Sequential Legislative Lobbying under Political Certainty[*]

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Abstract

In this paper, we analyse the equilibrium of a sequential game-theoretical model of lobbying, based on Groseclose and Snyder (1996), describing a legislature that votes on two alternatives and two opposing lobbies, lobby 0 and lobby 1, that compete by bidding for legislators' votes. In this model there is a strong second-mover advantage, so the lobbyist moving first will make offers to legislators only if he deters any credible counter-reaction from his opponent, i.e. if he anticipates winning the battle. Our main focus is on the calculation of the smallest budget that he needs to win the game and on the distribution of this budget across the legislators. We study the impact of game's key parameters on these two variables and show the connection of this problem with the combinatorics of sets and notions from cooperative game theory.

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In this paper, we consider a theoretical model of lobbying describing a legislature[1] that votes on two alternatives[2] and two opposing lobbies, lobby 0 and lobby 1, that compete by bidding for legislators' votes[3]. We examine how the voting outcome and the bribes offered to the legislators depend on the lobbies' willingness to pay, legislators' preferences and the decision-making process within the legislature.

There are many different ways to model the lobbying process. In this paper, we adopt the sequential model pioneered by Groseclose and Snyder (1996) and used by Banks (2000) and Diermeier and Myerson (1999). In this model, the competition between the two lobbies is described by a targeted offers game where each lobby gets to move only once and in sequence. For most of the paper, lobby 1 is pro-reform and moves first while lobby 0 is pro-status-quo and moves second. Votes are assumed to be observable. A strategy for each lobby is a profile of offers, where the offer made to each legislator is assumed to be based on his/her vote and to be honored irrespective of the voting outcome. The net payoff of a lobby is its gross willingness to pay minus the total amount of payments made to the legislators who ultimately vote for the policy advocated by this lobby. The legislators are assumed to care about how they cast their vote (independent of the income) and about monetary offers. Therefore, voters do not truly act strategically as their voting behavior is simply a best response to the pair of offers made by the lobbies and is independent of the decisions of other legislators. We focus on the complete-information environment, where the lobbies' and the legislators' preferences are known to the lobbies when they bid. We characterize the main features of the subgame perfect equilibrium of this game as a function of the following key parameters of the environment.

· Each lobby's maximal willingness to pay for winning[4] (i.e. to have their favorite policy selected). These two numbers represent the economic stakes under dispute and determine the intensity and asymmetry of the competition.

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[1] We depart from voluminous literature on the common agency setting in abandoning the assumption that policies are set by a single individual or by a cohesive, well-disciplined political party. In reality, most policy decisions are made not by one person but by a group of elected representatives acting as a legislative body. Even when the legislature is controlled by a single party (as is necessarily the case in a two-party system if the legislature consists of a single chamber), the delegation members do not always follow the instructions of their party leaders.

[2] Hereafter, we will often refer to the two alternatives as the status quo (alternative 0) versus the change or reform (alternative 1). While simplistic, many policy issues fit that formulation, like for instance: whether or not a free-trade agreement should be ratified, whether or not the free sale for guns should be banned, whether or not abortion should be allowed.

[3] By legislators, we mean all individuals who have a constitutional role in the process of passing legislation. This may include individuals from what is usually referred to as the executive branch, such as the president or the vice-president.

[4] Or, under an alternative interpretation, their respective budgets.

· The voting rule describing the legislative process.

· The heterogeneity in the legislators' preferences.

The binary setting considered in this paper is the simplest setting to tackle the joint influence of these three inputs on the final outputs. The first item consists of a single number per lobby, i.e. the amount of money this lobby is willing (able) to invest in this competition. The second item is also very simple. In this simplistic institutional setting, with no room for agenda setting or other sophisticated legislative action which would arise in the case of a large multiplicity of issues[5], we only need to know the winning coalitions, i.e. the coalitions of legislators able to impose the reform if the coalition unanimously supports this choice. Despite its apparent simplicity, this combinatorial object allows accommodating a wide diversity of legislatures. Banks and Groseclose and Snyder focus on the standard majority game, while Diermeier and Myerson consider the general case as we do. The third item describes the differences between the legislators other than those already attached to the preceding item if these legislators are not equally powerful or influential in the voting process. This "second" heterogeneity dimension refers to the differences between their intrinsic preferences for the reform versus the status quo. This difference, measured in monetary units, can be large or small and negative or positive. Diermeier and Myerson disregard this dimension by assuming that legislators are indifferent between the two policies, while Banks and Groseclose and Snyder consider the general situation but derive their results under some specific assumptions. We assume that legislators unanimously prefer the reform to the status quo but differ with respect to the intensity of their preference.

The first contribution consists in identifying the conditions under which the lobby moving first will make positive offers to some legislators. In this sequential game, the lobby moving last has an advantage as it can react optimally to its the opponent's offers without fear of response. If the asymmetry is too weak, lobby 1 will abandon the prospect of influencing the legislature as it will be rationally anticipating its defeat; in fact, it will make offers only if it is certain of success. If it does not make any offer, it is enough for lobby 0 to compensate a minimal winning coalition of legislators for their intrinsic preferences towards reform. Lobby 1 will participate if its willingness to pay or budget is larger than lobby 0's willingness to pay or budget. This minimal amount of asymmetry, which we call the *victory threshold*, defines by how much lobby 1's stake must outweigh lobby 0's stake to make sure that lobby 1 wins the game. Our first result states that the calculation of the victory threshold amounts to

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[5]Many formal models of the legislative process have been developed by social scientists to deal with more complicated choice environments. We refer to Grossman and Helpman (2001) for lobbying models with more than two alternatives.

calculating the supremum of a linear form over a convex polytope, which is closely related to the polytope of balanced families of coalitions introduced in cooperative game theory to study the core and other solutions. This result enables us to take advantage of the voluminous amount of work which has been done on the description of balanced collections. When heterogeneity in legislators' preferences is ignored, the victory threshold only depends upon the simple game describing the rules of the legislature. It corresponds to what Diermeier and Myerson have called the hurdle factor of the legislature. Quite surprisingly, this single parameter acts a summary statistic allowing us to predict the minimal budget that lobby 1 needs to invest to win the game.

Our second contribution consists in connecting the problem of computing the hurdle factor to the *covering problem,* which is one of the most famous, but also most difficult problems in the combinatorics of sets or hypergraphs. We provide a short introduction to this literature, show the connection with another famous parameter of a simple game, and calculate the hurdle factor of several simple games. Once, it is established that the hurdle factor is the fractional covering number of a specific hypergraph, we can take advantage of the enormous body of knowledge in that area of combinatorics.

The third contribution consists in showing that the hurdle factor can alternatively be calculated, surprisingly, as the maximum of specific equity criteria over the set of imputations of a cooperative game with transferable utility attached to the simple game of the legislature. The specific equity criterion is the minimum, across coalitions, of what the members of the coalitions will get in the imputation and what they could get on their own, i.e. the first component in the lexicographic order supporting the nucleolus. We use that result to show how to calculate the hurdle factor for the important class of weighted majority games. While there is a link between the weights of the legislators and the hurdle factor when the game is homogeneous, we show that the relation is more intricate in the general case.

The connection with cooperative game theory is even more surprising as it allows us to provide a complete characterization of the second dimension of lobby 1's optimal offer strategy. From what precedes, we know that the size of the lobbying budget is the hurdle factor times the willingness to pay (or budget) of lobby 0. It remains to be understood how this budget will be allocated across legislators. This is, of course, an important question as we would like to understand which legislator's characteristics determine lobby 1's willingness to buy his support and the amount that he will receive for selling his vote. As already discussed, legislators differ in two respects: the intensity of their preference for lobby 1 and their position/power in the legislature. The price of a legislator's vote is likely to be a function of both parameters. We show that the set of equilibrium offers is the least core of

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the cooperative game used to calculate the hurdle factor. It may contain multiple solutions, but the nucleolus[6] is always one of them. We illustrate the calculation of these prices in some important real world simple games and we revisit the model proposed by Diermeier and Myerson for the optimal determination of the hurdle factor of a legislative chamber given the other components of the legislative environment. One important conclusion is that these prices have little to do with a legislator's power as calculated through either the Banzhaf index (Banzhaf (1962), (1968)) or the Shapley-Shubik index (Shapley and Shubik (1954)). This suggests that the axiomatic theory of power measurement may not be fully relevant to predict players' payoffs in a game like this one[7].

Some legislators will not receive any offer from lobby 1. We may wonder about the identity and the number of legislators who will receive a positive offer. It is difficult to answer this question without being specific on either the legislators's preferences or the simple game. Our next contribution is to provide a characterization of this set in the case where the simple game is the standard majority game, i.e. we describe the conditions under which lobby 1 will target a minority, a minimal winning coalition or a supermajority and whether it will bribe in priority those who are more or less reluctant to support the reform[8]. The last contribution aims to show that the results of this paper hold, with some slight modifications, when we assume that legislators pay attention to the outcome rather than to their individual vote.

Related Literature

The literature on lobbying is very dispersed and voluminous[9]. The closest papers to ours are Banks (2000), Dekel, Jackson and Wolinsky (2006a, b), Diermeier and Myerson (1999), Groseclose and Snyder (1996), Young (1978 a, b, c) and Shubik and Young (1978d). Like us, they all consider the binary setting and assume that legislators care about their vote and money rather than about the outcome. As already mentioned, the two-round sequential vote buying model that we consider was constructed by Groseclose and Snyder. Banks as well as Diermeier and Myerson also consider this game. Their specific assumptions and focus, however, are quite different from ours. Banks and Groseclose and Snyder are primarily interested in identifying the number and the identity of the legislators who will receive an offer in the case of the simple majority game. By considering this important but specific symmetric game, they make it impossible to evaluate the impact of legislative power on the

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[6]The nucleolus also appears in a non-cooperative setting in Montero (2006) in a bargaining framework.

[7]This echoes Snyder, Ting and Ansolabehere (2005).

[8]Should the lobby seek to solidify support among those legislators inclined to support its positions anyway, or should it seek to win over those who might otherwise be hostile to its views?

[9]We refer to Grossman and Helpman (2001) for a description of the state of the art.

outcome. However, they consider more general profiles of legislators' preferences: instead of our unanimity assumption in favor of reform, Banks assumes that a majority of legislators has an intrinsic preference for the status quo. This implies that lobby 1 needs to bribe at least a majority to win; Banks provides conditions to the profile under which this majority will be minimal or maximal but does not determine the optimal size in the general case. Diermeier and Myerson assume instead that legislators do not have any intrinsic preference but consider an arbitrary simple game. Their main focus is on the architecture of multicameral legislatures and on the optimal behavior of each chamber under the presumption that it can select its own hurdle factor to maximize the aggregate offer made to its members. Our paper is very much related to the contributions of Young, who has analyzed a similar game and independently derived proposition 4. He can be credited with being the first to point out the relevance of the least core and the nucleolus in predicting some dimensions of the lobbyists' equilibrium strategies[10].

Dekel, Jackson and Wolinsky examine an open-ended sequential game where lobbies alternate in increasing their offers to legislators. By allowing lobbies to keep responding to each other with counter-offers, their game eliminates the asymmetry and the resulting second-mover advantage of Groseclose and Snyder's game. Several settings are considered, depending upon the type of offers that lobbies can make to legislators (up-front payments versus promises contingent upon the voting outcome) and upon the role played by budget constraints[11]. The difference in lobbies' budgets plays a critical role in determining which lobby is successful when lobbies are budget-constrained, while the difference in their willingness to pay plays an important role when they are not budget-constrained. When lobbies are budget-constrained, Dekel, Jackson and Wolinsky's main result states that the winning lobby is the one whose budget plus half the sum of the value that each legislator attaches to voting in favor of this lobby exceeds the corresponding magnitude calculated for the other lobby. In contrast, when lobbies are not budget-constrained, what matters are the lobbies' valuations and the intensity of the preferences held by a particular "near-median" group of legislators. The lobby with a-priori minority support wins when its valuation exceeds the

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[10]While we were working on this project, Ron Holzman pointed out to us the relevance of the notion of least core for our problem. After completing our paper, we discovered, while reading Montero (2006), that Young (1978a,b) reached the same conclusion a long time ago. In fact, he wrote four remarkable papers on this topic, containing many more results and insights. In (1978c), he presents a model of lobbying without opposition where the legislators are in charge (they "post" the price to which they are willing to sell their vote and, then, the lobby selects the coalition). Young and Shubik (1978) develop another version of the competitive model, which they call the session lobbying game, where the nucleolus is the equilibrium.

[11]These considerations, which are irrelevant in the case of our two-round sequential game, are important in their game.

other lobby's valuation by more than a magnitude that depends on the preferences of that near-median group. In our terminology, we can say that their main results are motivated by the derivation of the victory threshold(s). Once the value of these thresholds are known, the identity of the winner as well as the lobbying expenditures and the identity of bribed legislators follow. Note, however, that Dekel, Jackson and Wolinsky limit their analysis to the simple majority game and are not in a position to evaluate the intrinsic role of the simple game and the legislative power of legislators.

Note, finally, that the version of our game where the two lobbies make their offers simultaneously instead of sequentially has the features of a Colonel Blotto game. These games are notoriously difficult to solve and very little is known in the case of asymmetric players.

1　The Model and the Game

In this section, we formally describe the main components of the problem as well as the lobbying game which constitutes our model of vote-buying by lobbyists.

The external forces that seek to influence the legislature are represented by two players, whom we call lobby 0 and lobby 1. Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law. Lobby 0 is opposed to this bill and wants to maintain the status quo. Lobby 0 is willing to spend up to W_0 dollars to prevent passage of the bill, while lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we refer to these two policies in competition as being policies 0 and 1. We assume that $\Delta W \equiv W_1 - W_0 > 0$. While this assumption may receive different interpretations[12], we will assume here that the two lobbies faithfully represent the two opposite sides of society with regard to this binary social agenda and that policy 1 is therefore the socially efficient policy. We could consider that the two lobbies represent more private or local interests and that W_1 and W_0 ignore the implications of these policies on the rest of society: in that case the reference to social optimality should be abandoned. Finally, we could instead consider the budgets B_1 and B_0 of the two lobbies and assume that they are budget-constrained i.e. that $B_1 \leq W_1$ and $B_0 \leq W_0$. Under that interpretation, the ratio $\frac{W_1}{W_0}$ should be replaced by the ratio $\frac{B_1}{B_0}$. This ratio, which is (by assumption) larger than 1, will be a key parameter in our equilibrium analysis. Depending upon the interpretation, it could measure the intensity of reform's superiority as compared to the status quo or the ex ante advantage of lobby 1

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[12] As explained forcefully in Dekel, Jackson and Wolinsky (2006), in general, equilibrium predictions will be sensitive to the type of offers that can be made by the lobbies and whether they are budget-constrained or not. As explained later, these considerations are not relevant in the case of our lobbying game.

over lobby 0 in terms of budgets.

The legislature is described by a *simple game*[13] i.e. a pair $(N, \mathcal{W})$, where $N = \{1, 2, ..., n\}$ is the set of legislators and $\mathcal{W}$ the set of *winning* coalitions, satisfies: $S \in \mathcal{W}$ and $S \subseteq T$ implies $T \in \mathcal{W}$. The interpretation is the following. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions. A coalition C is *blocking* if $N \backslash C$ is not winning: some legislators (at least one) from C are needed to form a winning coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions[14]; by definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition. The simple game is called *proper* if $S \in \mathcal{W}$ implies $N \backslash S \notin \mathcal{W}$. The simple game is called *strong* if $S \notin \mathcal{W}$ implies $N \backslash S \in \mathcal{W}$ and *constant-sum* if it is both proper and strong, i.e. equivalently if $\mathcal{B} = \mathcal{W}$[15]. The set of minimal (with respect to inclusion) winning (blocking) coalitions will be denoted $\mathcal{W}_m$ ($\mathcal{B}_m$). A legislator is a *dummy* if he is not part of any minimal winning coalition, while a legislator is a *vetoer* if he belongs to all blocking coalitions. A group of legislators forms an *oligarchy* if a coalition is winning iff it contains that group, i.e. each member of the oligarchy is a vetoer and the oligarchy does not need any extra support to win, i.e. legislators outside the oligarchy are dummies. When the oligarchy consists of a single legislator, the game is called dictatorial.

In this paper, all legislators are assumed to be biased towards policy 1, i.e. all of them will vote for policy 1 against policy 0 if no other event interferes with the voting process. Under the interpretation offered in this paper, this assumption simply means that legislators vote for the policy maximizing aggregate social welfare. This assumption is of course controversial[16] and becomes even more so when we abandon the interpretation in terms of social efficiency. It is introduced here for the sake of simplicity as, otherwise, we would have to consider an additional parameter of differences among the legislators that we prefer to ignore for the time being. Indeed, in contrast to Banks (2000) and Groseclose and Snyder (1996), our assumption on the preferences of legislators rules out the existence of horizontal heterogeneity. However, legislators also value money, so we introduce instead some form of vertical heterogeneity. More specifically, we assume that legislators differ with regard to their willingness to depart from social welfare. The type of legislator i, denoted by α_i, is the minimal amount of dollars

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[13]In the social sciences it is sometimes called a committee or a voting game. In computer science, it is called a quorum system (Holzman, Marcus and Peleg (1997)) while in mathematics, it is called a hypergraph (Berge (1989), Bollobas(1986)). An excellent reference is Taylor and Zwicker (1999).

[14]In game theory, $(N, \mathcal{B})$ is often called the dual game.

[15]When the simple game is constant-sum, the two competing alternatives are treated equally.

[16]It is, however, very common in the recent literature on lobbying (Grossman and Helpman (2001)).

that he needs to receive in order to sacrifice one dollar of social welfare. Therefore, if the policy adopted generates a level of social welfare equal to W, legislator i's payoff if he receives a transfer t_i is:

$$t_i + \alpha_i W.$$

This payoff formulation is compatible with two behavioral assumptions. A first possibility is that the component W appears as soon as the legislator has voted for a policy generating a level of social welfare W regardless of whether this policy has ultimately been selected: we will refer to this model as *behavioral model P*, where P stands for procedural. Alternatively, the component W appears whenever the policy ultimately selected generates a level of social welfare W regardless of whether the legislator has voted for or against this policy: we will refer to this model as *behavioral model C*, where C stands for consequential. In this paper, we will focus exclusively on behavioral model P.

To promote passage of the bill, lobby 1 can promise to pay money to individual legislators conditional on their supporting the bill. Similarly, lobby 0 can promise to pay money to individual legislators conditional on their opposing the bill. We denote by $t_{i0} \geq 0$ and $t_{i1} \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted by t_0 and t_1 respectively.

The timing of actions and events that we consider to describe the lobbying game is the following[17].

    1. Nature determines the type of each legislator.

    2. Lobby 1 make contingent monetary offers to individual legislators.

    3. Lobby 0 observes the offers made by lobby 1 and makes contingent monetary offers to individual legislators

    4. Legislators vote.

    5. Payments (if any) are implemented.

This game has $n + 2$ players. A strategy for a lobby is a vector in $\Re_+^n$. Each legislator can choose among two (pure) strategies: to oppose or to support the bill.

The important thing to note is that the two lobbies move in sequence. Following Banks (2000), Diermeier and Myerson (1999) and Groseclose and Snyder (1996), we assume that lobby 1, the advocate for change, makes the first move and announces its offers first, and that lobby 1's offers are known to lobby 0 when lobby 0 makes its offers to induce legislators to oppose the bill. This sequential version of the lobbying game should be contrasted with the version where both lobbies move simultaneously or where lobbies make offers in an

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[17]Specific details and assumptions will be provided in due time.

open-ended sequential and alternating bidding process. As pointed out by Dekel, Jackson and Wolinsky (2006), in such a case the detailed specification of the type of offers as well as the budget constraints (if any) may matter. For instance, we can assume that lobbies' offers are either up-front payments or campaign promises honored only if the policy supported by the lobby is ultimately selected. If the moves are simultaneous budget-constrained campaign promises, the lobbying game belongs to the family of Colonel Blotto games, a class of discontinuous two-player zero-sum games which are notoriously difficult to solve; existence and characterization of equilibria in mixed strategies has been proved in the symmetric case, i.e. when $W_1 = W_0$ and for some very specific simple games, like the simple majority game. In our case, where each lobby moves only once and in sequence, these differences do not matter. The specificity of the sequential game which is considered here has been criticized by several authors, including Dekel, Jackson and Wolinsky (2006) and Grossman and Helpman (2001). In particular, in this game, there is a strong second-mover advantage. Note however, that alternatively the results of this paper answer the following questions: how asymmetric must the budgets or valuations of the two lobbies be to ensure the existence of a pure strategy Nash equilibrium in the simultaneous version of the game, i.e. in this generalized Colonel Blotto game? If it exists, what do the offers made to legislators in such an equilibrium look like and how do they depend upon their personal characteristics?

To complete the description of the game, we should specify the information held by the players when they act. In this paper, we have already implicitly assumed that the votes of the legislators are observable, i.e. by open voting, and that the vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of legislators' types is common knowledge and without loss of generality such that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n..$ We refer to this informational environment as *political certainty*. This has two implications: first, the lobbies know the types of legislators when they make their offers and second, each legislator knows the type of any other legislator when voting. The environment where the type α_i of legislator i is private information, to which we refer as *political uncertainty*, is analyzed in Le Breton and Zaporozhets (2007) in the context of the two lobbies moving simultaneously.

2 The Victory Threshold

In this section, we begin our examination of the *subgame perfect Nash equilibria* of the lobbying game. Hereafter, we will refer to them simply as equilibria. Our first objective is to calculate a key parameter of the game, which we call the *victory threshold*. Once calculated, this parameter leads to the following preliminary description of the equilibrium. Either the

ratio $\frac{W_1}{W_0}$ is larger than or equal to the victory threshold and then lobby 1 makes an offer and wins the game, or $\frac{W_1}{W_0}$ is smaller than the victory threshold and then lobby1 does not make any offer and lobby 0 wins the game. The victory threshold depends both upon the vector of types α and the simple game $(N, \mathcal{W})$. Given the second-mover advantage, the victory threshold is larger than or equal to 1. Therefore,while necessary, $W_1 > W_0$ is not sufficient in general to guarantee the victory of lobby 1. The victory threshold provides the smallest value of the relative differential leading to such a victory.

The equilibrium of the lobbying game can easily be described. Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re_+^n$ be lobby 1's offers. Lobby 0 will find it profitable to make a counter offer[18] if there exists a blocking coalition S such that:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i W_1\right) < \sum_{i \in S} \alpha^i W_0 + W_0.$$

Indeed, in such a case, there exists a vector $t_0 = (t_{10}, t_{20}, ..., t_{n0})$ of offers such that:

$$t_{i1} + \alpha^i W_1 < t_{i0} + \alpha^i W_0 \text{ for all } i \in S \text{ and } \sum_{i \in S} t_{i0} < W_0.$$

The first set of inequalities implies that legislators in S will vote against the bill, while the last one simply says that the operation is beneficial from the perspective of lobby 0. Therefore , if lobby 1 wants to make an offer that cannot be cancelled by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S \in \mathcal{B}.$$

The cheapest offers t_1 meeting these constraints are the solutions of the following linear program:

$$\underset{t_1}{Min} \sum_{i \in N} t_{i1}$$
$$\text{subject to the constraints} \tag{1}$$
$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S \in \mathcal{B}$$
$$\text{and } t_{i1} \geq 0 \text{ for all } i \in N.$$

Let t_1^* be any optimal solution of problem (1). Lobby 1 will find it profitable to offer t_1^* if the optimal value to this linear program is less than W_1. It is then important to be able

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[18]We assume that a legislator who is indifferent between the two offers will vote for the reform.

to compute this optimal value. To do so, we first introduce the following definition from combinatorial theory.

Definition 1. A family of coalitions $\mathcal{C}$ is *(sub)balanced* if there exists a vector $\delta \in \Re^{\#\mathcal{C}}$, called *(sub)balancing coefficients*, such that:

$$\sum_{S \in \mathcal{C}_i} \delta(S) \leq (=1) \text{ for all}[19] i \in N$$
$$\text{and } \delta(S) \geq 0 \text{ for all } S \in \mathcal{C}.$$

The following result summarizes the equilibrium analysis of the sequential game.

Proposition 1. *Either (i)* $W_1 \geq \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$ *for all vectors of sub-balancing coefficients δ attached to B. Then lobby 1 offers t_1^*, lobby 0 offers nothing and the bill is passed. Or (ii)* $W_1 < \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$ *for at least one vector of subbalancing coefficients δ attached to B. Then lobby 1 does not make any offer, lobby 0 offers t_0^* with $t_{i0}^* = \alpha^i \Delta W + \varepsilon$ for all $i \in S$ and $t_{i0}^* = 0$ otherwise where ε is an arbitrarily small positive[20] number and S is any coalition in the family of coalitions $T \in B$ such that $\sum_{i \in S} \alpha^i \Delta W < W_0$ and the bill is not passed.*

Proof. Let $v^* (\mathcal{B}, \boldsymbol{\alpha})$ be the optimal value of problem (1). From the duality theorem of linear programming, $v^* (\mathcal{B}, \boldsymbol{\alpha})$ is the optimal value of the following linear program:

$$\underset{\delta}{Max} \sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$$
$$\text{subject to the constraints}$$
$$\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N$$
$$\text{and } \delta(S) \geq 0 \text{ for all } S \in \mathcal{B}.$$

The conclusion follows. $\square$

This result[21] leads to several conclusions. If $W_0 - \sum_{i \in S} \alpha^i \Delta W \leq 0$ for all $S \in \mathcal{B}$, then $\delta = 0$ is a solution and therefore $v^* (\mathcal{B}, \boldsymbol{\alpha}) = 0$. We are in case (i), but lobby 1 promises nothing. If, instead, $W_0 - \sum_{i \in S} \alpha^i \Delta W > 0$ for at least one $S \in \mathcal{B}$, then $v^* (\mathcal{B}, \boldsymbol{\alpha}) > 0$. Note

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[20] To take care of indifferences.

[21] Note that we could replace $\mathcal{B}$ by $\mathcal{B}_m$ in the statement of proposition 1.

further that for any vector of subbalancing coefficients δ attached to $\mathcal{B}$:

$$
\begin{aligned}
\sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha_i \Delta W \right] &= W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{S \in \mathcal{B}} \delta(S) \sum_{i \in S} \alpha^i \\
&= W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{i \in N} \alpha^i \sum_{S \in \mathcal{B}_i} \delta(S) \\
&\geq W_0 \sum_{S \in \mathcal{B}} \delta(S) - \Delta W \sum_{i \in N} \alpha^i.
\end{aligned}
$$

We deduce:

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\sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right] \geq W_0 \gamma^*(\mathcal{B}) - \Delta W \sum_{i \in N} \alpha^i,
$$

and therefore

$$
v^* (\mathcal{B}, \boldsymbol{\alpha}) + \Delta W \sum_{i \in N} \alpha^i \geq W_0 \gamma^*(\mathcal{B}), \tag{2}
$$

where $\gamma^*(\mathcal{B}) \equiv v^*(\mathcal{B}, \mathbf{0})$, called hereafter the *hurdle factor*[22], is the value of the problem:

$$
\underset{\delta}{Max} \sum_{S \in \mathcal{B}} \delta(S)
$$

subject to the constraints

$$
\sum_{S \in \mathcal{B}_i} \delta(S) \leq 1 \text{ for all } i \in N
$$

and $\delta(S) \geq 0$ for all $S \in \mathcal{B}$.

After simplifications, we deduce that if we are in case (i), then:

$$
\frac{W_1}{W_0} \geq \frac{\gamma^*(\mathcal{B}) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i}. \tag{3}
$$

Inequality (3) is simply a necessary condition for case (i) to prevail. It is also sufficient for any problem where it can be shown that all the coordinates of t_1^*, the solution to problem (1), are strictly positive. Indeed, in that case, we deduce from the complementary slackness condition, that:

$$
\sum_{S \in \mathcal{B}_i} \delta(S) = 1 \text{ for all } i \in N,
$$

and (2) becomes an equality.

---

[22] This terminology is based on Diermeier and Myerson (1999).

The force of proposition 1 is to reduce the derivation of the victory threshold to the exploration of the geometry of a convex polytope: the polytope of the vector of subbalancing coefficients attached to $\mathcal{B}$. To use it efficiently, it may be appropriate to consider an arbitrary family of balanced coalitions, i.e. with edges not necessarily in $\mathcal{B}$. If we define the function Φ over coalitions of N as follows:

$$\Phi(S) = \begin{cases} W_0 - \sum_{i \in S} \alpha^i \Delta W & \text{if } S \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

then the duality argument used in the proof of proposition 1 shows that in the statement we can trivially replace "$\sum_{S \in \mathcal{B}} \delta(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right]$ for all vectors of subbalancing coefficients δ attached to $\mathcal{B}$" by "$\sum_{S \subset N} \delta(S) \Phi(S)$ for all vectors of balancing coefficients δ". The first formulation is useful if we can characterize the vector of subbalancing coefficients attached to $\mathcal{B}$[23]. This amounts first to exploring the combinatorics of simple games. A classification of simple games was first provided by Morgenstern and von Neumann (1944) and further explored by Isbell (1956)(1959). The second formulation takes advantage of the tremendous volume of research accomplished in cooperative game theory. Indeed, it is well known since Bondareva (1963) and Shapley (1967) that a game with transferable utility has a nonempty core iff it is balanced. As pointed out by Shapley, this amounts to checking the balancedness inequalities for the extreme points of the polytope of balanced collections of coalitions. He demonstrated that vector δ is an extreme point of the polytope of balanced collections iff the collection of coalitions $\{S\}_{S \subseteq N : \delta(S) > 0}$ is minimal in terms of inclusion within the set of balanced collections of coalitions. A minimal balanced collection has n sets at most[24]. Peleg (1965) has given an algorithm for constructing the minimal balanced sets inductively. We illustrate the mechanical use of proposition 1 through a sequence of simple examples .

Example 1. Consider the simple majority game with 3 legislators, i.e. $S \in \mathcal{B}_m$ iff $\#S = 2$ i.e. $S = \{1,2\}$, $\{1,3\}$ and $\{2,3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points

$$(0,0,0), (1,0,0), (0,1,0), (0,0,1), \left(\frac{1}{2},\frac{1}{2},0\right), \left(\frac{1}{2},0,\frac{1}{2}\right), \left(0,\frac{1}{2},\frac{1}{2}\right) \text{ and } \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

From the ordering of the α_i and proposition 1, we deduce that

$$v^*(\mathcal{B}, \alpha) = Sup\left(W_0 - (\alpha_1 + \alpha_2)\Delta W, \frac{3W_0 - 2(\alpha_1 + \alpha_2 + \alpha_3)\Delta W}{2}, 0 \right),$$

---

[23]Holzman, Marcus and Peleg (1997) contains results on the polytope of balancing coefficients for an arbitrary proper and strong simple game.

[24]We refer to Owen (2001) or Peleg and Sudhölter (2003) for a complete and thorough exposition of this material.

$$\gamma^*(\mathcal{B}) = \frac{3}{2}W_0.$$

The first (respectively second) term is the largest whenever $(\alpha_1 + \alpha_2)\,\Delta W \le W_0 \le 2\alpha_3\Delta W$ (respectively $W_0 \ge 2\alpha_3\Delta W$) and $v^*(\mathcal{B}, \alpha) = 0$ whenever $(\alpha_1 + \alpha_2)\,\Delta W \ge W_0$.

We will examine later how to derive the optimal (offers) of lobby 1 and in particular the personal characteristics of the legislators who are offered some positive amount. This will obviously depend on two main features: α_i i.e. his/her personal propensity to vote against social welfare and also his position in the family of coalitions. If legislator i is a dummy, then obviously, $t_{i1} = 0$. But if he is not a dummy, then in principle all situations are conceivable: he may receive something in all optimal offers, in some of them or in none of them. It will be important to know the status of a legislator according to this classification in three groups. In example 1, no legislator is a dummy. However, if $(\alpha_1 + \alpha_2)\,\Delta W \le W_0 \le 2\alpha_3\Delta W$, then the relevant extreme point is $(1, 0, 0)$. Since $\sum_{S \in \mathcal{B}_3} \delta(S) < 1$ in that case, we deduce from complementary slackness that $t_{31} = 0$.

Example 2. Consider the simple game with 4 legislators[25] defined as follows: $S \in \mathcal{B}_m$ iff $S = \{1, 2\}, \{1, 3\}, \{1, 4\}$ or $\{2, 3, 4\}$. According to Shapley (1967), besides partitions, the minimal balanced families of coalitions are (up to permutations):

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \{\{1, 2\}\{1, 3\}\{1, 4\}\{2, 3, 4\}\},$$
$$\{\{1, 2\}\{1, 3\}\{2, 3\}\{4\}\}, \{\{1, 2\}\{1, 3, 4\}\{2, 3, 4\}\}$$

with the following respective vectors of balancing coefficients $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. We deduce from proposition 1 that:

$$
\begin{aligned}
v^*(\mathcal{B}, \alpha) &= Sup\left(\frac{\frac{4W_0 - 3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\Delta W}{3}, \frac{5W_0 - 3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\Delta W}{3},}{\frac{2W_0 - (2\alpha_1 + \alpha_2 + \alpha_3)\Delta W}{2}, \frac{3W_0 - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)\Delta W}{2}, W_0 - (\alpha_1 + \alpha_2)\,\Delta W, 0}\right) \\
&= Sup\left(\frac{5W_0 - 3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\,\Delta W}{3}, W_0 - (\alpha_1 + \alpha_2)\,\Delta W, 0\right) \\
\gamma^*(\mathcal{B}) &= \frac{5}{3}W_0.
\end{aligned}
$$

Example 3. Consider the simple game with 3 legislators defined as follows: $S \in \mathcal{B}_m$ iff $S = \{1, 2\}$ or $\{1, 3\}$. The set of vectors of subbalancing coefficients is the polytope described by the set of extreme points $(0, 0), (1, 0), (0, 1)$. We deduce from proposition 1 that:

$$v^*(\mathcal{B}, \alpha) = Sup\left(W_0 - (\alpha_1 + \alpha_2)\,\Delta W,\ 0\right),$$

---

[25] As demonstrated by Von Neumann and Morgenstern ((1944), 52C), this is the only strong simple four-person game without dummies.

$$\gamma^*(\mathcal{B}) = W_0.$$

Example 4. Consider the simple game with 5 legislators defined as follows: $S \in \mathcal{B}_m$ iff $S = \{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}$ or $\{2, 3, 5\}$. The geometry of the polytope becomes more intricate. We will demonstrate later, through a different technique, that when $\alpha = \mathbf{0}$, the relevant extreme point is the vector $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5} \right)$ i.e.

$$\gamma^*(\mathcal{B}) = \frac{9}{5} W_0.$$

3   Complements and Extensions

Proposition 1 constitutes an important element of the toolkit to determine the victory threshold. In this second section, we continue this exploration of the problem aiming to add more elements to the toolkit. In the first subsection, we show that in the special case where $\alpha = 0$, our problem is strongly connected to one of the most famous problem in the combinatorics of sets. We elaborate on the relationship with this branch of applied mathematics and show how we can take advantage of this body of knowledge to get a better understanding of our own questions, which include the determination of the hurdle factors attached to a simple game. In the second subsection, we find, quite surprisingly, that the set of equilibrium offers to the legislators made by the first mover lobby coincides with the least core (and always contains the nucleolus) of the simple game. We then explore, within the important class of weighted majority games, how legislators' personal positions in the simple game translate into personal prices and we evaluate, through real world examples, the differences among their respective prices. Finally, in the third subsection, we characterize the size and the composition of the coalitions of legislators receiving an offer in the case of the simple majority game and for an arbitrary α.

3.1   *Fractional Matchings and Coverings*

The main purpose of this section is to connect our problem to the covering problem, which is considered to be one of the most famous problems in the combinatorics of sets. As pointed out by Füredi (1988), "the great importance of the covering problem is supported by the fact that apparently all combinatorial problems can be reformulated as the determination of the covering number of a certain hypergraph". A *hypergraph* is an ordered pair $H = (N, \mathcal{H})$ where N is a finite set of n vertices and $\mathcal{H}$ is a collection of subsets of N called edges. The *rank* of H is the integer $r(H) \equiv Max\{\#E : E \in \mathcal{H}\}$. If every member of $\mathcal{H}$ has r elements, we call it $r-$uniform. An $r-$uniform hypergraph H is called $r-$partite if there exists a

partition $\{N_k\}_{1\leq k\leq K}$ of N such that $\#(N_k \cap E) = 1$ holds for all $E \in \mathcal{H}$ and all $k = 1, ...K$. The maximum *degree* of the hypergraph H, denoted $D(H)$, is the number $Max_{i\in N}Deg_H(i)$ where $Deg_H(i) \equiv \#\{E \in \mathcal{H} : i \in E\}$. A hypergraph is $D-$regular if $Deg_H(i) = D(H)$ for all $i \in N$. Given an integer k, a hypergraph is $k-$*wise intersecting* if any of its k edges have a non-empty intersection; intersecting is used in place of 2-intersecting. An $(r, \lambda)-$design is a hypergraph $(N, \mathcal{H})$ such that for all $i \in N$, $Deg_H(i) = r$ and for all $\{i, j\} \subseteq N$, $\#\{S \in \mathcal{H} : \{i, j\} \subseteq S\} = \lambda$. It is called symmetric if $n = \#\mathcal{H}$. Then, $n = \frac{r^2-r+\lambda}{\lambda}$. A projective plane of order n, denoted by $PG(2, n)$ is a symmetric $(n + 1, 1)$ design. More generally, a $t-$dimensional finite projective space of order q, denoted by $PG(t, q)$, where q is a primepower, is an $(r, \lambda)-$design with $r = q^t + q^{t-1} + ... + 1$ and $\lambda = q^{t-1} + ... + q + 1$.

Given an integer k, a k-cover of H is a vector $t \in \{0, 1, ..., k\}^n$ such that:

$$\sum_{i\in S} t_i \geq k \text{ for all } S \in \mathcal{H}. \tag{4}$$

A $k-$matching of H is a collection $\{E_1, ..., E_s\}$(repetitions are possible) such that $E_j \in \mathcal{H}$ for all $j = 1, ...s$ every $i \in N$ is contained in at most k of E_j. A $1-$cover ($1-$matching) is simply called a cover (matching) of H. Note that a cover is simply a set T intersecting every edge of H i.e. $T \cap E \neq \varnothing$ for all $E \in \mathcal{H}$ while a matching is a collection of pairwise disjoint members of $\mathcal{H}$. A $k-$cover t^* minimizing $\sum_{i\in N} t_i$ subject to the constraints (4) is called an optimal $k-$cover and $\gamma_k^*(H) \equiv \sum_{i\in N} t_i^*$ is called the $k-$covering number. A $k-$matching δ^*maximizing $\sum_{S\subseteq N} \delta(S)$ is called an optimal $k-$matching and $\mu_k^*(H) \equiv \sum_{S\subseteq N} \delta^*(S)$ is called the $k-$matching number. When $k = 1$, $\gamma_1^*(H)$ is the minimum cardinality of the covers and is called the *covering number* of H while $\mu_1^*(H)$ is the maximum cardinality of a matching and is called the *matching number* of H. A hypergraph H is $\gamma-$*critical* if each of its subfamilies has a smaller covering number i.e. $\gamma_1^*((N, \mathcal{H} - \{E\})) < \gamma_1^*(H)$ for all $E \in \mathcal{H}$.

A *fractional cover* of H is a vector $t \in \Re^n$ such that:

$$\sum_{i\in S} t_i \geq 1 \text{ for all } S \in \mathcal{H} \tag{5}$$
$$\text{and } t_i \geq 0 \text{ for all } i \in N.$$

A fractional matching of H is a vector $\delta \in \Re^{\#\mathcal{H}}$ such that:

$$\sum_{S\in\mathcal{H}_i} \delta(S) \leq 1 \text{ for all } i \in N \tag{6}$$
$$\text{and } \delta(S) \geq 0 \text{ for all } S \in \mathcal{H}.$$

A fractional cover t^* minimizing $\sum_{i \in N} t_i$ subject to the constraints (5) is called an optimal fractional cover and $\gamma^*(H) \equiv \sum_{i \in N} t_i^*$ is called the *fractional covering number*. A fractional matching δ^* maximizing $\sum_{S \subseteq N} \delta(S)$ subject to the constraint (6) is called an optimal fractional matching and $\mu^*(H) \equiv \sum_{S \subseteq N} \delta^*(S)$ is called the *fractional matching number*.

It follows from these definitions that the hurdle factor of the simple game $(N, \mathcal{W})$ is the fractional covering number of $H = (N, \mathcal{B})$. If , in contrast to what has been assumed in the preceding section, money is available in indivisible units, then the appropriate parameter becomes $\gamma_{W_0}^*(H)$ where the integer W_0 is the value of policy 0 for lobby 0 (when $\mathcal{C} = \mathcal{B}$ i.e. when lobby 0 is the follower) expressed in monetary units. The case where $W_0 = 1$ is of particular interest, as it describes the situation where lobby 0 has a single money unit to spend in the process. The problem is now purely combinatorial: who should be the legislators on which lobby 1 should spend one unit to prevent lobby 0 from a targeting a single pivotal legislator[26]. Hereafter, the integer $\gamma_1^*(H)$ will be called the *integral hurdle factor*. While we will mostly focus on the divisible case, it is also interesting to note the implications of indivisibilities on the equilibrium outcome of the lobbying game. Note, finally, that if we invert the order of moves between the two lobbies, then the relevant simple game is the *dual game* $(N, \mathcal{B})$ and the corresponding hurdle factor, which we will call the *dual hurdle factor*, is the fractional covering number of $H = (N, \mathcal{W})$. The following developments apply equally to both hurdle factors and we will often use the symbol $\mathcal{H}$ without specifying whether $\mathcal{H} = \mathcal{B}$ or $\mathcal{H} = \mathcal{W}$. For an arbitrary hypergraph H, we have the inequalities:

$$\mu_1^*(H) \leq \frac{\mu_k^*(H)}{k} \leq \mu^*(H) = \gamma^*(H) \leq \frac{\gamma_k^*(H)}{k} \leq \gamma_1^*(H). \tag{7}$$

We immediately deduce from these inequalities that the value of the hurdle factor increases with the "degree" of indivisibility; indivisibilities act as additional integer constraints in the linear program describing the determination of the optimal fractional matchings and coverings. The calculation of the covering number[27] of an arbitrary hypergraph is an NP-hard problem, in contrast to the determination of the fractional covering number which amounts to solving a linear program. The examples presented below arise from the theory of simple games. In some cases, the hypergraph $\mathcal{H}$ describes the family of minimal winning coalitions, while in some others it represents the family of minimal blocking coalitions.

Example 5 (Qualified Majorities/minorities). Consider the case of an arbitrary symmetric simple game i.e. $S \in \mathcal{H}$ iff $\#S = q$ where q is a fixed integer. In that case, it

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[26] To support that interpretation, we need to assume, however, that a legislator who is indifferent breaks the tie in the direction of lobby 0.

[27] It has been demonstrated by Chung, Furedi, Garey and Graham (1988) that for any rational number x, there exists a, hypergraph $H = (N, \mathcal{H})$ such that $\mu^*(\mathcal{H}) = x$.

is easy to show that $\gamma^*(\mathcal{H}) = \frac{n}{q}$. For instance, in the case of the winning coalitions of the majority game ($q = \frac{n+1}{2}$ is n is odd and $q = \frac{n+2}{2}$ is n is even), we obtain:

$$\gamma^*(\mathcal{H}) = \begin{cases} \frac{2n}{n+1} & \text{if n is odd,} \\ \frac{2n}{n+2} & \text{if n is even.} \end{cases}$$

which tends to 2 when n tends to infinity. In contrast:

$$\gamma_1^*(\mathcal{H}) = \begin{cases} \frac{n+1}{2} & \text{if n is odd,} \\ \frac{n+2}{2} & \text{if n is even.} \end{cases}$$

When n is odd, the family of blocking coalitions of the majority game coincides with the family of winning coalitions. Instead, when n is even, the family of minimal blocking coalitions $\mathcal{H}$ is the family of subsets of cardinality $\frac{n}{2}$ and then $\gamma^*(\mathcal{H}) = 2$ while $\gamma_1^*(\mathcal{H}) = \frac{n+2}{2}$.

Example 6 (Symmetric Simple Games). The games considered in example 5 display a total symmetry[28], in the sense that the group of automorphisms of the simple game is the entire group of permutations. We can consider simple games exhibiting some regularity without displaying such a level of symmetry[29]. This is the case of the $(r, \lambda)-$design and in particular the projective planes of order n, $PG(2, n)$ which were defined earlier. Simple calculations show that $\gamma^*(r, \lambda) = \frac{r-1}{\lambda} + \frac{1}{r}$ and therefore $\gamma^*(PG(2, n)) = n + \frac{1}{n+1}$.

Another example of a hypergraph displaying some symmetry in line with Erdös and Lovasz (1975) is the following[30]. Consider a set S with $2\gamma - 2$ elements where γ is a given integer. For each partition $\pi = (P, P')$ of S where $P \cup P' = S$ and $\#S = \#S' = \gamma - 1$, take a new element i_π. Let $N \equiv S \cup_\pi \{i_\pi\}$ and let $\mathcal{H}$ be the collection of all $\gamma-$tuples of the form $P \cup \{i_\pi\}$ where $\pi = (P, P')$ is a partition. Then, it is easy to verify that $\gamma^*(H) = 2$ and $\gamma_1^*(H) = \gamma$.

Example 7 (Compound Simple Games). The following class of hypergraphs describes an important class of voting procedures. Let $(N_r, \mathcal{W}_r)_{1 \le r \le R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{W})$ be such that $N = \cup_{r=1}^R N_r$ and $S \in \mathcal{W}$ iff $S \cap N_r \in \mathcal{W}_r$ for all $r = 1, ..., R$. This is the definition of a multicameral legislature as defined by Diermeier and Myerson (1999): a reform is approved if it is approved in all the different R chambers according to the rules (possibly different)

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[28] This hypergraph is often called the complete $q-$graph.

[29] Von Neumann and Morgenstern (1944) offer a clear definition of symmetry based on the group of automorphisms of the game, i.e. the group of permutations leaving the winning coalitions invariant. We may for instance require this group to be $k-$transitive for some integer k. With such a definition, the symmetry of the game increases along with the value of k.

[30] The nucleus coterie constructed by Holzman, Marcus and Peleg (1997) and the symmetric hypergraph considered by Le Breton (1989) bear similarities with that hypergraph.

being used in the chambers. It is easy to show that:

$$\gamma^*(\mathcal{B}) = \sum_{r=1}^{R} \gamma^*(\mathcal{B}_r).$$

This multicameral system is a special case of a compound simple game as first defined by Shapley (1962). Let $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ be a hypergraph on the set of chambers: $\widetilde{\mathcal{H}}$ describes the power of coalitions of chambers (Diermeier and Myerson (1999)'s definition corresponds to the case where $\widetilde{\mathcal{H}} = \{\{1, ..., R\}\}$, i.e. each chamber has a veto power). In general, $S \in \mathcal{H}$ iff:

$$\{r \in \{1, ..., R\} : S \cap N_r \in \mathcal{W}_r\} \in \widetilde{\mathcal{W}}.$$

The computation of $\gamma^*(\mathcal{W})$ is now more intricate. If $\left(\{1, ..., R\}, \widetilde{\mathcal{W}}\right)$ is uniform as well as $(N_r, \mathcal{W}_r)$ for all $r = 1, ..., R$, then $(N, \mathcal{W})$ is also uniform. Füredi (1981)'s inequality gives an upper bound on $\gamma^*(\mathcal{W})$.

Consider the case where $R = 2K + 1$ and $\#N_r = 2n_r + 1$ for all $r = 1, ..., R$ where K, $n_1, ...n_R$ are integers and assume that $\left(\{1, ..., R\}, \widetilde{\mathcal{W}}\right)$ and $(N_r, \mathcal{W}_r)$ for all $r = 1, ..., R$ are the simple majority games. Exploiting the symmetry of the game, the determination of an optimal fractional cover is equivalent to the determination of a vector $(t_1, ..., t_R) \in \Re_{+}^{R}$ minimizing $\sum_{r=1}^{R}(2n_r + 1)t_r$ subject to the constraints:

$$\sum_{r \in S}(n_r + 1)\, t_r \geq 1 \text{ for all } S \subset \{1, ..., R\} \text{ such that } \#S = K + 1.$$

With the change of variables $T_r = (n_r + 1)\, t_r$, the problem is equivalent to the minimization of $2\sum_{r=1}^{R}(\frac{n_r + \frac{1}{2}}{n_r + 1})T_r$ subject to the constraints:

$$\sum_{r \in S} T_r \geq 1 \text{ for all } S \subset \{1, ..., R\} \text{ such that } \#S = K + 1.$$

This problem is almost identical to the covering problem attached to the majority game. The only difference lies in the fact that the weights on the variables do not need to be the same if the populations in the chambers differ in size. When they are identical, using the calculation in example 1, we deduce that:

$$\gamma^*(\mathcal{W}) = \frac{(2K + 1)(2n + 1)}{(n + 1)(K + 1)},$$

which tends to 4 when n becomes large.

In general it is difficult to derive the exact value of $\gamma^*(\mathcal{H})$ when $\mathcal{H}$ is the family of minimal blocking or winning coalitions describing the decision making process of the legislature.

In the above examples, we have examined the covering numbers of either the family of winning or the family of blocking coalitions. In the case of the hypergraph of winning coalitions, i.e. when we want to calculate the dual hurdle factor, it is traditionally assumed that it is intersecting, i.e. that the simple game $(N, \mathcal{W})$ is proper. In such case, it should be clear from what precedes that the intersection pattern of winning coalitions plays some role in the determination of the integral and fractional hurdle factors. A cover is a set which intersects every edge. When the simple game is proper, the set of minimal winning coalitions is an intersecting family. Any set in $\mathcal{W}$ is therefore a cover. This implies that the integral covering number is smaller than $Min_{E \in \mathcal{W}} \# E$. The knowledge of the integral hurdle factor provides useful information of the smallest size of a group of legislators able to collectively control the legislative process. When it is equal to 1, we have the familiar notion of a vetoer. When the number is equal to k, this means that there is a subset of k legislators which is represented in any winning coalition and that no smaller subset has this property. When the game is strong, the optimal cover is itself a winning coalition: a vetoer is then a dictator.

The following proposition relates the integral hurdle factor to another key parameter of a simple game known as the Nakamura number (Nakamura (1978)). The Nakamura number provides the exact largest possible cardinality of the set of alternatives, such that the core of the voting game resulting from the list of winning coalitions is non-empty for every conceivable preference profile. This parameter has attracted a lot of attention in the theory of voting and committees.

Definition. Let $G = (N, \mathcal{W})$ be a simple game. The Nakamura number of G, is the integer:
$$\nu(G) = \begin{cases} Min_{\mathcal{W}' \subseteq \mathcal{W}} \ \#\mathcal{W}' \text{ such that: } \cap_{S \in \mathcal{W}'} S = \varnothing, \\ +\infty \text{ if } \cap_{S \in \mathcal{W}} S \neq \varnothing. \end{cases}$$

Proposition 2. *For any simple game*

$$\gamma^*(\mathcal{H}) \leq \gamma_1^*(\mathcal{H}) \leq 1 + \frac{(Min \ \#S : S \in \mathcal{H}) - 1}{\nu(G) - 2} \text{ if } \nu(G) \neq \infty$$

and

$$\gamma^*(\mathcal{H}) = \gamma_1^*(\mathcal{H}) = 1 \text{ if } \nu(G) = \infty.$$

Proof. If $\nu(G) < \infty$, it follows from the definition of the Nakamura number that the collection $\mathcal{H}$ of minimal winning coalitions is a $(\nu(G) - 1)$-intersecting family. The conclusion follows from an inequality established in Lovasz (1979). If $\nu(G) = \infty$, then any $\{i\}$ with $i \in T \equiv \cap_{S \in \mathcal{W}} S$ is obviously an optimal cover.$\square$

There is an obvious trade-off between the number of minimal winning coalitions and the magnitude of the hurdle factors. If we have many coalitions in $\mathcal{W}_m$, the hurdle factor is more likely to be a large number[31]. The hurdle factor is also often sensitive to the addition or the deletion of a coalition from $\mathcal{W}_m$, i.e. situations defined above as $\gamma-$critical hypergraphs. For instance, it is easy to check that the hypergraphs attached to the simple games in example 1, the Erdos-Lovasz's simple game in example 2 and the cyclic majority game are $\gamma-$critical.

3.2   *Weighted Majority Games*

In this section, we focus on the important class of weighted majority games. A simple game is a weighted majority game if there exists a vector $\omega = (\omega_1, ..., \omega_n, q)$ of $(n+1)$)non negative numbers such that a coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega_i \geq q$; $\omega_i > 0$ is the weight attached to legislator[32] i. The vector $\omega \equiv (\omega_1, ..., \omega_n)$ is called a *representation* of the simple game. It is important to note that the same game may admit several representations. A simple game is *homogeneous* if there exists a representation ω such that $\sum_{i \in S} \omega_i = \sum_{i \in T} \omega_i$ for all $S, T \in \mathcal{W}_m$. This representation is called *the* homogeneous representation of the simple game, as Isbell (1956)[33] has demonstrated that an homogeneous simple game admits a unique (up to multiplication by a constant). The homogeneous representation ω for which $\sum_{i \in N} \omega_i = 1$ is called the homogeneous normalized representation and a homogeneous representation ω for which ω_i is an integer for all $i \in N$ is called an integral representation.

Consider an arbitrary cooperative game with transferable utility (N, V) and let $x \in X_n \equiv \left\{ y \in \Re_+^n : \sum_{i=1}^n y_i = V(N) \right\}$. Let $\theta(x)$ be the 2^n dimensional vector[34] whose components are the numbers $V(S) - \sum_{i \in S} x_i$ arranged according to their magnitude, i.e. $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq 2^n$. The *nucleolus* of (N, V) is the unique vector $x^* \in X_n$ such that $\theta(x^*)$ is the minimum, in the sense of the lexicographic order, of the set $\{\theta(y) : y \in X_n\}$. The *least core*[35] is the subset of X_n consisting of the vectors x such that $\theta_1(x) = \theta_1(x^*)$. It will be denoted $LC(V, N)$; by construction $x^* \in LC(V, N)$

To any simple game, we attach the cooperative game with transferable utility (N, V)

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[31] For instance, when all minimal winning coalitions are of the same size r i.e. $\mathcal{W}_m$ is $r-$uniform, we may ask how small the hypergraph can be if we want the covering number to be at least equal to r?

[32] In many applications, it is more relevant (if party discipline is strong) to assume that the players in the legislature are the different parties to which the legislators belong rather than the legislators themselves; in such a case, ω_i denotes the number of legislators affiliated to party i.

[33] See also the generalization by Ostmann (1987).

[34] This vector is called the vector of excesses attached to x.

[35] This notion was first introduced by Maschler, Peleg and Shapley (1979).

defined as follows:

$$V(S) = \begin{cases} 1 \text{ if } S \in \mathcal{W}, \\ 0 \text{ if } S \notin \mathcal{W}. \end{cases}$$

In such case, only minimal winning coalitions matter in "minimizing" the vector of excesses. The least core consists in the subset of vectors x such that

$$x \in ArgMax_{y \in S_n} Min_{S \in \mathcal{W}_m} \sum_{i \in S} y_i,$$

where $S_n \equiv \{y \in \Re_+^n : \sum_{i=1}^n y_i = 1\}$. Let:

$$C^* \equiv Max_{y \in S_n} Min_{S \in \mathcal{W}_m} \sum_{i \in S} y_i.$$

The following simple assertion holds true for any simple game.

Proposition 3. *Let (N, W) be a simple game. Then, $\gamma^*(W) = \frac{1}{C^*}$.*

Proof. By definition of C^*, there exists $y \in \Re_+^n$ such that:

$$\sum_{i=1}^n y_i = 1 \text{ and } \sum_{i \in S} y_i \geq C^* \text{ for all } S \in \mathcal{W}_m.$$

Therefore, the vector z such that $z_i \equiv \frac{y_i}{C^*}$ for all $i = 1, ..., n$ verifies:

$$\sum_{i=1}^n z_i = \frac{1}{C^*} \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m$$

implying that $\gamma^*(\mathcal{W}) \leq \frac{1}{C^*}$.

Assume that $\gamma^*(\mathcal{W}) < \frac{1}{C^*}$. This means that there is a vector $z \in \Re_+^n$ such that:

$$\sum_{i=1}^n z_i = \gamma^*(\mathcal{W}) \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m.$$

Therefore, the vector y such that $y_i \equiv \frac{z_i}{\gamma^*(\mathcal{W})}$ for all $i = 1, ..., n$ verifies:

$$\sum_{i=1}^n y_i = 1 \text{ and } \sum_{i \in S} y_i \geq \frac{1}{\gamma^*(\mathcal{W})} \text{ for all } S \in \mathcal{W}_m.$$

Since $\frac{1}{\gamma^*(\mathcal{W})} > C^*$, this contradicts our definition of C^*.$\square$

The proof is also quite instructive by itself as it also demonstrates that the set of optimal fractional covers of $(N, \mathcal{W})$ is, up to a division by C^*, the least core of the game induced by the simple game. Since the set of optimal fractional covers is, up to the multiplication by

W_0, the set of offers to legislators made by lobby 1 at equilibrium, the least core provides a complete characterization of lobby 1's equilibrium behavior.

Proposition 3 raises a number of questions. First, is it simple to calculate the quantity C^* for some particular families of simple games? Second, what does the least core look like, i.e. how are the different legislators treated? We answer these questions when $(N, \mathcal{W})$ is a weighted majority game

Peleg (1968) has demonstrated[36] that the normalized homogeneous representation of an homogeneous strong weighted majority game $(N, \mathcal{W})$ coincides with the nucleolus x of (N, V). Similarly, the integral representation of the nucleolus (which is well defined) is the minimum representation of the game, i.e. the unique minimal integral representation of the game. Since the nucleolus is an element of the least core, proposition 3, combined with Peleg's result, provides a straightforward way to calculate $\mu^*(\mathcal{W})$ for strong homogeneous weighted majority games. The task amounts to discovering the weight of each minimal winning coalition in the normalized homogeneous representation. For instance, the weighted majority game resulting from a legislature with 4 parties where the number of representatives of each party is described by the vector $\omega = (49, 17, 17, 17)$ is exactly the apex game considered in example 2. It is easy to see that the normalized homogeneous representation is here $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. It follows that $\gamma^*(\mathcal{W}) = \frac{5}{3}$.

The task is more intricate, however, when the simple game is not homogeneous[37]. Peleg has also proved that the minimal integral representation of the nucleolus is a minimal integral representation of the game if some condition is fulfilled, and has disproved by means of a counterexample of size 12 that the assertion holds true in general. He asks whether this assertion holds true when the simple game has a *minimum* integral representation. This conjecture has been disproved by Isbell (1969) by means of a counterexample of size 19. Therefore, within the class of non homogeneous weighted majority games[38], the relationship between the nucleolus (and then covering) and the set of minimal representations is less transparent. In such a case, the computation of $\gamma^*(\mathcal{W})$ can exploit the general algorithms which have been developed to calculate the nucleolus.

As already pointed out, besides the knowledge of $\gamma^*(\mathcal{W})$, it is of interest to know how the amount of money $\gamma^*(\mathcal{W})W_0$ is allocated across legislators or parties. This question is of

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[36] See also Peleg and Rosenmüller (1992).

[37] Several authors, including among others (Ostmann (1987), Peleg and Rosenmuller (1992), Rosenmuller (1987) and Sudhölter (1996)), have investigated the class of homogeneous weighted majority games, which are not necessarily strong. Sudhölter has introduced a notion of nucleolus (called the modified nucleolus), which is a representation of the game when it is homogeneous.

[38] The question becomes even more complicated when we move outside the world of weighted simple games, as exemplified by the calculation of the nucleolus of compound simple games (Meggido (1974)).

course very important as we would like to know which characteristics of legislator i (parties) besides α_i determine the "price" of that legislator from the perspective of lobby 1 (in fact, on the market for votes where the two lobbies compete). This is related to the position of i in the set of minimal winning coalitions[39]. In a weighted majority game, we intuitively expect this price to be positively correlated with the weight of the legislator, if not even exactly proportional to that weight. We have just seen that this intuition is correct in the case of a homogeneous weighted majority game (for an appropriate vector of weights), but that the exact relationship between weights and price is less clear otherwise[40].

Proposition 3 establishes an "unexpected" connection between the least core which has been developed in cooperative game theory and the set of equilibrium monetary offers in this non-cooperative game. The exploration of the least core of the simple game reveals all that we expect to discover about the payoffs of the legislators in the lobbying game. We know that the nucleolus is one of those vectors, but as we will see, the least core may also contain some other vectors. It is obvious that legislators who are dummies will never receive any offer from the lobbies. However, as illustrated through some of the following examples, there are situations where legislators who are not dummies do not receive any offer.

Example 3 Revisited. The simple game considered in example 3 is a weighted majority game; $\omega = (2, 1, 1)$ is a representation. It is easy to see that the core (and therefore the least core and the nucleolus) is equal to the vector $(1, 0, 0)$. The first legislator gets all the money despite the fact that neither legislator 2 nor legislator 3 is a dummy. Legislator 1 needs one of them to pass or to block (depending upon the interpretation of the hypergraph) the proposal. Legislators 2 and 3, however, are perfect substitutes and in excess supply on this market. Their internal competition drives down their price to 0. Note otherwise that this game is not strong and therefore Peleg's theorem does not apply. In fact the nucleolus is not a representation of the game while the modified nucleolus $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ is.

Example 8 (Vector Weighted Majority Games). In many situations, the type

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[39]The pattern of these positions defines, in some sense, the power of legislator i. There is an extensive literature on the measurement of the power of players in simple games with a prominent place occupied by the Banzhaf index (1965)(1968) and the Shapley-Shubik index (1954). The view that any of this one-dimensional power measures helps in predicting the legislators' payoffs in strategic environments where their votes can be bought has been disputed by several authors (see, for instance Snyder, Ting and Ansolabehere (2005) in the context of a legislative bargaining model).

[40]In some cases it will be possible to order, partially or totally, the legislators according to *desirability* as defined by Maschler and Peleg (1966). Legislator $i \in N$ is at least as desirable as legislator $j \in N$ if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \setminus \{i, j\}$. Legislators i and j are symmetric or interchangeable if $S \cup \{j\} \in \mathcal{W}$ iff $S \cup \{i\} \in \mathcal{W}$ for all $S \subset N \setminus \{i, j\}$. Peter Sudhölter has drawn our attention to the fact that the least core does not necessarily preserve the desirability relation. Krohn and Sudhölter (1995) illustrate many simple games with such a feature. Note, however, that the nucleolus preserves desirability.

of legislator, defined by a vector of traits or attributes, is an important parameter in the explicit description of winning or blocking coalitions: for instance, the type of legislator may consist of the chamber to which he belongs (in a multicameral system), the legislator's gender, geographic area (east, west, north, south), and so on. Let us assume that there are K mutually exclusive possible types and denote by n_k the number of legislators of type k. A coalition is a $K-$tuple of integers $m \equiv (m_1, ..., m_K)$ where $m_k \leq n_k$ for all $k = 1, ..., K$. We consider the setting where there exist J vectors $\left(a_1^j, ..., a_K^j, b^j\right) \in \Re_+^{K+1}$ such that a coalition m is winning iff:

$$\sum_{k=1}^K a_k^j m_k \geq b^j \text{ for } j = 1, ..., J.$$

This framework generalizes[41] the concept of a weighted majority game: in the case of a strong weighted majority game, the set of different weights is the set of types, $J = 1$, $a = (1, 1, ..., 1)$ and $b = \left\lfloor \frac{\sum_{1 \leq k \leq K} n_k \omega_k}{2} + 1 \right\rfloor$[42]. The minimal winning coalitions are the lower vertices of the polyhedron described by the above inequalities and the hypercube $\prod_{k=1}^K \{0, 1, ...n_k\}$. We illustrate the calculation of the hurdle and dual hurdle factors and the least core in the following specific case, which describes the U.S. federal legislative system[43]. Let $K = 4$, $J = 2$, $a^1 = \left(0, 1, \frac{1}{2}, \frac{33}{2}\right)$, $a^2 = (1, 0, 0, 72)$, $b^1 = 67$, $b^2 = 290$, $n_1 = 435$, $n_2 = 100$, $n_3 = 1$ and $n_4 = 1$. This simple game represents a bicameral system (the House of Representatives and the Senate) with two additional players: the vice president and the president. A coalition is winning if it contains either more than half the house and more than half the senate (with the vice president playing the role of tie-breaker in the senate), together with the support of the president or two-thirds of both the house and the senate (to override a veto by the president). The determination of the least core is reduced to the minimization of $435x_1 + 100x_2 + x_3 + x_4$ with respect to $(x_1, x_2, x_3, x_4) \in \Re_+^4$ under the constraints:

$$218x_1 + 50x_2 + x_3 + x_4 \geq 1$$
$$218x_1 + 51x_2 + x_4 \geq 1$$
$$290x_1 + 67x_2 \geq 1.$$

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[41] Taylor and Zwicker (1999) call vector weighted games such simple games. Among the real world voting systems which are vector-weighed, we can cite the system to amend the Canadian constitution and the different decision rules for the council of ministers of the EU like those prescribed by the treaty of Nice.

[42] $\lfloor x \rfloor$ denotes the integer part of x.

[43] This representation of the U.S. federal legislative system appears in Taylor and Zwicker (1999).

Note that if $x_3 + x_4 = 0$, then the relevant inequality is $218x_1 + 50x_2 \geq 1$. Since $\frac{435}{218} \simeq 1.9954 < \frac{100}{50} = 2$, the minimum is obtained in $\left(\frac{1}{218}, 0\right)$ leading to the value 1.9954. If instead $x_3 + x_4 > 0$, then the inequalities $290x_1 + 67x_2 \geq 1$ and $218x_1 + 50x_2 + x_3 + x_4 \geq 1$ are the relevant inequalities. This follows from the fact that necessarily at the optimum, $x_3 \leq x_2$. Indeed, if instead $x_3 > x_2$, then the first inequality is implied by the second. Since $x_3 > 0$, we could reduce it by some small positive amount without violating the constraints as x_3 does not appear in any active constraint, contradicting our assumption of a minimum. Then $x_4 = 72x_1 + 17x_2$ and the objective to be minimized is $507x_1 + 117x_2$. The minimum is obtained for $x_1 = 0$, $x_2 = \frac{1}{67}$ and $x_3 + x_4 = \frac{17}{67}$. Since the lower bound on x_4 is $1 - \frac{51}{67} = \frac{16}{67}$, we obtain that the upper bound on x_3 is $\frac{1}{67}$. For any such solution, the value of the program is $\frac{117}{67} \simeq 1.746 < 1.9954$. We have just proved that the dual hurdle factor of the US federal legislative game is 1.746 and that the least core is a one dimensional convex set, namely the convex hull of the vectors $\left(0, \frac{1}{67}, \frac{1}{67}, \frac{16}{67}\right)$ and $\left(0, \frac{1}{67}, 0, \frac{17}{67}\right)$. Interestingly, the members of the House do not get any offer although they are not dummies and the offer made to the president is around 16 times larger than the offer made to any single senator or to the vice president.

Let us now look at the U.S. federal legislative game from the point of view of the blocking coalitions, i.e. at the hurdle factor. It is easy to see that the minimal blocking coalitions (m_1, m_2, m_3, m_4) are the following:

$$m_1 = 146, \ m_2 = m_3 = 0, \ m_4 = 1$$
$$m_1 = 0, \ m_2 = 34, \ m_3 = 0, \ m_4 = 1$$
$$m_1 = 218, \ m_2 = m_3 = m_4 = 0$$
$$m_2 = 51, \ m_1 = m_3 = m_4 = 0$$
$$m_2 = 50, \ m_3 = 1, \ m_1 = m_4 = 0.$$

In such case, we obtain that the least core consists of the unique vector $\left(\frac{1}{218}, \frac{1}{51}, \frac{1}{51}, \frac{17}{51}\right)$ (which is the nucleolus) and that the hurdle factor is approximately 4.31[44].

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[44]This result should be contrasted with the claims formulated by Diermeier and Myerson (1999) in their footnote 9. They write "....The veto-override provision is not significant. The $\frac{2}{3}$ veto override option allows that lobby 1 can get a bill passed by paying $3W_0$ in the house and $3W_0$ in the senate, rather than paying W_0 to the president plus $2W_0$ in the house and $2W_0$ to the senate. So the alternative legislative path that is allowed by the $\frac{2}{3}$ veto override has a hurdle factor of 6, which is higher than the hurdle factor of 5 that is available without it. *Thus our analysis predicts that lobbyists for change should generally ignore the more expensive option of overriding a presidential veto,* and should lobby just as they would if the congress were a purely serial bicameral legislature with a presidential veto...". This prediction differs from ours, as for instance, they predict that the president will receive 20% of the "cake" while we predict that he will receive only 7.74%. Additionally, they predict that the bribe offered to the president will be 50 times larger than the bribe offered to any single senator, while we predict that it will be 17 times larger.

Example 9 (The United Nations Security Council). The voters are the 15 countries that make up the Security Council, 5 of which are called permanent members whereas the other 10 are called nonpermanent members. Passage of a bill requires a total of at least 9 votes, subject to approval from any one of the 5 permanent members. It is easy to show that this simple game is a weighted majority game: assigning a weight of 7 to each permanent member, a weight of 1 to any nonpermanent member and a quota equal to 39 provides a representation. If lobby 1 acts to pass a reform (here a resolution), the problem of determining the least core is reduced to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \Re_+^2$ under the constraints:

$$x_1 \geq 1 \text{ and } 7x_2 \geq 1.$$

We deduce that the least core consists of the unique vector $\left(1, \frac{1}{7}\right)$ (which is the nucleolus) and that the hurdle factor $5 + \frac{10}{7}$ is approximately equal to 6.43.

If instead lobby 0 acts to block a reform, the problem of determination of the least core is reduced to the minimization of $5x_1 + 10x_2$ with respect to $(x_1, x_2) \in \Re_+^2$ under the constraint:

$$5x_1 + 4x_2 \geq 1.$$

Now we obtain that the least core consists of the unique vector $\left(\frac{1}{5}, 0\right)$ (which is the nucleolus) and that the dual hurdle factor is equal to 1. Here, only the permanent members receive an offer and with a hurdle factor equal to 1, lobbying expenditures by lobby 1 remain moderate.

Example 10 (Diermeier and Myerson's Multicameral Systems). The main objective in Diermeier and Myerson's paper is to determine the optimal hurdle factor of one of the chambers (say the House) in a multicameral system, given the hurdle factors of the other chambers where optimal means maximizing the expected aggregate amount of bribes received by the members of the house. They assume that W_0 and W_1 are independent and identically distributed random variables and they offer detailed illustrations of the optimization problem in the case where the marginals are either lognormal or uniform. It is important to bear in mind that they conduct their analysis under the assumption that there is no uncertainty about which lobby will move first: lobby 1 always moves first. Let t be the sum of the hurdle factors of the other chambers and s be the hurdle factor of the house. Lobby 1 makes offer when $\frac{W_1}{W_0} \geq s + t$. In such a case, the house receives sW_0. When instead $\frac{W_1}{W_0} < s + t$, the house does not receive any transfer. Let $F(s, t)$ be the corresponding expected income of the house. Diermeier and Myerson's central result asserts that the best response $s^*(t)$ of

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the house, which can be implemented by choosing of an appropriate simple game $(N, \mathcal{W})$, increases as the external hurdle factor t increases.

Conceivably, in some circumstances, the lobby which wants the status quo to be preserved acts first. If that is the case, the relevant simple game is the dual game and the relevant hurdle factor is the dual hurdle factor. As is demonstrated below, if lobby 0 makes an offer, then the member of the house receives a fraction of the total bribe (in fact the totality) iff their hurdle factor is smaller than the hurdle factor of the other chamber. Consider the case of a bicameral system and let $\widehat{t}$ be the dual hurdle factor of the other chamber and $\widehat{s}$ be the dual hurdle factor of the house[45]. Let $\widehat{F}(\widehat{s}, \widehat{t})$ be the corresponding expected income of the house. If we assume that the two situations occur with probabilities[46] p and $1 - p$, then in the simple case where there is no other chamber (unicameral legislature), the expected income is now:

$$pF(s, 0) + (1 - p)F(\widehat{s}, 0)$$

as $F = \widehat{F}$ in the unicameral case. A new trade-off appears as increasing s now has two effects: a direct effect like before (as active lobbying becomes less likely) and an indirect effect through a decrease of $\widehat{s}$. Of course, in the above expressions, there is a one to one relationship between the two hurdle factors $\widehat{s} = \gamma^*(\mathcal{W})$ and $s = \gamma^*(\mathcal{B})$. If we limit the implementation to symmetric quota games i.e. $S \in W$ iff $\#S \geq q$, we deduce from example 5 that $\widehat{s} = \gamma^*(\mathcal{W}) = \frac{n}{q}$ and $s = \gamma^*(\mathcal{B}) = \frac{n}{n-q+1}$. If n is large, we deduce that:

$$\frac{q}{n}\gamma^*(\mathcal{W}) = \left(1 - \frac{q}{n}\right)\gamma^*(\mathcal{B}) \text{ i.e. } \widehat{s} = \frac{s}{s-1}$$

This leads to the first order condition :

$$p\frac{\partial F}{\partial s}(s, 0) = \frac{1 - p}{(s - 1)^2}\frac{\partial F}{\partial s}\left(\frac{s}{s - 1}, 0\right)$$

The following table provides the value of the optimal hurdle factor for different values of the parameters p and σ in the lognormal case.

The first line of table 1 is of course similar to the first line of table 3 in Diermeier and Myerson. An interesting observation is that moving from $p = 1$ to the more balanced assumption $p = \frac{1}{2}$ leads to the optimality of the standard majority game for a large range of values of σ (approximately when σ is less than 1.57). Exploiting the symmetry for $p = \frac{1}{2}$, we know that if s is a solution then $\frac{s}{s-1}$ is also a solution. In table 1, we have reported the largest

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[45]i.e. $\widehat{s} = \gamma^*(\mathcal{W})$. The above examples show that unless the simple game is constant-sum (in which case $\widehat{s} = s$), the two factors behave quite differently.

[46]Diermeier and Myerson assume $p = 1$.

Table 1: *Optimal Hurdle Factor in Lognormal Model.*

					σ					
p	0.6	0.8	1.0	1.2	1.3	1.5	1.6	1.7	2.0	3.0
1	1.0	1.0	1.2414	1.8516	2.3392	3.9872	5.3765	7.4056	21.9487	3122.3942
0.75	1.0	1.5486	1.6915	1.9293	2.189	3.5283	4.8525	6.8422	21.3253	3121.9111
0.5	2.0	2.0	2.0	2.0	2.0	2.0	3.0271	5.4101	20.01	3120.9446

of the two solutions[47]. Interestingly enough, it is larger than Diermeier-Myerson's optimal hurdle factor for small enough values of σ and smaller afterwards. When σ gets larger than 1.57, the optimal[48] hurdle factor increases but stays smaller than Diermeier-Myerson's one.

In a multicameral legislature, F's second argument is no longer equal to 0. As defined in example 7, let $(N_r, \mathcal{W}_r)_{1 \leq r \leq R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{W})$ be such that $N = \cup_{r=1}^{R} N_r$ and $S \in \mathcal{W}$ iff $S \cap N_r \in \mathcal{W}_r$ for all $r = 1, ..., R$ i.e. a reform is approved if it is approved in all the different R chambers according to the rules (possibly different) in use in the chambers. Given the hurdle factors $\gamma^*(\mathcal{W}_r)$ of each chamber $r = 1, ..., R$, let us calculate $\gamma^*(\mathcal{W})$. It is the value of the linear program:

$$Min_{t \in \Re_+^n} \sum_{r=1}^{R} \sum_{i \in N_r} t_{ir}$$

under the constraints

$$\sum_{r=1}^{R} \sum_{i \in S_r} t_{ir} \geq 1 \text{ for all } R - \text{tuple } (S_r)_{1 \leq r \leq R} \text{ such that } S_r \in \mathcal{W}_r \text{ for all } r = 1, ..., R.$$

Let $\theta \in \Re_+^R$ be such that $\sum_{r=1}^{R} \theta_r = 1$. The value of the above program is less than the value of the program:

$$Min_{t \in \Re_+^n} \sum_{r=1}^{R} \sum_{i \in N_r} t_{ir}$$

under the constraints

$$\sum_{i \in S_r} t_{ir} \geq \theta_r \text{ for all } R - \text{tuple } (S_r)_{1 \leq r \leq R} \text{ such that } S_r \in \mathcal{W}_r \text{ for all } r = 1, ..., R.$$

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[47] The largest is the unique solution when p is slightly on the right of $\frac{1}{2}$.
[48] The function which is maximized displays interesting nonconvexities.

But this new problem is decomposable into R disjoint minimization programs. We deduce from that argument that:

$$\gamma^* \left(\mathcal{W} \right) \leq \sum_{r=1}^{R} \theta_r \gamma^* \left(\mathcal{W}_r \right) \text{ for all } \theta \in \Re_+^R \text{ such that } \sum_{r=1}^{R} \theta_r = 1.$$

Since the above inequality is an inequality for any vector θ attached to a solution of the initial problem, we deduce:

$$\gamma^* \left(\mathcal{W} \right) = Min_{\theta \in \Re_+^R} \sum_{r=1}^{R} \theta_r \gamma^* \left(\mathcal{W}_r \right) \text{ under the constraint } \sum_{r=1}^{R} \theta_r = 1,$$

and therefore:

$$\gamma^* \left(\mathcal{W} \right) = Min_{1 \leq r \leq R} \gamma^* \left(\mathcal{W}_r \right).$$

This result has important implications for the determination of the optimal dual hurdle factor by the house. Indeed, in the case where the first-mover lobby is the lobby which wants to block the passage of the reform, the amount of money received by the house will critically depend upon its dual hurdle factor compared to the dual hurdle factors of the other chambers. If it is larger than the smallest one, then the house will not be approached by the lobby. The game describing the interaction between the chambers displays discontinuous payoff functions. In the case of two chambers and $p = \frac{1}{2}$, we obtain that the payoff of chamber 1^{49} is equal to:

$$\begin{cases} \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{\widehat{F}(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{2} & \text{if } \gamma^* \left(\mathcal{W}_1 \right) < \gamma^* \left(\mathcal{W}_2 \right), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} + \frac{\widehat{F}(\gamma^*(\mathcal{W}_1),\gamma^*(\mathcal{W}_2))}{4} & \text{if } \gamma^* \left(\mathcal{W}_1 \right) = \gamma^* \left(\mathcal{W}_2 \right), \\ \frac{F(\gamma^*(\mathcal{B}_1),\gamma^*(\mathcal{B}_2))}{2} & \text{if } \gamma^* \left(\mathcal{W}_1 \right) > \gamma^* \left(\mathcal{W}_2 \right), \end{cases}$$

if we assume that ties are broken equally. Interestingly enough, if both chambers were acting under the presumption that the lobby which will move first is the pro-status quo lobby, then the game becomes a Bertrand game[50] where behavioral responses converge to the Nash equilibrium $(1, 1)$. It would be interesting to know what we obtain in the general case. When it is taken for granted that the pro-reform lobby moves first, Diermeier and Myerson found convergence towards the Nash equilibrium $(2.20, 2.20)$ in the case of a bicameral legislature implemented by a quota of 54.5%; note that then $\gamma^* \left(\mathcal{W} \right) \simeq 1.835$.

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[49] The payoff of chamber 2 is obtained similarly.

[50] The game arising from the assumption considered by Diermeier and Myerson (1999) displays the features of a Cournot game.

3.3   *Buying Supermajorities*

In the two preceding subsections, we have assumed that $\alpha = 0$ and we have therefore ignored the impact of vector α on the equilibrium outcomes of the lobbying game. We have focused our attention on the implications of the rules governing the decision process within the legislature and the "power" derived by the legislators as a result of their status. We have covered a large class of simple games describing many alternative institutional legislative settings and isolated the influence of that component of the price of a legislator.

In this last subsection, we reintroduce vector α, but we focus our attention on a very special (while important) simple game: the classical majority game. In that respect, the analysis of this section is aligned with the framework of Banks (2000) and Groseclose and Snyder (1996)(2000). Given the symmetry of the simple game, all legislators are alike in terms of their power in the legislature. This means that if two legislators i and j receive different offers from the lobby, the rationale for this differential should be based on differences between α_i and α_j. We have seen, in the previous subsections, that some legislators endowed with limited power within the legislature were, sometimes, totally ignored by the lobby. Here, a legislator i with a large α_i will be cheap for lobby 1 and expensive for lobby 0. Finally, we have also observed that most of the time the lobby was bribing a coalition strictly larger than a minimal winning coalition. These considerations raise a number of questions:

· What will be the size of the coalition of legislators receiving offers from the lobby? Since the α_i are nonnegative numbers, lobby 1 may bribe a submajority coalition (at the extreme, nobody at all), a minimal majority or a supermajority (at the other extreme, everybody) depending upon the profile α. Which legislators will be part of that coalition and, in particular, when the cheapest strategy of lobby 1 consists in bribing the whole legislature?

· Which legislators will be part of the bribed coalition? Will we observe a flooded coalition as in Banks (2000) or a nonflooded coalition as in Groseclose and Snyder (1996)(2000), where flooded refers to the fact that lobby 1 bribes in priority the legislators more willing to support the reform.

· What are the differences between the offers received by the legislators who are in the coalition?

The following proposition answers these three questions in the case where $(N, \mathcal{W})$ is the classical majority game and n is odd[51] i.e. $n = 2k - 1$ for some integer $k \geq 2$.

Proposition 4. *If W_1 is large enough, there exists an optimal offer $t_1^* = (t_{11}^*, t_{21}^*, \ldots, t_{n1}^*)$*

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[51] Therefore, this game is constant-sum.

by lobby 1 described by an integer $m^* \in \{0, 1, ..., n\}$ and [such such that $t^*_{i1} > 0$ and $t^*_{i1} + \alpha^i \Delta W = t^*_{j1} + \alpha^j \Delta W$ for all $i, j = 1, ..., m^*$. Further, either $\frac{W_0}{k} > \alpha^k \Delta W$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any, and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \alpha^k \Delta W$ and m^* is the smallest value of $m \leq k - 1$ such that $W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m \alpha^{m+1} \right].$]

Proof. [Assume without loss of generality that $\alpha^1 \leq \alpha^2 \leq .... \leq \alpha^n$. Let $t^*_1 = (t^*_{11}, t^*_{21}, ..., t^*_{n1})$ be an optimal solution to problem (1)] and $x^*_{i1} \equiv t^*_{i1} + \alpha^i \Delta W$ for all $i = 1, ..., n$. Problem (1) can be expressed equivalently as follows:

$$\underset{t_1}{Min} \sum_{i \in N} x_{i1} - \Delta W \sum_{i \in N} \alpha^i$$

subject to the constraints $\qquad\qquad$ (8)

$$\sum_{i \in S} x_{i1} \geq W_0 \text{ for all } S \subseteq N \text{ such that } \#S = k$$

and $x_{i1} \geq \alpha^i \Delta W$ for all $i \in N$.

Let $v^*(k, \boldsymbol{\alpha})$ be the optimal value of problem (8). From the duality theorem of linear programming, $v^*(k, \boldsymbol{\alpha})$ is the optimal value of the following linear program:

$$\underset{\delta}{Max} \sum_{S \in \mathcal{H}} \delta(S) W(S) - \Delta W \sum_{i \in N} \alpha^i$$

subject to the constraints

$$\sum_{S \in \mathcal{H}_i} \delta(S) \leq 1 \text{ for all } i \in N$$

and $\delta(S) \geq 0$ for all $S \in \mathcal{H}$.

where $\mathcal{H} = \{S \subseteq N$ such that either $\#S = k$ or $\#S = 1\}$ and:

$$W(S) = \left\{ \begin{array}{l} W_0 \text{ if } \#S = k \\ \alpha^i \Delta W \text{ if } S = \{i\} \end{array} \right.$$

It can be shown[52] that to solve (10) the relevant families δ of balanced collections of coalitions attached to $\mathcal{H}$ are in one to one correspondence with the sets S such that either $\#S \geq k$ or $S = \varnothing$.

For all S such that $\#S = m \geq k: \quad \delta(T) = \dfrac{1}{C^{k-1}_{m-1}}$ for all $T \subseteq S$ with $\#T = k$ and $\delta(\{i\}) = 1$ for all $i \in N \backslash S$

For $S = \varnothing: \delta(\{i\}) = 1$ for all $i \in N$

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[52] A detailed proof can be obtained from the authors upon request.

We obtain then that the victory threshold $v^*(k, \boldsymbol{\alpha})$ is equal to:

$$\sup\left(0, \sup_{k \leq m \leq n} \frac{m}{k} W_0 - \Delta W \sum_{i=1}^{m} \alpha^i\right)$$

The function $\frac{m}{k} W_0 - \Delta W \sum_{i=1}^{m} \alpha^i$ is "concave" as a function of the variable m. Therefore, it admits a unique maximizer defined as the largest value m^* of m such that:

$$\frac{W_0}{k} \geq \Delta W \alpha^m$$

or

$$\frac{W_0}{k} < \Delta W \alpha^k$$

Case 1: $\frac{W_0}{k} \geq \Delta W \alpha^m$ for some $m \in \{k+1, ..., n\}$.

Let

$$t_{i1}^* = \begin{cases} \frac{W_0}{k} - \Delta W \alpha^i \text{ for all } i = 1, ..., m^* \\ 0 \text{ for all } i = m^* + 1, ..., n \end{cases}$$

This solution is optimal since the attached budget is $v^*(k, \boldsymbol{\alpha}) = \frac{m^*}{k} W_0 - \Delta W \sum_{i=1}^{m^*}$ and $\frac{W_0}{k} < \Delta W \alpha^i$ for all $i = m^* + 1, ..., n$. Further, it satisfies, by construction, the conditions of the proposition.

Case 2: $\Delta W \alpha^k \leq \frac{W_0}{k} < \Delta W \alpha^{k+1}$.

Let

$$t_{i1}^* = \begin{cases} \frac{W_0}{k} - \Delta W \alpha^i \text{ for all } i = 1, ..., k \\ 0 \text{ for all } i = k+1, ..., n \end{cases}$$

This solution is optimal since the attached budget is $v^*(k, \boldsymbol{\alpha}) = \frac{m^*}{k} W_0 - \Delta W \sum_{i=1}^{m^*}$ and $\frac{W_0}{k} < \Delta W \alpha^i$ for all $i = k+1, ..., n$. Further, it satisfies, by construction, the conditions of the proposition.

Case 3: $\frac{W_0}{k} < \Delta W \alpha^k$ and $W_0 > \Delta W \sum_{l=1}^{k} \alpha^l$

From what precedes, the budget attached to an optimal offer is equal to $W_0 - \Delta W \sum_{i=1}^{k} \alpha^i$. Let $m^* \leq k - 1$ and define t_1^* as follows:

$$t_{i1}^* = \begin{cases} x^* - \Delta W \alpha^i \text{ for all } i = 1, ..., m^* \\ 0 \text{ for all } i = m^* + 1, ..., n \end{cases},$$

where

$$x^* \equiv \frac{W_0 - \Delta W \sum_{l=m^*+1}^{k} \alpha^l}{m^*}.$$

It is straightforward to check that $\sum_{1 \leq i \leq n} t_{i1}^* = v^*(k, \boldsymbol{\alpha})$. This solution which satisfies the conditions of the proposition is optimal if:

$$x^* \geq \Delta W \alpha^{m^*} \text{ and } x^* \leq \Delta W \alpha^{m^*+1},$$

i.e.

$$\Delta W \sum_{l=m^*+1}^{k} \alpha^l + \Delta W m^* \alpha^{m^*+1} \geq W_0 \geq \Delta W \sum_{l=m^*+1}^{k} \alpha^l + \Delta W m^* \alpha^{m^*}.$$

Since the function $\Delta W \sum_{l=m+1}^{k} \alpha^l + \Delta W m \alpha^m$ is increasing in m and takes the value $k \Delta W \alpha^k$ when $m = k-1$, we deduce from the inequalities $\frac{W_0}{k} < \Delta W \alpha^k$ and $W_0 > \Delta W \sum_{l=1}^{k} \alpha^l$ that there is a largest value of $m^* \geq 1$ such that the above inequalities hold.

Case 4: $W_0 \leq \Delta W \sum_{l=1}^{k} \alpha^l$.

In such case $v^*(k, \boldsymbol{\alpha}) = \mathbf{0}$ i.e. $m^* = 0$ $\square$

In proposition 4 we have exhibited an optimal offer such that all the legislators bribed by lobby 1 end up with an identical net payoff (Groseclose and Snyder call these strategies leveling strategies). There may exist other optimal strategies, even when $m^* > 0$. For example, in the case where $k = 3$, $\alpha^1 = \alpha^2 = \alpha^3 = 0$, $\alpha^4 = \alpha^5 = \beta$ with $W_0 < \beta \Delta W$, we derive easily that any offer $t_1 \in \mathbb{R}_+^5$ such that $\sum_{1 \leq i \leq 3} t_1^i = W_0$ and $t_1^4 = t_1^5 = 0$ is optimal.

Let us examine how proposition 5 answers the three questions formulated at the beginning of the subsection. Note first that if:

$$\frac{W_0}{k} > \Delta W \alpha_n, \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k \alpha^n},$$

then lobby 1's cheapest offer consists in bribing all the legislators. The corresponding cost is $\frac{n W_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l$ and lobby 1 will therefore find it profitable to do so iff:

$$W_1 \geq \frac{n W_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l, \text{ i.e. } \frac{W_1}{W_0} \geq \frac{\left(\frac{2k-1}{k}\right) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i},$$

i.e. inequality (3) since $\mu^*(\mathcal{B}) = 2 - \frac{1}{k}$. For lobby 1 to bribe at least a majority of legislators, it is necessary and sufficient that:

$$\frac{W_0}{k} > \Delta W \alpha^k, \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k \alpha^k}.$$

It will bribe a minimal majority if:

$$1 + \frac{1}{k \alpha^{k+1}} \leq \frac{W_1}{W_0} < 1 + \frac{1}{k \alpha^k}.$$

The corresponding cost is $W_0 - \Delta W \sum_{l=1}^{k} \alpha^l$ and lobby 1 will therefore always find it profitable to do so. At the other extreme, if:

$$W_0 < \Delta W \sum_{l=1}^{k} \alpha^l,$$

then, lobby 1 does not offer any bribe.

While derived under quite different assumptions, proposition 4 shares some common features with Banks's main result[53]. He assumes that there is a majority of legislators who have an intrinsic preference for the status quo: without lobbying, the reform is rejected by the legislature. We assume instead that legislators' intrinsic preferences are unanimously oriented towards the reform side. Under his assumption, lobby 0 has a double advantage: it is second mover in the game and has a majority of partisans, while, in our case, the second advantage is entirely eliminated. Both Banks and our analysis prove the optimality of levelling schedules, but his coalition is nonflooded while it is flooded in our case. We provide a complete characterization of the optimal size m^*, while Banks provides necessary and sufficient conditions for this coalition to be minimal winning on the one hand and universalistic on the other hand[54].

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[53]The comparison with Groseclose and Snyder is more difficult as most of the analysis in their 1996 paper assumes a continuum of voters.

[54]Under his assumptions, lobby 1 cannot consider a submajority coalition. Further, in his framework, a full characterization of m^* seems out of reach.

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