at a solution of the problem defining $V^k_{\mathrm{co}}(t^{-k})$ and at least one type must trade at any kink of t.

holds: because $p \leq c_i$, market maker k would like to reduce the quantity he trades with these types.

Step 3 One way for market maker k to achieve these objectives is to reduce the slope of the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ on quantities between $\underline{s}^k(p_i)$ and $\overline{q}_i$ and to increase it between $\overline{q}_i$ and $\overline{s}^k(p_i)$. Consider accordingly a small positive ε and let $\tilde{t} \equiv t_{\boldsymbol{p}-\varepsilon\boldsymbol{1}_I,\overline{\boldsymbol{q}}}$, where $\boldsymbol{1}_I \equiv (1,\ldots,1) \in \mathbb{R}^I$ and $\overline{\boldsymbol{q}} \equiv (\overline{q}_1,\ldots,\overline{q}_I)$. Note that, for any type i, we have $\partial^-\tilde{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i < \partial^+\tilde{t}(\overline{q}_i)$, so that slopes were changed in the right directions (see Figure 3). Let $(\tilde{q}_1,\ldots,\tilde{q}_I)$ be any best response of the insider to the tariff $\tilde{t}$, given the equilibrium tariffs t^{-k} of the market makers other than k. Given the definition of $\overline{q}_i$, two cases must be distinguished:

(i) If $p_i > c_i$, then $\underline{s}^k(p_i) \leq q_i \leq \overline{q}_i$. Then, because for each $q \leq q_i$ the tariff $\tilde{t}$ satisfies

$$\partial^-\tilde{t}(q) \leq \partial^-\tilde{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i$$

and type i has quasilinear preferences, one must have $\tilde{q}_i \geq q_i$.

(ii) If $p_i \leq c_i$, then $\overline{q}_i \leq q_i \leq \overline{s}^k(p_i)$. Then, because for each $q \geq q_i$ the tariff $\tilde{t}$ satisfies

$$\partial^+\tilde{t}(q) \geq \partial^+\tilde{t}(\overline{q}_i) > p_i$$

and type i has quasilinear preferences, one must have $\tilde{q}_i \leq q_i$.

Step 4 Finally, for all q and ε, we have $\tilde{t}(q) = t_{\boldsymbol{p}-\varepsilon\boldsymbol{1}_I,\overline{\boldsymbol{q}}}(q) \geq t_{\boldsymbol{p},\boldsymbol{q}}(q) - O(\varepsilon)$ (see Figure 3). Thus, for any best response $(\tilde{q}_1,\ldots,\tilde{q}_I)$ of the insider to the tariff $\tilde{t}$, given the equilibrium tariffs t^{-k} of the market makers other than k, we have

$$\begin{aligned}\sum_i m_i[\tilde{t}(\tilde{q}_i) - c_i\tilde{q}_i] &\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(\tilde{q}_i) - c_i\tilde{q}_i] - O(\varepsilon)\\ &\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(q_i) - c_i q_i] - O(\varepsilon)\\ &\geq \sum_i m_i[t(q_i) - c_i q_i] - O(\varepsilon),\end{aligned}$$

where the second inequality follows from the fact that $\tilde{q}_i \leq q_i$ if $p_i \leq c_i$ and $\tilde{q}_i \geq q_i$ if $p_i > c_i$ by Step 3, and the third inequality follows from Step 1. Hence, by posting the tariff $\tilde{t}$, market maker k can secure an expected profit within $O(\varepsilon)$ of $\sum_i m_i[t(q_i) - c_i q_i]$, where ε is arbitrarily small. The result follows. ■

Proof of Proposition 2. As a preliminary remark, observe that, if $(t^1,\ldots,t^K,s)$ is an equilibrium with nondecreasing individual quantities, then, from Lemma 2, each market maker k must earn an expected profit $V^k_{\text{co}}(t^{-k})$. Thus the tariff t^k is a solution to the problem that defines $V^k_{\text{co}}(t^{-k})$. According to Footnote 18, this implies that the tariff t^k is piecewise linear and that at least one type trades at any kink of t^k. Recall from Footnote 13 that $\underline{s}^k(p) = \inf\{q : p \in \partial t^k(q)\}$, $\overline{s}^k(p) = \sup\{q : p \in \partial t^k(q)\}$, and $\overline{S}(p) = \sum_k \overline{s}^k(p)$ for all k and p. Similarly let $\underline{S}(p) \equiv \sum_k \underline{s}^k(p)$ for all p. The proof consists of four steps.

Step 1 For each $Q \geq 0$, define $T(Q)$ as in (2) to be the minimal aggregate transfer the insider has to make in return for the aggregate quantity Q and let $p \equiv \partial^- T(Q_I)$ be the highest price at which trade takes place. Because any market maker k who supplies quantities beyond $\underline{s}^k(p)$ at price p to types i such that $u'_i(Q_i) \geq p$ must have an incentive to do so, one must have

$$\sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)[q^k_i - \underline{s}^k(p)] \geq 0 \tag{18}$$

for all k. We now show that, for each k, (18) holds as an equality. To this end, note that any market maker k could deviate by posting a tariff equal to t^k up to $\underline{s}^k(p)$, and then offering to sell any additional quantity between $\underline{s}^k(p)$ and $\overline{S}(p)$ at price p. A best response for the insider is to continue purchasing the equilibrium quantity q^k_i from market maker k if $u'_i(Q_i) < p$ and to purchase Q_i from market maker k if $u'_i(Q_i) \geq p$. One can thus apply Lemma 2 to conclude that

$$\begin{aligned}\sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{s}^k(p)] &\leq \sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)[q^k_i - \underline{s}^k(p)] \\ &\leq \sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{S}(p)] \end{aligned}\tag{19}$$

for all k, where the second inequality follows from the inequalities (18). Summing the inequalities (19) over k yields

$$\sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)(K - 1)\underline{S}(p) \leq 0$$

and this inequality is strict as soon as (18) is strict for some k. If this were the case, then, as $K > 1$, we would have $\sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i) < 0$, which, because $Q_i \geq \underline{S}(p)$ for all i such that $u'_i(Q_i) \geq p$ and because Q_i and c_i are nondecreasing in i, would contradict the fact that $\sum_{\{i\,:\,u'_i(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{S}(p)] > 0$ when (18) holds for all k with at least one strict inequality. It follows that all the inequalities (18) are in fact equalities, as claimed.

Step 2 From now on, suppose by way of contradiction that some trades take place at a price strictly lower than p and let p' be the highest such price, that is, $p' \equiv \partial^- T(\underline{S}(p))$

and $\overline{s}^k(p') = \underline{s}^k(p)$ for all k. We follow the same procedure as in Step 1. First, because any market maker k that supplies quantities beyond $\underline{s}^k(p')$ at price p' to types i such that $p > u_i'(Q_i) \geq p'$ must have an incentive to do so and because, according to Step 1, he does not make any additional profit trading at price p in equilibrium, one must have

$$\sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{q_i^k, \overline{s}^k(p')\} - \underline{s}^k(p')] \geq 0 \tag{20}$$

for all k. Second, in analogy with (18), we show that, for each k, (20) holds as an equality. To this end, note that any market maker k could deviate by posting a tariff equal to t^k up to $\underline{s}^k(p')$, and then offering to sell any additional quantity between $\underline{s}^k(p')$ and $\overline{S}(p') = \underline{S}(p)$ at price p'. A best response for the insider is to continue purchasing the equilibrium quantity q_i^k from market maker k if $p' > u_i'(Q_i)$, to purchase Q_i from market maker k if $p > u_i'(Q_i) \geq p'$, and to purchase $\overline{S}(p')$ from market maker k if $u_i'(Q_i) \geq p$. Because, according to Step 1, market maker k does not make any additional profit trading at price p in equilibrium, one can thus apply Lemma 2 to conclude that, in analogy with (19),

$$\begin{aligned}
&\sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{Q_i, \overline{S}(p')\} - \underline{s}^k(p')] \\
&\qquad\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{q_i^k, \overline{s}^k(p')\} - \underline{s}^k(p')] \\
&\qquad\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{Q_i, \overline{S}(p')\} - \underline{S}(p')].
\end{aligned}$$

One can then proceed as in Step 1 to show that the inequalities (20) are in fact equalities, as claimed.

Step 3 The upshot of Steps 1–2 is that, if trades take place at prices p and p', no market maker can make additional profits on these trades. We now show that this leads to a contradiction, thereby establishing that all trades must take place at price p. Note that according to our preliminary remark, there exists at least one type who exhausts supply at price p', that is, who purchases $\overline{s}^k(p')$ from each market maker k and thus has a unique best response to the equilibrium tariffs $(t^1, \ldots, t^K)$. Let i_0 be the lowest such type; all types $i_0, \ldots, I$ then exhaust supply at price p'. It follows from Step 2 that $p' \leq \mathbf{E}[c_i \mid i \geq i_0] \equiv \sum_{i\geq i_0} m_i c_i / \sum_{i\geq i_0} m_i$. This must hold as an equality, for, otherwise, some market maker k would have an incentive to offer less than $\overline{s}^k(p')$ at price p'. Now, either $i_0 = 1$ and, for each k, $q_{i_0-1}^k \equiv 0$ by convention, or $i_0 > 1$ and, by definition of i_0, there exists some k such that

$$\max\{\underline{s}^k(p'), q_{i_0-1}^k\} < \overline{s}^k(p').$$

Take any such k and let $\underline{q}^k \equiv \max\{\underline{s}^k(p'), q^k_{i_0-1}\}$, so that $t^k(\overline{s}^k(p')) = t^k(\underline{q}^k) + p'[\overline{s}^k(p') - \underline{q}^k]$. Because type i_0 has a unique best response to $(t^1, \ldots, t^K)$, there exists $\varepsilon > 0$ such that

$$z^{-k}_{i_0}(\overline{s}^k(p'), t^k(\overline{s}^k(p')) + \varepsilon[\overline{s}^k(p') - \underline{q}^k]) > z^{-k}_{i_0}(q, t^k(q)) \tag{21}$$

for all $q \leq \underline{q}^k$. Define

$$\overline{q}^k \equiv \max\{\arg\max\{z^{-k}_{i_0}(q, t^k(\underline{q}^k) + (p' + \varepsilon)[q - \underline{q}^k]) : q \in [\underline{q}^k, \overline{s}^k(p')]\}\} \tag{22}$$

Then, because

$$t^k(\overline{s}^k(p')) + \varepsilon[\overline{s}^k(p') - \underline{q}^k] = t^k(\underline{q}^k) + (p' + \varepsilon)[\overline{s}^k(p') - \underline{q}^k],$$

it follows from (21) that $\underline{q}^k < \overline{q}^k \leq \overline{s}^k(p')$. Market maker k could deviate by posting a tariff equal to t^k up to $\underline{q}^k$, and then offering to sell any additional quantity between $\underline{q}^k$ and $\overline{q}^k$ at price $p' + \varepsilon$. A best response for the insider is to continue purchasing the equilibrium quantity q^k_i from market maker k if $i \leq i_0 - 1$ and, according to (22) along with the single-crossing property (4), to purchase the quantity $\overline{q}^k$ from market maker k if $i \geq i_0$. Because, according to Step 1, market maker k does not make any additional profit trading at price p in equilibrium, one can thus apply Lemma 2 to conclude that

$$\sum_{i \geq i_0} m_i(p' + \varepsilon - c_i)(\overline{q}^k - \underline{q}^k) \leq 0,$$

which, because $\varepsilon > 0$ and $\overline{q}^k > \underline{q}^k$, contradicts the above-noted fact that $p' = \mathbf{E}[c_i \,|\, i \geq i_0]$. Hence all trades must take place at price p, as claimed.

Step 4 At price p, either aggregate supply is zero and there is no trade; or aggregate supply is positive and the insider faces a linear tariff with slope p. To complete the proof, we must show that, in the latter case, each type i can purchase her unconstrained demand $D_i(p)$ at price p. Indeed, otherwise, some type would exhaust supply at price p and would thus have a unique best response to the equilibrium tariffs $(t^1, \ldots, t^K)$. Let i_0 be the lowest such type; all types $i_0, \ldots, I$ would then exhaust supply at price p. Arguing as in Step 3, we get that this leads to a contradiction. Hence each type can freely choose her preferred quantity $D_i(p)$ at price p. Hence the result. ■

Proof of Lemma 3. Define $\nu^k_i(q) \equiv v^k_i(q, pq)$ for all q. In this proof, we more generally assume that the insider's type i is distributed over some subset $\mathcal{I}$ of $\mathbb{R}$. The corresponding distribution $\mathbf{m}$ may be discrete, continuous, or mixed. We also assume that the appropriate generalization of SC-v holds and that $\sup\{D_i(p) : i \in \mathcal{I}\} < \infty$. Now, observe that, if the nondecreasing quantities $(q_i)_{i\in\mathcal{I}}$ satisfy the constraints in (12)–(13), so do the quantities $(\min\{q_i, \overline{q}\})_{i\in\mathcal{I}}$ for all $\overline{q}$. Hence we can restrict our quest for a solution to (12)–(13) to the set of nondecreasing quantities $(q_i)_{i\in\mathcal{I}}$ such that (13) holds for each i and such that

$$\int \nu_i^k(\overline{q})1_{\{q_i\geq\overline{q}\}}\,\mathbf{m}(di) \leq \int \nu_i^k(q_i)1_{\{q_i\geq\overline{q}\}}\,\mathbf{m}(di) \tag{23}$$

for all $\overline{q} \in [0, \|\tilde{q}\|_\infty)$, where $\|\tilde{q}\|_\infty \equiv \inf\{q : \mathbf{m}[\{i \in \mathcal{I} : q_i \leq q\}] = 1\}$ is the essential supremum of the quantities $(q_i)_{i\in\mathcal{I}}$. Note that this set is nonempty as it contains $(0)_{i\in\mathcal{I}}$. We now show that any $(q_i)_{i\in\mathcal{I}}$ in this set yields seller k an expected profit at most equal to that provided by $(\min\{D_i(p), \|\tilde{q}\|_\infty)_{i\in\mathcal{I}}$. This is obvious if $\|\tilde{q}\|_\infty = 0$. If $\|\tilde{q}\|_\infty > 0$, then, for each $\varepsilon \in (0, \|\tilde{q}\|_\infty]$, applying (23) for $\overline{q} = \|\tilde{q}\|_\infty - \varepsilon$ yields that there exists a type i' such that $q_{i'} \geq \|\tilde{q}\|_\infty - \varepsilon$ and

$$\nu_{i'}^k(\|\tilde{q}\|_\infty - \varepsilon) \leq \nu_{i'}^k(q_{i'}).$$

Applying the contraposition of SC-v yields[19]

$$\nu_i^k(\|\tilde{q}\|_\infty - \varepsilon) \leq \nu_i^k(q_{i'})$$

for any type $i \leq i'$. Because the quantities $(q_i)_{i\in\mathcal{I}}$ are nondecreasing, this actually holds for any type i such that $q_i < \|\tilde{q}\|_\infty - \varepsilon$. As the functions ν_i^k are weakly quasiconcave, it follows that, for any type i such that $q_i < \|\tilde{q}\|_\infty - \varepsilon$, ν_i^k is nondecreasing over $[0, \|\tilde{q}\|_\infty - \varepsilon]$. Because this is true for all $\varepsilon > 0$, we have shown that, for any type i such that $q_i < \|\tilde{q}\|_\infty$, the function ν_i^k is nondecreasing over $[0, \|\tilde{q}\|_\infty]$. Hence market maker k could choose quantities equal to $(\min\{D_i(p), \|\tilde{q}\|_\infty\})_{i\in\mathcal{I}}$ without reducing his expected profit relative to $(q_i)_{i\in\mathcal{I}}$, as claimed. This implies that problem (12)–(13) reduces to

$$\sup\left\{\int \nu_i^k(\min\{D_i(p), \overline{q}\})\,\mathbf{m}(di) : \overline{q} \in [0, \sup\{D_i(p) : i \in \mathcal{I}\}]\right\},$$

which has a solution as the objective function is continuous in $\overline{q}$, owing to the fact that the functions $(\nu_i^k)_{i\in\mathcal{I}}$ are continuous along with Lebesgue's dominated convergence theorem. Hence (12)–(13) has a solution with limit-order quantities at price p. Finally, if the mappings ν_i^k are strictly quasiconcave, any solution to (12)–(13) is of this form. ■

Proof of Lemma 4. Consider a profile $(t^1, \ldots, t^K)$ of convex tariffs. Recall that the resulting optimal aggregate trade (Q_i, T_i) is uniquely determined for each type i and that

[19] Strictly speaking, the contraposition of SC-v gives only that $v_{i'}^k(q', t') > v_{i'}^k(q, t)$ implies $v_i^k(q', t') > v_i^k(q, t)$. But because the profit functions are continuous and monotonic in transfers, one can easily show as in Step 2 of the proof to Property SC-z that $v_{i'}^k(q', t') \geq v_{i'}^k(q, t)$ implies $v_i^k(q', t') \geq v_i^k(q, t)$, which is the implication we use here.

one can associate to each type i a price p_i as in Step 0 of the proof of Lemma 2. To find an efficient risk-sharing allocation, one may first solve for each i

$$\max\left\{\sum_k v_i^k(q_i^k, t^k(q_i^k)) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\}$$

subject to

$$\sum_k q_i^k = Q_i$$

and

$$\sum_k t^k(q_i^k) = T_i.$$

Because the market makers' profit functions are identical and quasilinear, this problem can be rewritten as:

$$\min\left\{\sum_k c_i(q_i^k) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\}$$

subject to

$$\sum_k q_i^k = Q_i$$

and

$$\underline{s}^k(p_i) \leq q_i^k \leq \overline{s}^k(p_i)$$

for all k, where the latter constraints ensure that the vector of trades $(q_i^1, \ldots, q_i^K)$ is indeed a best response of type i. We want to show that this family of problems indexed by i admits a family of solutions with nondecreasing individual quantities.

To do so, notice first that each of these problems is well behaved, with a nonempty compact set of solutions. Hence there exists a family of solutions $(q_1^1, \ldots, q_1^K, \ldots, q_I^1, \ldots, q_I^K)$ that minimizes the following measure of violations

$$\sum_k \sum_{i<I} \max\{q_i^k - q_{i+1}^k, 0\}. \tag{24}$$

Let us proceed by contradiction and suppose that this minimum is strictly positive. Then, at the minimum, one has

$$q_i^k > q_{i+1}^k \tag{25}$$

for some k and $i < I$. Given that $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ are nondecreasing in i, this implies that

$$\underline{s}^k(p_i) \leq \underline{s}^k(p_{i+1}) \leq q_{i+1}^k < q_i^k \leq \overline{s}^k(p_i) \leq \overline{s}^k(p_{i+1}). \tag{26}$$

Because the intervals $[\underline{s}^k(p_i), \overline{s}^k(p_i)]$ and $[\underline{s}^k(p_{i+1}), \overline{s}^k(p_{i+1})]$ have a nontrivial intersection, it must be that $p_i = p_{i+1}$. Therefore, for any market maker k' we have $\underline{s}^{k'}(p_i) = \underline{s}^{k'}(p_{i+1})$ and $\overline{s}^{k'}(p_i) = \overline{s}^{k'}(p_{i+1})$. Moreover, because $q_i^k > q_{i+1}^k$ and $Q_i \leq Q_{i+1}$, we know that there exists $k' \neq k$ such that

$$q_i^{k'} < q_{i+1}^{k'}. \tag{27}$$

Using the equalities we have just shown, this implies that

$$\underline{s}^{k'}(p_i) = \underline{s}^{k'}(p_{i+1}) \leq q_i^{k'} < q_{i+1}^{k'} \leq \overline{s}^{k'}(p_i) = \overline{s}^{k'}(p_{i+1}). \tag{28}$$

Given (26) and (28), one can slightly reduce q_i^k and increase $q_i^{k'}$ by the same positive amount ε, so that all constraints are still verified. Such a transformation would reduce the criterion (24), so that the resulting trade cannot be a solution to the problem for type i. Hence

$$c_i(q_i^k - \varepsilon) + c_i(q_i^{k'} + \varepsilon) > c_i(q_i^k) + c_i(q_i^{k'}).$$

By convexity, this implies $q_i^k \leq q_i^{k'}$. Alternatively, one could slightly increase q_{i+1}^k and reduce $q_{i+1}^{k'}$ by the same positive amount ε. We obtain similarly

$$c_{i+1}(q_{i+1}^k + \varepsilon) + c_{i+1}(q_{i+1}^{k'} - \varepsilon) > c_{i+1}(q_{i+1}^k) + c_{i+1}(q_{i+1}^{k'}),$$

which implies $q_{i+1}^k \geq q_{i+1}^{k'}$. But it is easily seen that these last two inequalities together with (25) and (27) yield a contradiction. The result follows. ■

The Glosten (1994) Tariff as an Approximate Equilibrium Outcome. Suppose for simplicity that $I = 2$, that each type i has quasilinear preferences (Assumption QL-U), and that the market makers' profit functions are linear (Assumption L-v) with marginal costs $c_2 > c_1$. Let $c \equiv m_1 c_1 + m_2 c_2$ be the average cost and suppose, furthermore, that

$$0 < (u_1')^{-1}(c) < (u_2')^{-1}(c_2) < \infty.$$

Let $Q_1 \equiv (u_1')^{-1}(c)$ and $Q_2 \equiv (u_2')^{-1}(c_2)$. The Glosten (1994) aggregate tariff is the piecewise-linear tariff defined by

$$T^G(Q) \equiv 1_{\{Q \leq Q_1\}} cQ + 1_{\{Q > Q_1\}} [cQ_1 + c_2(Q - Q_1)].$$

When facing T^G, type 1 trades the aggregate quantity Q_1 and type 2 trades the aggregate quantity Q_2. The corresponding marginal prices are c and c_2. One can show along the lines of Glosten (1994) that the tariff T^G is robust to entry, in the sense that no additional market maker could enter and make a profit if the tariff T^G were already available.

Consider the following implementation of T^G. Suppose that each market maker k offers to sell any quantity in $[0, Q_1/K]$ at unit price c and then to sell any additional quantity at unit price c_2, which amounts to the tariff $t^k(q) = T^G(Kq)/K$. Clearly these convex tariffs lead to T^G in the aggregate. Each market maker k is indispensable for providing T^G and, therefore, has a profitable deviation. (This deviation is similar to that described in Step 3 of the proof of Proposition 2.) The question is how much k can gain by deviating. Because the market makers other than k offer convex tariffs, Property SC-z is satisfied. Observe that, as a result, we can assume that, following a deviation by k, the insider selects a best response in which she trades with him quantities that are nondecreasing in her type.

To get a bound on k's gain from deviating, note first that the most k can earn is obtained by offering a menu consisting of the no-trade contract and of a well-chosen pair of contracts, (q_1, t_1) and (q_2, t_2), respectively targeted at types 1 and 2, and resulting in an expected profit

$$t_1 - cq_1 + m_2[t_2 - t_1 - c_2(q_2 - q_1)]. \tag{29}$$

From the above observation, we can suppose that $q_1 \leq q_2$. Because type 2 always have the option to buy the nonnegative marginal quantity $q_2 - q_1$ at a unit price at most equal to c_2, one must have $t_2 - t_1 \leq c_2(q_2 - q_1)$ and, therefore, k's expected profit (29) is bounded above by $t_1 - cq_1$. Hence we may as well assume that k offers a single contract (q, t) distinct from the no-trade contract and which both types accept to trade. A necessary condition for type 1 to accept to trade (q, t) is that $z_1^{-k}(q, t) \geq z_1^{-k}(0, 0)$, which implies $z_2^{-k}(q, t) \geq z_2^{-k}(0, 0)$ by Property SC-z. Thus an upper bound for (29) is

$$\max \{t - cq : z_1^{-k}(q, t) \geq z_1^{-k}(0, 0)\}.$$

As the aggregate trade (Q^{-k}, cQ^{-k}), where

$$Q^{-k} \equiv (K - 1)\,\frac{Q_1}{K}\,,$$

remains available for trade following k's deviation, an even more generous upper bound is

$$\max \{t - cq : z_1^{-k}(q, t) \geq u_1(Q^{-k}) - cQ^{-k}\}. \tag{30}$$

Any optimal contract (q, t) solution to (30) is such that the constraint in (30) is binding and type 1 ends up purchasing Q_1 in the aggregate. A particular solution is such that

$q = Q_1 - Q^{-k}$ and then

$$q = \frac{Q_1}{K}$$

and

$$t = u_1(Q_1) - u_1\left((K-1)\,\frac{Q_1}{K}\right).$$

As $u_1'(Q_1) = c$, and assuming that u_1 is twice differentiable at Q_1, a Taylor–Young expansion yields the following approximation of (30):

$$t - cq = -\frac{u_1''(Q_1)}{2}\left(\frac{Q_1}{K}\right)^2 + o\left(\frac{1}{K^2}\right),$$

which is the desired result. ■

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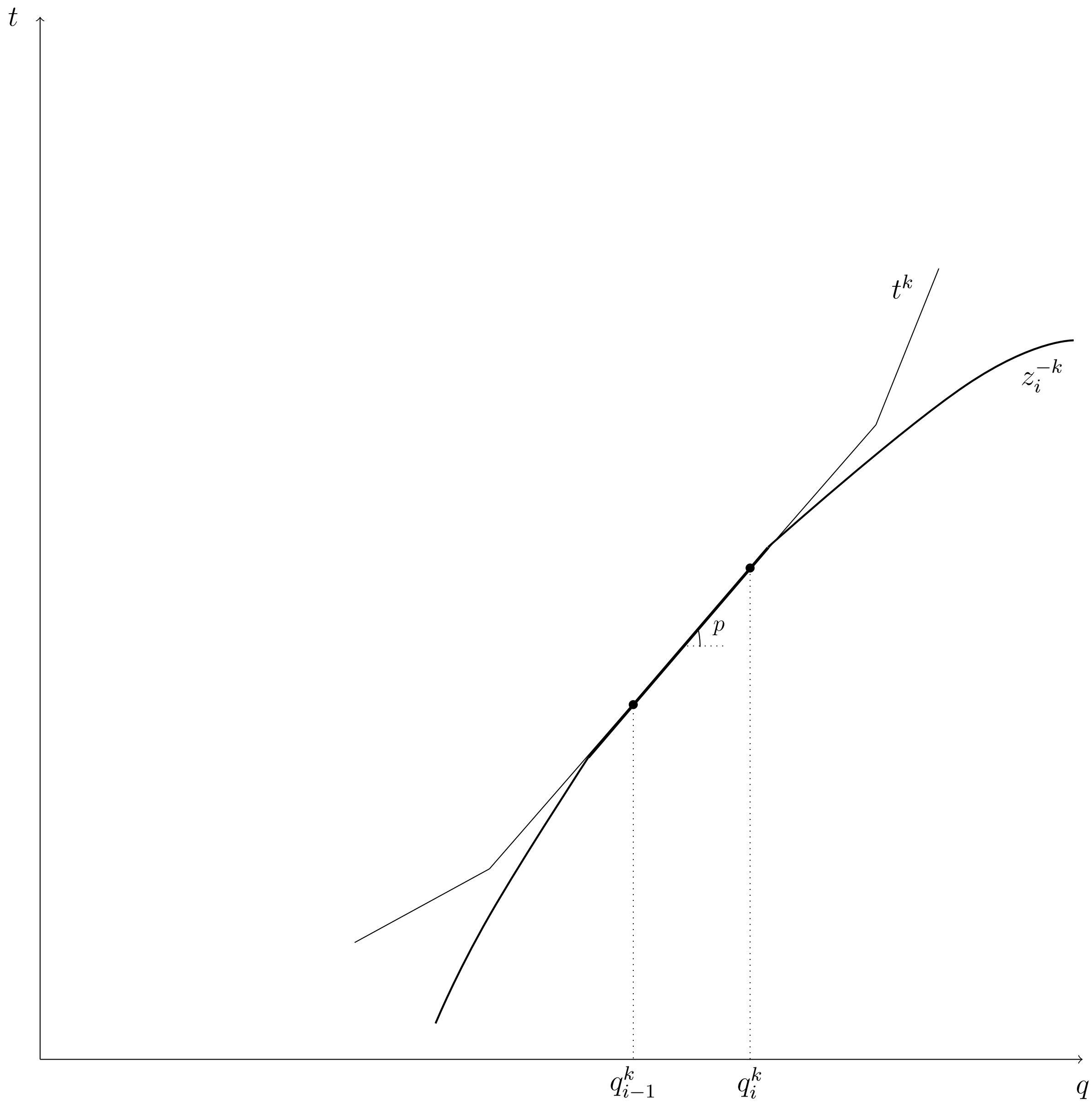

Figure 1 Binding downward local constraints and linearity.

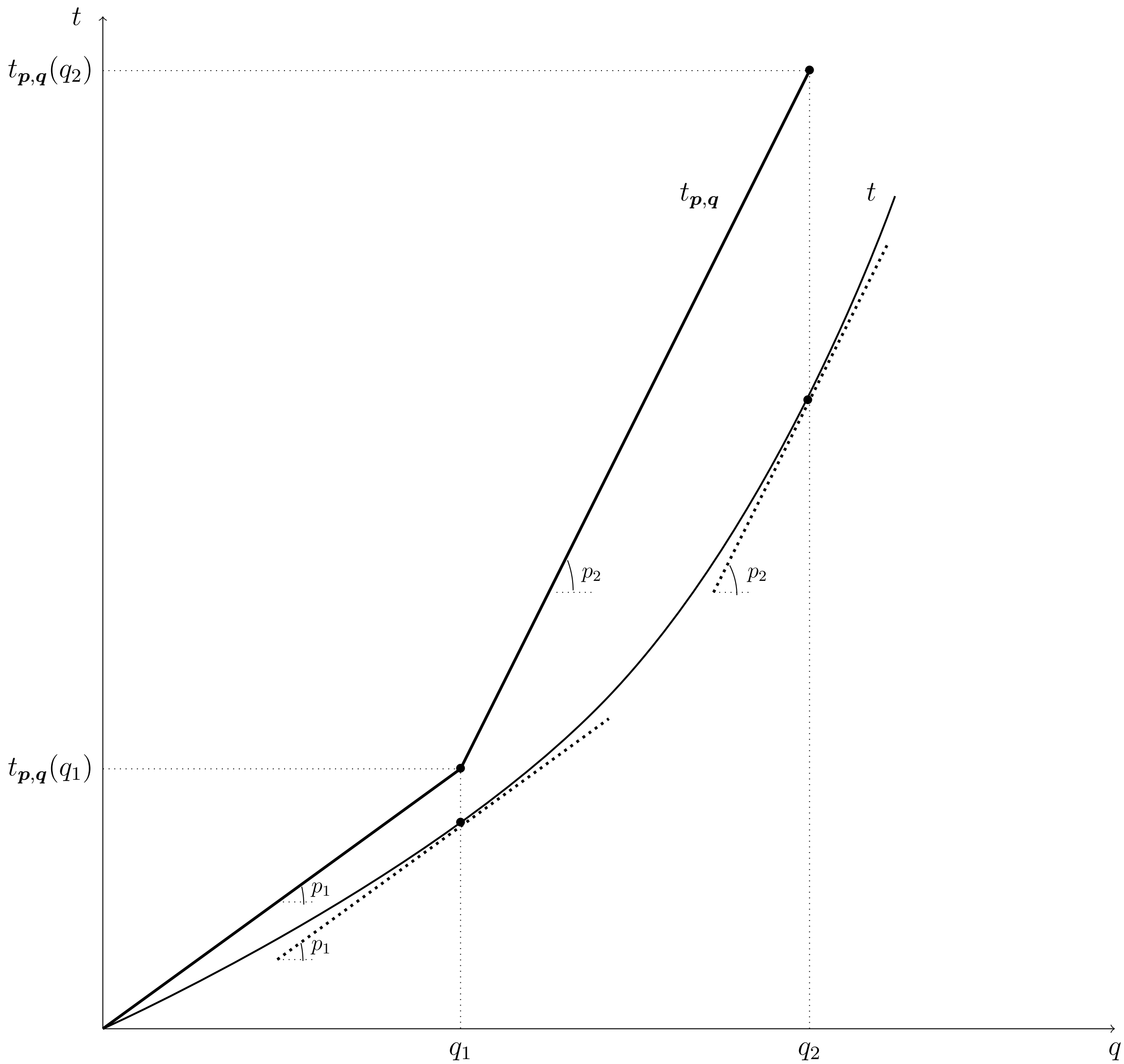

Figure 2 The $t_{\boldsymbol{p},\boldsymbol{q}}$ schedule in the case $I = 2$.

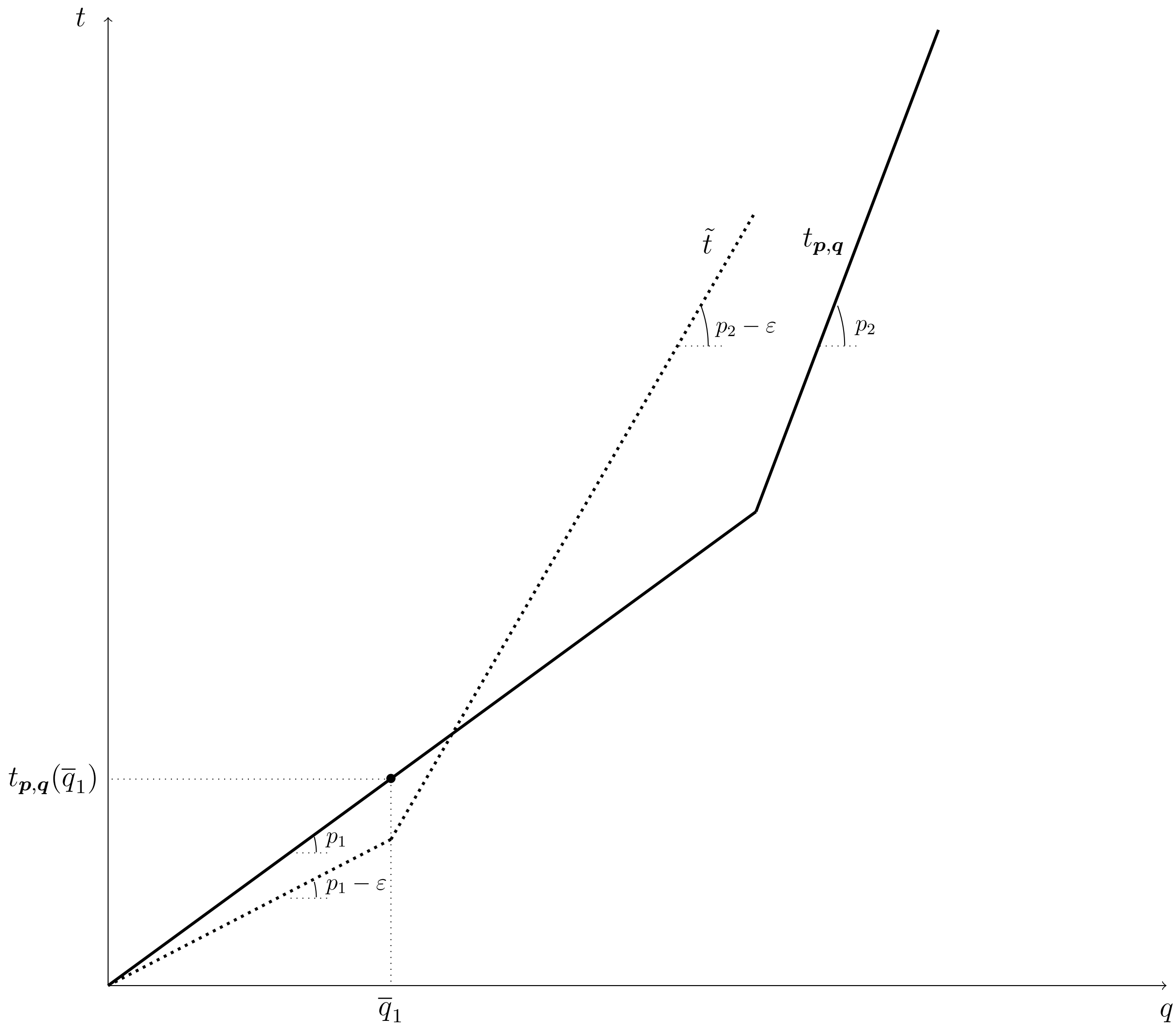

Figure 3 The $\tilde{t}$ schedule in the case $I = 2$.